Population dynamics, chaos and zombies

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Modelling populations sizes

Think about the number of bacteria in a petri dish:

A scientist records the number of bacteria in the dish each hour:

$$2000, \; 4000, \; 8000, \; 16000, \ldots$$

Modelling population sizes

We can represent how the size of a population changes with a **discrete time-series**.

This is just a sequence of values:

$$x_0, x_1, x_2, x_3, \ldots$$

x_0 is the starting value. The others could be values after 1, 2, 3 days/months/seasons/years/generations.

Our population of bacteria doubles every hour n:

$$x_0 = 2000, \;\; x_1 = 4000, \;\; x_2 = 8000, \;\; x_3 = 16000, \ldots$$

Malthusian growth

This rule can be represented by a recurrence equation called **Malthusian growth**, where we find the population x_{n+1} at the next time-step by multiplying the previous value x_n by a constant r:

Malthusian growth

$$x_{n+1} = r\, x_n$$

In the bacteria example we have $x_0 = 2000$ and $r = 2$, so:

$$\begin{aligned} x_1 &= 2x_0 = 2 \times 2000 = 4000 \\ x_2 &= 2x_1 = 2 \times 4000 = 8000 \\ x_3 &= 2x_2 = 2 \times 8000 = 16000 \ldots \end{aligned}$$

Malthusian growth

Malthusian growth

$$x_{n+1} = r\, x_n$$

So the population grows if $r = 2$. In general, what will happen if...

- $0 < r < 1$
- $r = 1$
- $r > 1$

Malthusian growth

Malthusian growth

$$x_{n+1} = r\, x_n$$

So the population grows if $r = 2$. In general, what will happen if. . .

- $0 < r < 1$ **Decline to zero**
- $r = 1$ **Equilibrium**
- $r > 1$ **Infinite growth**

Malthusian growth

So in the Malthusian model, populations either get larger forever (or die out, or stay exactly the same).

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But real populations can't grow forever due to limited resources...

Logistic growth

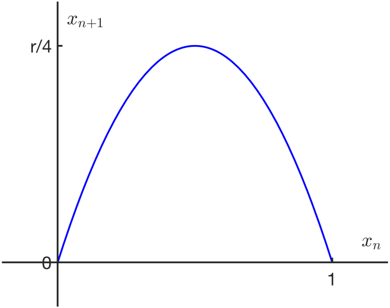
Let's modify our growth model to introduce a carrying capacity:

Logistic growth

$$x_{n+1} = rx_n(1 - x_n)$$

x_n could represent the human population density of a region (max. value of 1) every generation.

If x_n gets too big, the next step will be a decrease:



How does the outcome depend on the growth rate r?

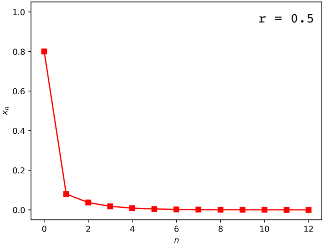
Now it's less obvious to predict what the overall outcome will be.

Computers are really good at performing simple calculations (like the logistic growth recurrence relation) many times:

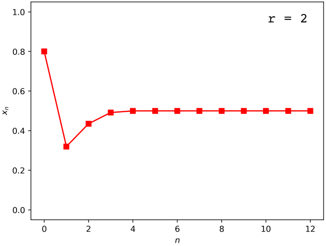
```
for step in range(num_iterations):
    x = max(0, r * x * (1.0 - x))
    output_x[step + 1] = x
```

Let's see what happens when we start with $x_0 = 0.8$,and try different values of r...

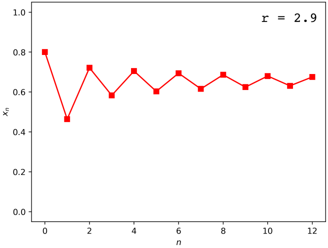
How does the outcome depend on the growth rate r?



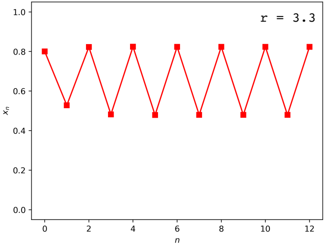
(a) $r = 0.5$



(b) $r = 2$

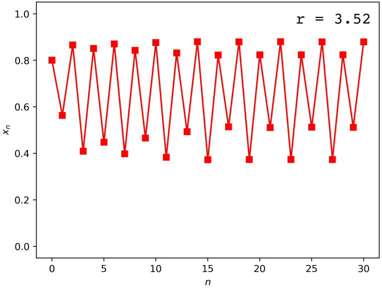


(c) $r = 2.9$

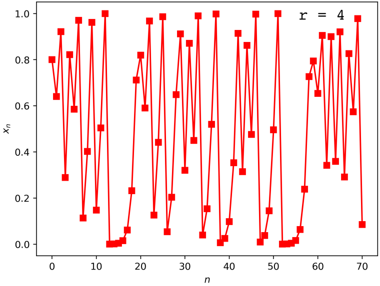


(d) $r = 3.3$

How does the outcome depend on the growth rate r?

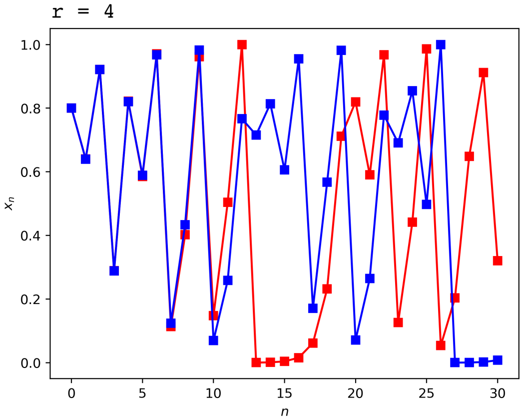


(e) $r = 3.52$



(f) $r = 4$

How does the outcome depend on the starting value?

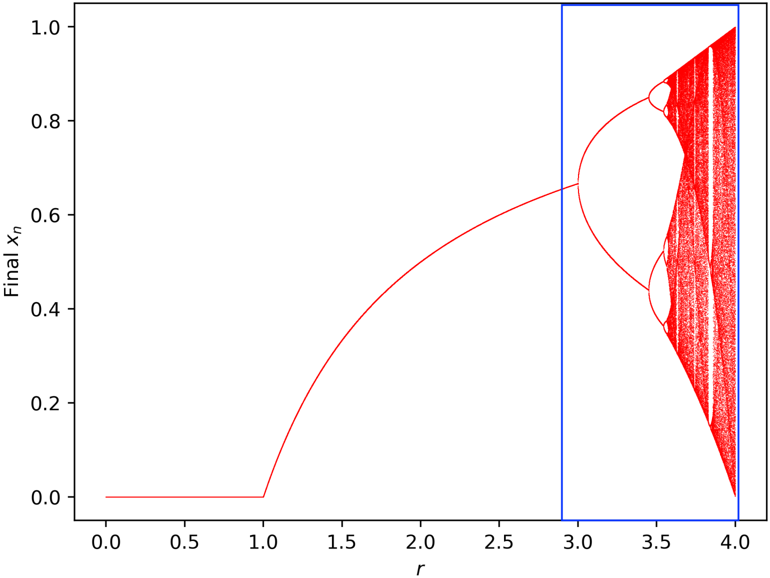


$x_0 = 0.8$

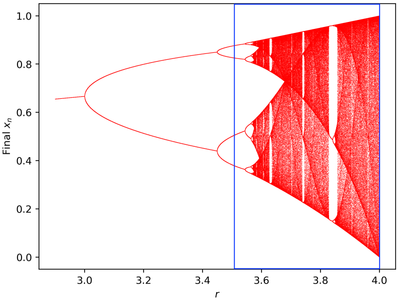
$x_0 = 0.8001$

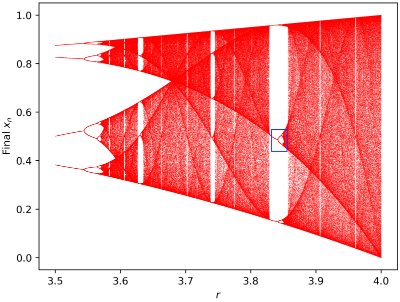
With chaos, small differences matter!

Feigenbaum diagrams

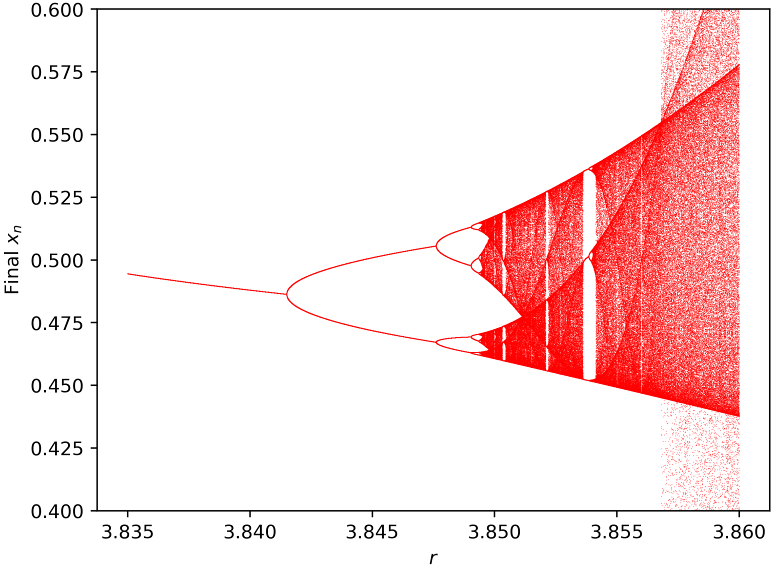


Feigenbaum diagrams





Feigenbaum diagrams



Fractals

Self-similar patterns that repeat at different scales.

- Snowflakes
- Fern leaves
- Coastlines

ZOMBIES


EVER FELT LIKE YOU WERE SURROUNDED BY ZOMBIES?
SHAUN OF THE DEAD
A ROMANTIC COMEDY. WITH ZOMBIES.

ZOMBIES

30 seconds:

- Turn to the person next to you.
- Sheffield is attacked by zombies!
- What's your plan to survive?

Two-species population models

Let's introduce a second population of zombies to our mathematical model:

x_n is the number of humans at time-step n

y_n is the number of zombies at time-step n

How can we represent how the size of these populations might behave over time?

Two-species population models: Version 1

$$x_{n+1} = rx_n(1 - x_n) - cx_n y_n$$

$$y_{n+1} = y_n + \lambda c x_n y_n$$

Two-species population models: Version 1

Next number of humans: $x_{n+1} = rx_n(1 - x_n) - cx_ny_n$

Next number of zombies: $y_{n+1} = y_n + \lambda cx_ny_n$

- Humans reproduce (with rate r) as before...
- ...but some are killed by each zombie (with rate c)!
- A fraction $0 < \lambda < 1$ of whom become new zombies...
- ...joining the existing ranks of the undead.

Without conducting any simulations, can we figure out what must eventually happen in this model?

Two-species population models: Version 2

To make things more realistic (and give humanity a chance!), perhaps a fraction $0 < d < 1$ of the zombies die at each step:

$$x_{n+1} = rx_n(1 - x_n) - cx_n y_n$$

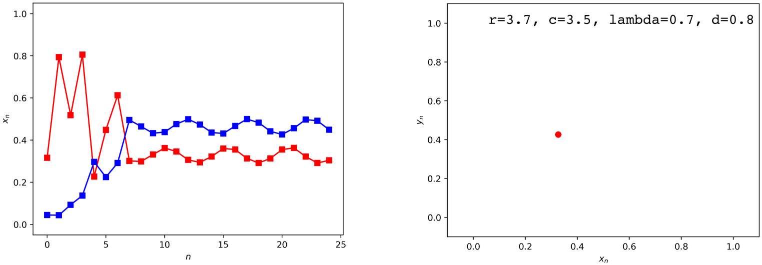
$$y_{n+1} = (1 - d)y_n + \lambda c x_n y_n$$

To predict the outcome, we need to know/choose values of r, c, λ, d and starting values of x_0 and y_0.

More "realistic" models are more complicated.

What kinds of behaviour are possible in 2-D?

Like the logistic map, there can be fixed points - where the population of humans and zombies remains the same forever:

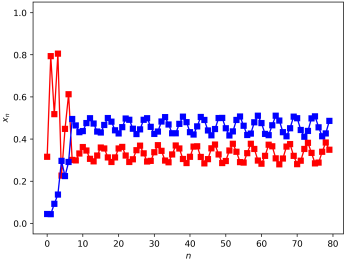


(a) Time-series

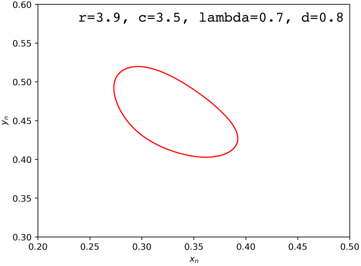
(b) Final value

What kinds of behaviour are possible in 2-D?

Increasing the human growth rate r can give a new kind of cyclic behaviour (around the now “unstable” fixed point):



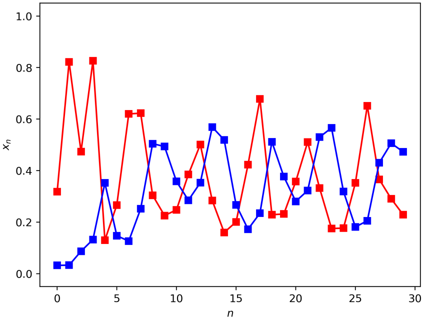
(a) Time-series



(b) Final value

What kinds of behaviour are possible in 2-D?

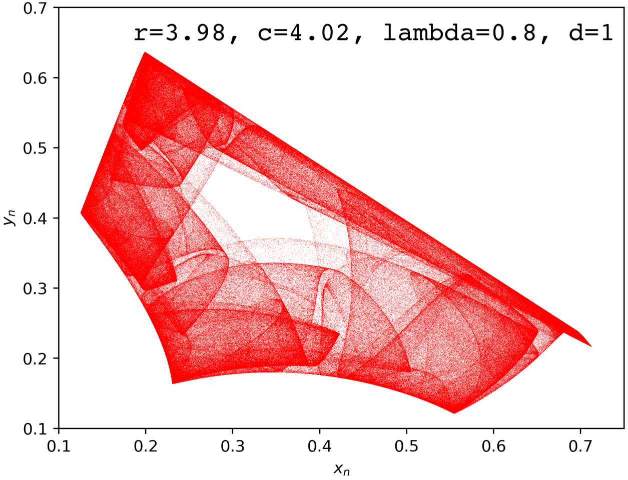
Like in the one-dimensional logistic map, we can also choose values that result in a chaotic sequence of points:



These paired values of human and zombie populations will **never** exactly repeat. What does it look like if we plot them?

What kinds of behaviour are possible in 2-D?

This is a **strange attractor**, plotting 5,000,000 points!



$$x_{n+1} = 3.98x_n(1 - x_n) - 4.02x_n y_n$$

$$y_{n+1} = 3.216x_n y_n$$

All from these two equations! ↑

Humans and zombies

- Chaotic behaviour, fractal images, population biology - all interconnected!
- Can you think of other ways to modify the model?
- Perhaps the death rate of the zombies should depend on the humans - but are humans better at fighting when there are more of us, or when we are more desperate?
- Modelling social behaviour (like epidemic modelling for predicting Covid-19 case numbers) is not as clear-cut as representing physical laws. We might need to make choices when making our model!