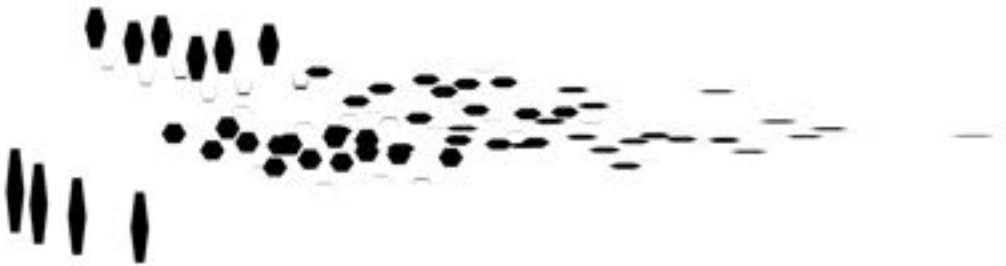
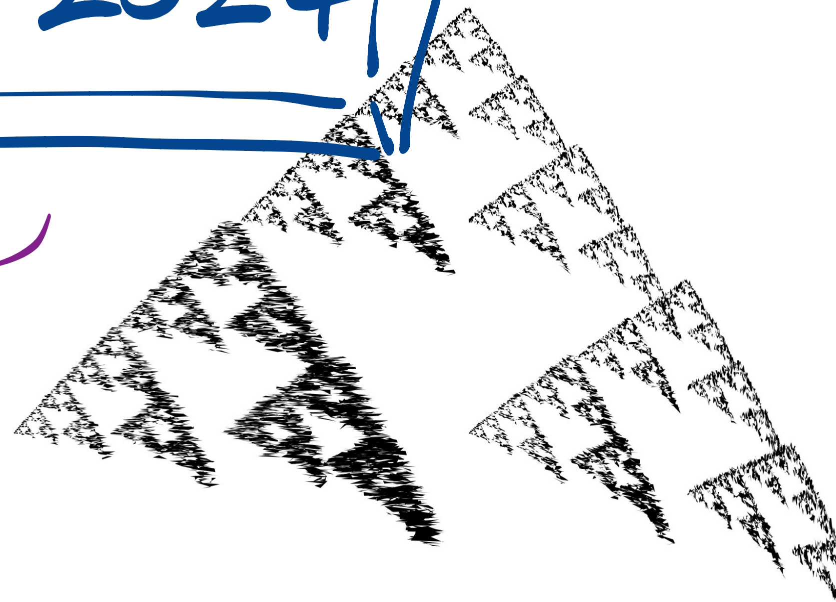
Introduction to Fractal Geometry

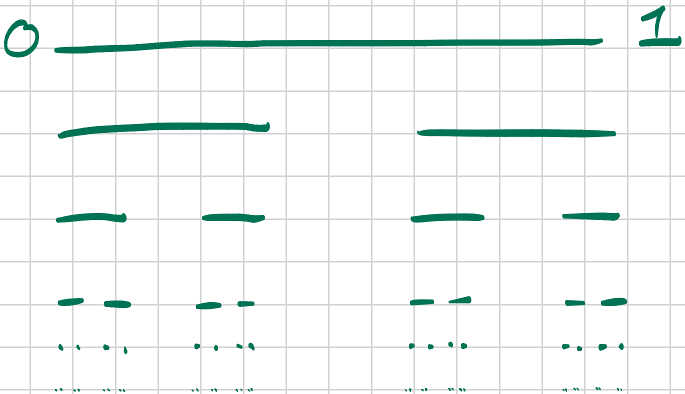
$\boxed{\text{SUMS 2024}}$

GMJ

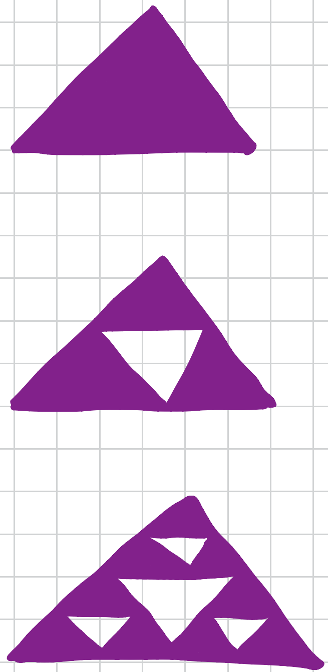




1 Fractal Objects



Middle-third Cantor Set



Sierpinski triangle

Mandelbrot Set

colour = rate at which $z_n \to +\infty$, where $z_0 = 0$, $z_{n+1} = z_n^2 + c$

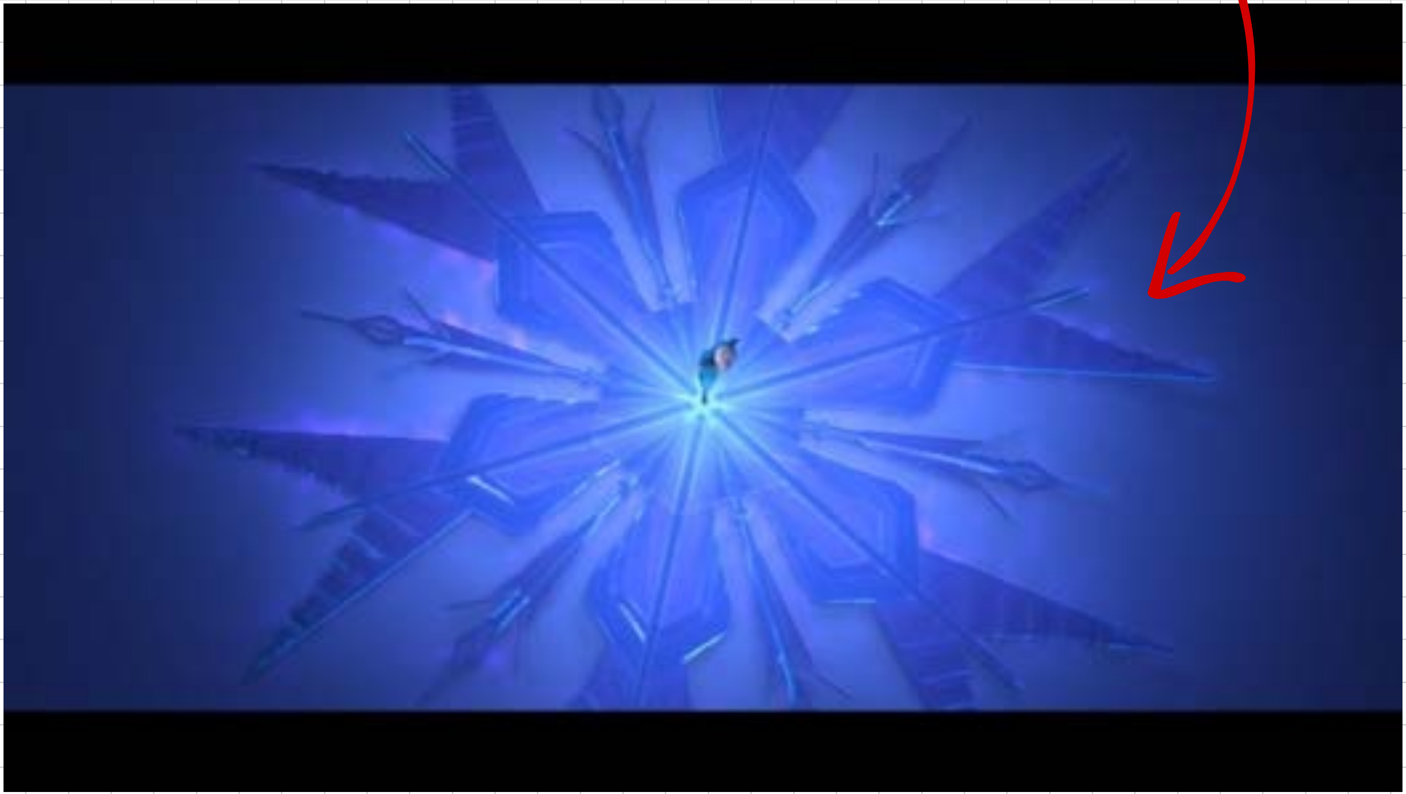
② Fractal Experts

Benoit B. Mandelbrot

Prof Kenneth Falconer (Regius Professor of Mathematics, St.A)

Queen Elsa

"Let it go" — fractal snowflakes

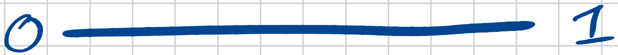


③ Koch curve

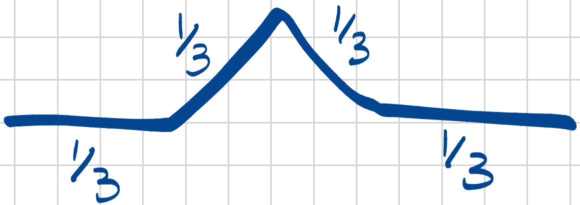
Define the fractal as $F = \bigcap_{k=0}^{\infty} E_k$

Let $L(k)$ be the length of E_k, and $A(k)$ the area enclosed by it and $y=0$.

E_0



E_1



E_2

$L(0) = 1$, $\quad A(0) = 0$

$L(1) = 4/3 \qquad A(1) = \frac{\sqrt{3}}{36}$

$L(2) = \left(\frac{4}{3}\right)^2 \qquad A(2) = \frac{\sqrt{3}}{36}\left(1 + \frac{4}{9}\right)$

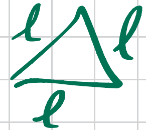
$\vdots \qquad \vdots$

$L(k) = (4/3)^k \qquad A(k) = \frac{\sqrt{3}}{36} \sum_{i=0}^{k} \left(\frac{4}{9}\right)^i$

$\longrightarrow +\infty$

$\longrightarrow \frac{\sqrt{3}}{36}\left(\frac{1}{1-\frac{4}{9}}\right) = \frac{\sqrt{3}}{20}$

Area of



is $\frac{\sqrt{3}}{4}\ell^2$

④ Natural fractals

self-similar

How long is this coast?

(Will return to this!)

What is the Scale?

branching structure

fine detail

5 What is a fractal?

all of them!

- fine structure / detail at multiple scales

koch curve

- too "irregular" for length, area etc. to satisfy

and scale-invariance

fern

- self-similarity (small parts look like the whole)
- usually also simple in someway (eg. recursive definition)

cantor set, koch curve

- "fractal dimension" greater than $\underbrace{\text{topological dimension}}$

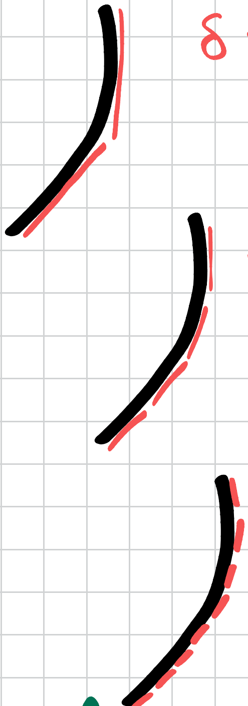
?

0 = disconnected points
1 = line
2 = area etc.

(Falconer, "Fractal Geometry" 2014)

6 Dimension

$$N \approx \delta^{-d}$$

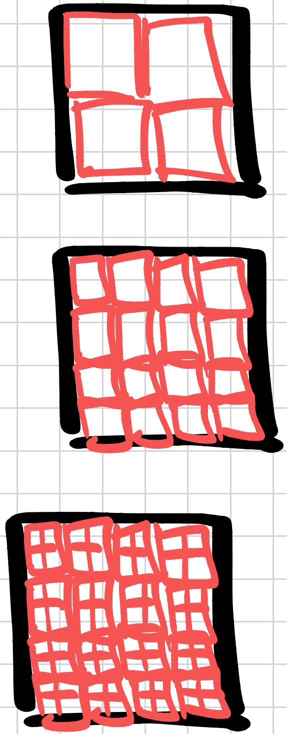


$\delta = 1/2$
$N = 2$

$\delta = 1/4$
$N = 4$

$\delta = \frac{1}{8}$
$N = 8$

$d = 1$

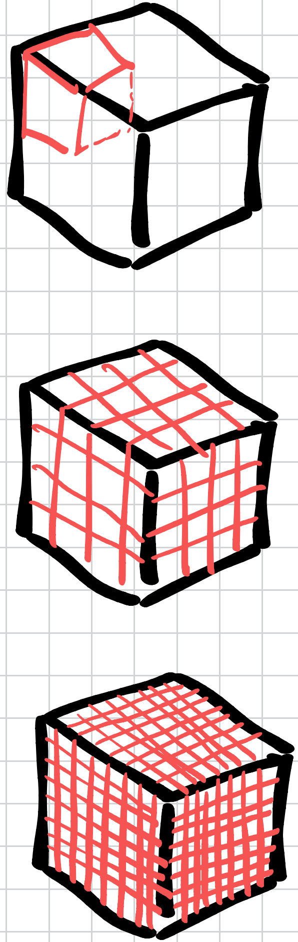


$\delta = \frac{1}{2}$
$N = 4$

$\delta = 1/4$
$N = 16$

$\delta = \frac{1}{8}$
$N = 64$

$d = 2$



$\delta = \frac{1}{2}$
$N = 8$

$\delta = \frac{1}{4}$
$N = 64$

$\delta = \frac{1}{8}$
$N = 512$

$d = 3$

7 Box-counting Dimension

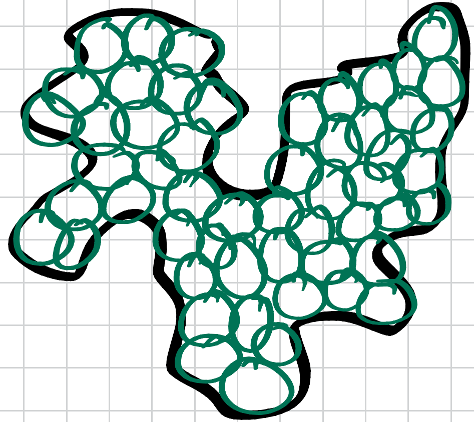
"How many boxes of size δ do we need to cover the set?"

Let $F \subseteq \mathbb{R}^n$ be non-empty and bounded.

For $\delta > 0$, $\{U_i\}_i$ is a δ-cover of F if:

$$F \subseteq \bigcup_i U_i \quad \text{and} \quad 0 \leq |U_i| \leq \delta \;\; \forall i$$

(they cover F) (they are each δ-wide)



Let $N_\delta(F)$ be the least number of sets in a δ-cover of F.

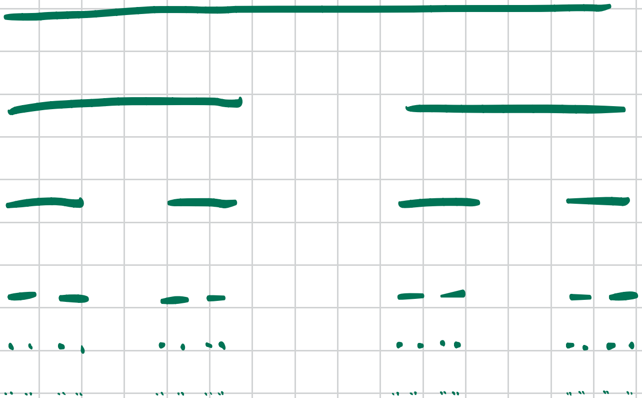
Then

$$\boxed{\dim_B(F) = \lim_{\delta \to 0} \frac{\log(N_\delta(F))}{\log(\delta^{-1})}}$$

if this limit exists.

diameter $\leq \delta$

Example: Middle-third Cantor Set



$E_0 = [0,1]$

$E_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$

Level E_k consists of 2^k intervals of length 3^{-k}!

$$F = \bigcap_{n=0}^{\infty} E_n \subseteq E_k \text{ for any } k \in \mathbb{N}_0$$

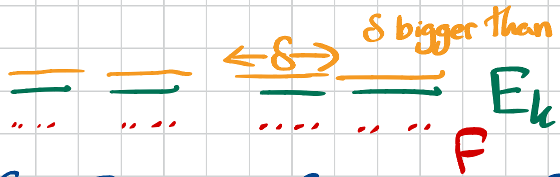
Idea:

$\boxed{\text{Obtain upper and lower bounds for the minimum } \delta\text{-cover at a given level. Then take the limit as } \delta \to 0}$

Upper bound: ("How many sets would suffice?")

Let $0<\delta<1$, then $\exists$ unique k s.t.

$$3^{-k} < \delta \le 3^{-k+1}$$



So E_k is a δ-cover of F, with 2^k sets. Hence $N_\delta(F) \le 2^k$

$$\therefore \frac{\log(N_\delta(F))}{-\log(\delta)} \le \frac{\log(2^k)}{-\log(3^{-k+1})}$$

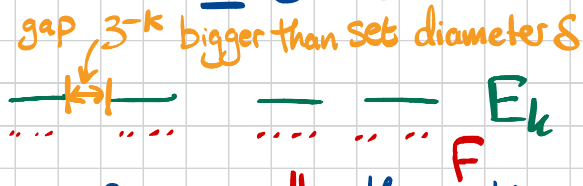
$$= \frac{k\log(2)}{(k-1)\log(3)} \longrightarrow \frac{\log(2)}{\log(3)} \text{ as } \delta \to 0$$

$(\Rightarrow k \to +\infty)$

Lower bound: ("How many MUST I need?")

Let $0<\delta<1$, then $\exists$ unique k s.t.

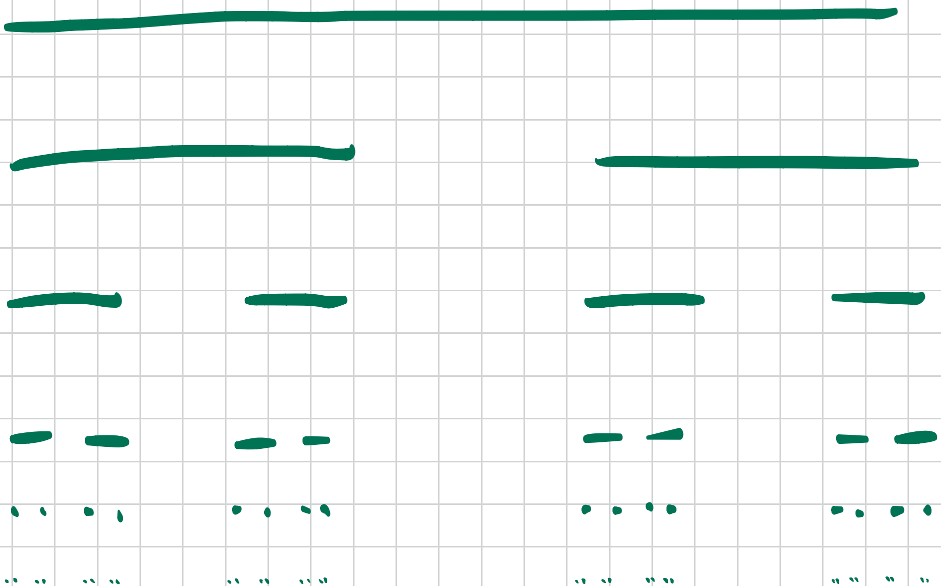
$$3^{-k-1} \le \delta < 3^{-k}$$



Since δ is smaller than the gap, any set in a δ-cover intersects only one interval in E_k – so at least one is required for each: $N_\delta(F) \ge 2^k$

$$\therefore \frac{\log(N_\delta(F))}{-\log(\delta)} \ge \frac{\log(2^k)}{-\log(3^{-k-1})} = \frac{k\log(2)}{(k+1)\log(3)}$$

$$\longrightarrow \frac{\log(2)}{\log(3)} \text{ as } \delta \to 0.$$



Hence in the limit $\delta \to 0$,

$$\frac{\log(2)}{\log(3)} \leq \dim_B(F) \leq \frac{\log(2)}{\log(3)}$$

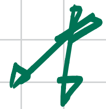
So the box-counting dimension is:

$$\dim_B(F) = \frac{\log(2)}{\log(3)} \approx 0.631$$

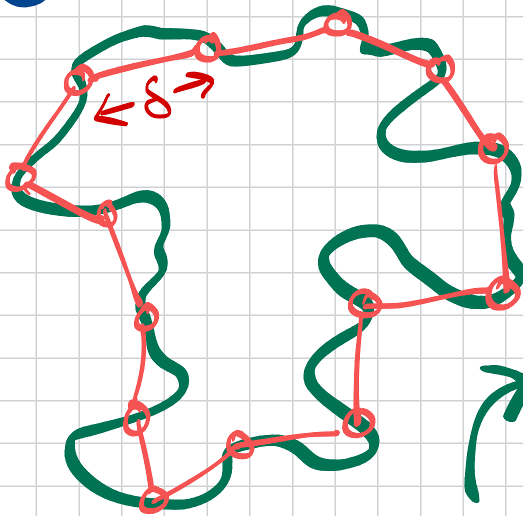
(11) Divider dimension

(There are many related definitions of "dimension"!)

1. Given a curve C, set "divider" width δ and measure how many $M_\delta(C)$ points δ-apart on the curve.

yardstick or compass  for a map

2. Repeat for smaller δ.



3. Estimate log-log gradient:

$$\dim_D(C) = \lim_{\delta \to 0} \frac{\log(M_\delta(C))}{\log(\delta^{-1})}$$

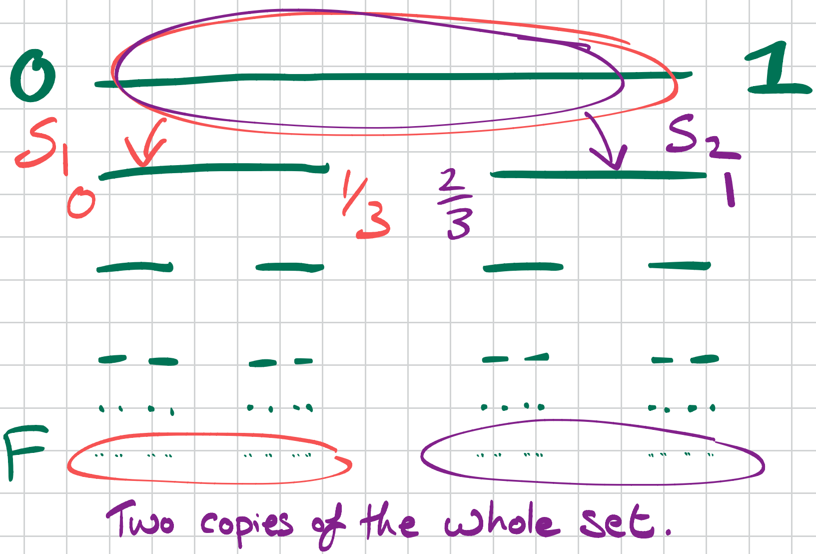
The Divider Dimension of the Irish Coast

The simplest way to define a fractal is as an object which is self similar across all scales of magnification. That is, a fractal is an object which, no matter how closely we 'zoom in' on it, will look the same, or similar to the whole.

1.23 for Ireland (but higher on the West coast)

(12) Iterated Function Systems

Alternatively, the non-empty compact invariant set associated with a system of contracting functions: $\boxed{F = \bigcup_{i=1}^{N} S_i(F)}$



For middle-third Cantor set:

$$S_1(x) = \frac{1}{3}x, \quad S_2(x) = \frac{1}{3}x + \frac{2}{3}$$

Then F is uniquely such that:

$$F = S_1(F) \cup S_2(F)$$

Let

$S: \mathbb{R}^n \to \mathbb{R}^n$ be

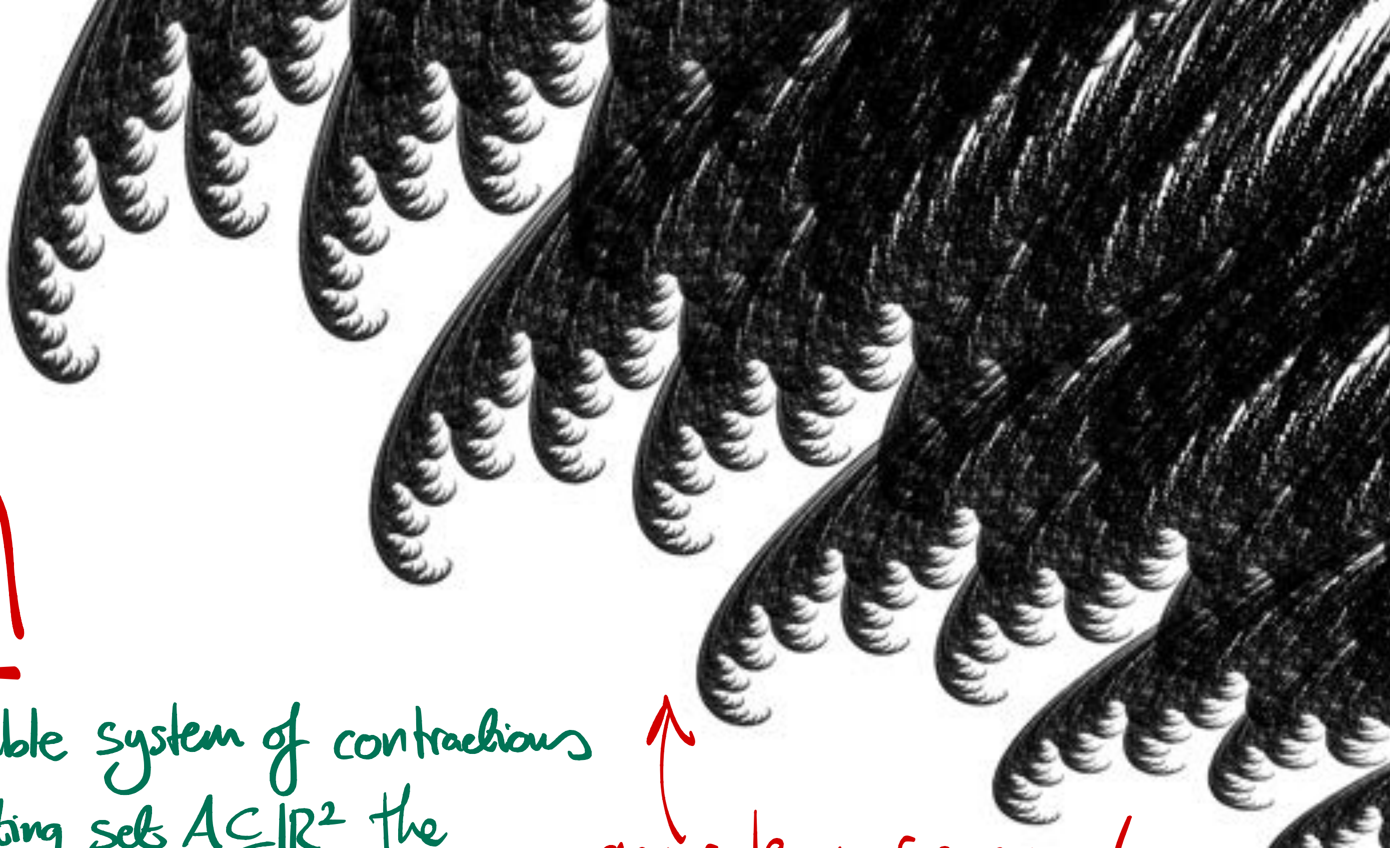
$S(X) = \bigcup_{i=1}^{N} S_i(X)$

Then for suitable

$A \subseteq \mathbb{R}^n$,

$$\boxed{F = \bigcap_{k=1}^{\infty} S^k(A)}$$

So choosing a suitable system of contractions $S_1, \dots, S_N$ and starting sets $A \subseteq \mathbb{R}^2$ the system converges to a fractal.



generate your own at:

github.com/gavinabernethy/gifs_generator

References

key textbook

Falconer, K. (2014) *Fractal geometry: mathematical foundations and applications.* John Wiley & Sons.

McCartney, M. (2020). The area, centroid and volume of revolution of the Koch curve. *International Journal of Mathematical Education in Science and Technology*, 52(5), 782–786.