Limiting probability of solitude in infinite prayer groups

Gavin M Abernethy

November 29, 2024

Abstract

"*And the Lord added to the church daily those who were being saved*" (Acts of the Apostles, 2:47b; NKJV). Assuming that the number of humans to be added is a positive integer and strictly greater than the number of daily deaths, such that net daily growth is at least as great as 1, this presents a range of logistical challenges associated with congregational membership tending to infinity. In particular, how will intercessory prayer be managed at the hypothetical denouement of a meeting of your infinitely-large small group? Consider a group of n members, such that one person begins the process by selecting another at random to pray for. That person subsequently selects a member (who could be the previous) to pray for, and so on. No member of the group may pray or be prayed for more than once. We show that in a group of n, the probability of there being one person left over at the end who has no choice but to pray for themselves converges to e^{-1} as $n \to +\infty$.

The evening Bible study draws to a close, and Alice, Bob, Caoimhe and Derek share prayer requests. Starting off, Alice prays for Bob, then Bob prays for Caoimhe, and then Caoimhe prays for Alice. A difficult silence follows, as everyone slowly realises that Derek has no choice but to pray for himself, lest an entire second round of prayers need begin. So how likely is this situation to arise in a group of n individuals, if prayer occurs in sequence and with a randomly-selected target from the available pool? This can be formulated as a combinatoric problem, and an immediate solution obtained by means of derangements: given n individuals, fix precisely one individual to have to pray for themselves. Then there are $!(n-1)$ ways that all of the other individuals can be permuted with each other so that none of them are matched with themselves - leaving this fixed one isolated. This occurs as a fraction of the $(n-1)!$ overall ways of permuting them, so that the probability of this occurrence is $\frac{!(n-1)}{(n-1)!}$.

However, what if one was not aware of existing results concerning derangements[1]? We shall show how to arrive at this solution by considering the problem as a probabilistic graph theory question - namely, how many ways are there to sequentially draw edges between n vertices to form a collection of disconnected cyclic directed graphs where all vertices have degree 2 and there are no loops except for a single isolated vertex? (Note that we *do* require the restriction of drawing the edges in sequence.)

In general, consider a set of n vertices, and we let $P(n)$ denote the probability of a lonely member arising from a group of this size. Thus, trivially $P(1) = 1$ and $P(2) = 0$ since a prayer pair will experience no issues, then $P(3) = \frac{1}{2}$ since in a triplet after the first person prays there is a 1 in 2 chance of their target praying for that first person, completing the 2-cycle and leaving the third individual isolated. Clearly then this is not simply the proportion of unordered ways to choose groups (as there are three divisions into $2-1$ versus only one way to select all three), however there are 3! ordered ways to choose a group of 3 and similarly to choose a group of 2 followed by a group of 1. $P(4) = \frac{1}{3}$ since the only way that an isolated vertex occurs is if second individual chooses someone other than the first (probability $\frac{2}{3}$) who then closes the 3-cycle by selecting the initiating individual with probability $\frac{1}{2}$.

[1](and consequently spent a very long time deriving the results presented hereafter)

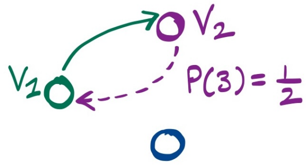

Figure 1: $n = 3$ system: the selection of the first individual v_1, and their choice of target v_2 is without loss of generality. The second individual v_2 must choose between two options, each with probability $\frac{1}{2}$, one of which results in a final isolated individual.

In the general case, one vertex v_1 is (arbitrarily) selected to start the process, they select a different member v_2 from their $n-1$ options with uniform probability. From the perspective of v_2, everyone *they* can see remains unchosen, so they also have $n-1$ possible targets to choose from. Observe that precisely one of these choices (selecting v_1) will close the 2-cycle, leaving the process to start again for $n-2$ members, and we call this Scenario I.

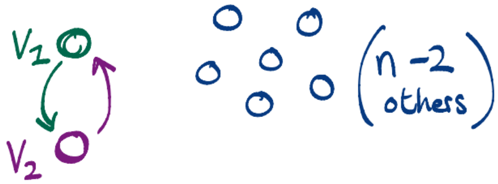

Figure 2: Scenario I

Alternatively (Scenario II), v_2 selects a new person with probability $\frac{n-2}{n-1}$. But in this case, observe that v_2 has already been effectively eliminated from the process having both chosen and been chosen. Thus we can collapse the chain $v_1 v_2 v_3$ into a single amalgamated vertex v'. Then the situation is also similar to that of a system of $n-2$ individuals, except that the loop $v' \sim v'$ is permitted.

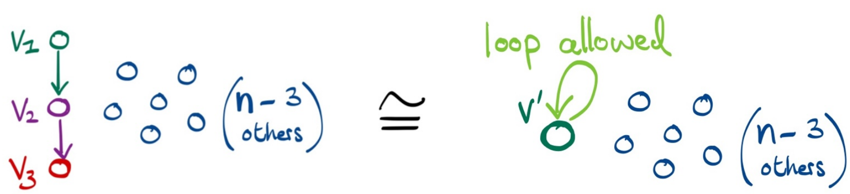

Figure 3: Scenario II

In this second scenario, then there is probability $\frac{1}{n-2}$ of v_3 selecting v_1 to complete the cycle and removing v' so that the process resets with the remaining $n-3$ individuals. Alternatively (probability $\frac{n-3}{n-2}$) they select a new individual v_4 and we add that vertex to the chain so that $v' = v_1 v_2 v_3 v_4$. Then we repeat the argument above, so that:

$$P(n) = \frac{1}{n-1} P(n-2) + \frac{n-2}{n-1} \left\{ \frac{1}{n-2} P(n-3) + \frac{n-3}{n-2} \left\{ \frac{1}{n-3} P(n-4) + \frac{n-4}{n-3} \left\{ \dots \right\} \right\} \right\} \tag{1}$$

This summation telescopes to:

$$
\begin{aligned}
P(n) &= \frac{1}{n-1}\big\{P(n-2)+P(n-3)+\cdots+P(3)+P(2)+P(1)\big\} &(2)\\
&= \frac{1}{n-1}\sum_{k=1}^{n-2}P(k) &(3)\\
&= \frac{1}{n-1}\left\{P(n-2)+\sum_{k=1}^{n-3}P(k)\right\} &(4)\\
&= \frac{1}{n-1}P(n-2)+\frac{n-2}{n-1}\left\{\frac{1}{n-2}\sum_{k=1}^{n-3}P(k)\right\} &(5)\\
&= \frac{1}{n-1}P(n-2)+\frac{n-2}{n-1}\left\{\frac{1}{(n-1)-1}\sum_{k=1}^{(n-1)-2}P(k)\right\} &(6)
\end{aligned}
$$

Or alternatively, simply notice from the original expansion that:

$$P(n) = \frac{1}{n-1}P(n-2)+\frac{n-2}{n-1}\Bigg\{\underbrace{\frac{1}{n-2}P(n-3)+\frac{n-3}{n-2}\left\{\frac{1}{n-3}P(n-4)+\frac{n-4}{n-3}\left\{\ldots\right\}\right\}}_{=P(n-1)}\Bigg\} \quad (7)$$

So that either way, we derive the following second-order linear homogeneous recurrence relation with non-constant coefficients:

$$P(n) = \frac{1}{n-1}P(n-2)+\frac{n-2}{n-1}P(n-1) \quad (8)$$

Thus the sequence given by $p_n = P(n)$ for $n \geq 1$ is:

$$(p_n)_{n=1}^{\infty} = \left(1, 0, \frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{11}{30}, \ldots\right) \quad (9)$$

which when plotted has the appearance of an alternating convergent sequence.

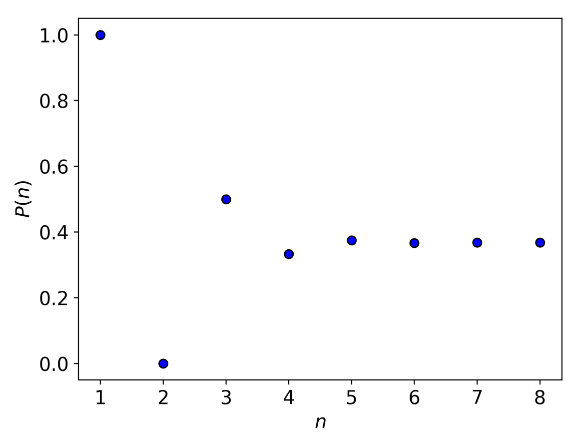

Figure 4: Sequence $(p_n)_n$

Now, let $\Delta P(n) = P(n+1) - P(n)$, and observe that:

$$\begin{aligned} \Delta P(n) &= \frac{1}{n}P(n-1) + \frac{n-1}{n}P(n) - P(n) && (10) \\ &= \frac{1}{n}P(n-1) - \frac{1}{n}P(n) && (11) \\ &= -\frac{1}{n}\Delta P(n-1) && (12) \end{aligned}$$

Then recall that $P(1) = 1$ and $P(2) = 0$ so that $\Delta P(1) = -1$. Thus we can establish by induction that our second-order recurrence yields:

$$\begin{aligned} P(n) &= P(n-1) + \left(\frac{-1}{n-1}\right)\left(\frac{-1}{n-2}\right)\left(\frac{-1}{n-3}\right)\cdots\left(\frac{-1}{3}\right)\left(\frac{-1}{2}\right)\Delta P(1) && (13) \\ &= P(n-1) + \left(\frac{-1}{n-1}\right)\left(\frac{-1}{n-2}\right)\left(\frac{-1}{n-3}\right)\cdots\left(\frac{-1}{3}\right)\left(\frac{-1}{2}\right)\left(\frac{-1}{1}\right) && (14) \\ &= P(n-1) + \frac{(-1)^{n-1}}{(n-1)!} && (15) \end{aligned}$$

So we have formulated our sequence of probabilities as a series:

$$P(n) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(k-1)!} \tag{16}$$

Then by a simple shift of indices:

$$P(n) = \sum_{k=0}^{n-1} \frac{(-1)^{k}}{k!} \tag{17}$$

This partial sum is then related to the number of derangements by $P(n) = \frac{!(n-1)}{(n-1)!}$, so that the probability of one isolated prayer group member is precisely the probability that all of the other members of the group are deranged. We could establish the convergence of this series using a variety of tactics in the analysis of infinite series, for example the alternating series test (since the sequence $b_k = (k!)^{-1} \to 0$ and is decreasing), or by showing that the series is absolutely convergent using the comparison test since:

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^k}{k!}\right| \le \sum_{k=0}^{n-1} \left|\frac{(-1)^k}{k!}\right| = \sum_{k=0}^{n-1} \frac{1}{k!} \le 1 + \sum_{k=0}^{n-2} \frac{1}{2^k} \tag{18}$$

and the standard geometric argument for convergence of $\sum_k \frac{1}{2^k}$ by filling up the unit square.

However, the summation (17) is also well-known as the n^{th} partial sum of the Taylor series expansion for e^x about zero with $x = -1$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{19}$$

so that:

$$\lim_{n\to+\infty} P(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} \tag{20}$$

Thus we may deduce that as the size of the group grows to infinity, the probability of one person being left out will converge to $e^{-1} = 0.367879\ldots$