Functions and Calculus

Contents

1 Week 1

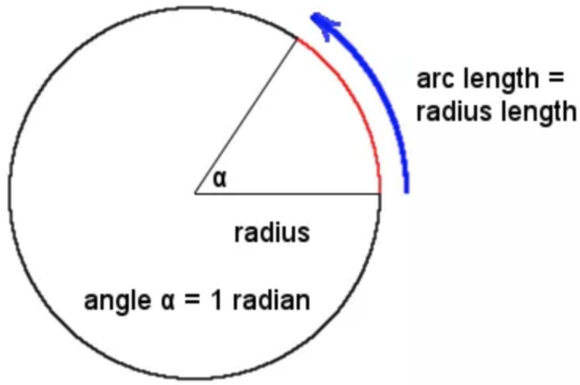
Advice:

- This material requires a solid understanding of algebra and transposition of equations.
- My suggestions for success:
  - There are two key aspects to succeeding in mathematics at this level. You need to **know** what the fundamental theory says, and you need to **understand** how to apply it correctly.
  - To know the first, you should be familiar with using the formula sheet (actually read all of it!), but you should also make your own set of concise notes on what you think is most important about each topic: rules, theorems, special equations. Do your best to commit these to memory. You should also make your own notes during class - do not rely on the online backup or those of a friend. How you interpret and record your own notes is part of how you will learn it.
  - For the second, you *must* frequently practice tutorial questions from all parts of the course. Try a question at least twice before checking the solution if stuck, then make sure you can do it yourself without using the solution. If you don't understand, work through the questions in your lecture notes (this means that you try to answer the question yourself rather than reading the entire solution first). If you still don't understand something, you must talk to me about it as soon as possible.
  - Finally, take part! I will ask you to help me with examples in class - just give it a go! We are all here to learn, and we don't expect to get everything right the first time, as with developing any skill. So think about the problems as we are working through them, ask questions, and try to answer even if you aren't sure.

1.1 Lecture 1: Radians, Trigonometric, Exponential and Logarithmic functions

1.1.1 Radians and Degrees

- Definition of radian: the angle subtended at the centre of a circle by an arc of equal length to the radius of the circle. Denoted by a superscript lowercase "r" or "c".



- Taking an arc to be the whole circumference of length $2\pi r$, the angle of a whole revolution is therefore 2π radians.

- To convert degrees to radians, multiply by $\pi/180$. To convert radians to degrees, multiply by $180/\pi$.

- **Examples**

  Converting between radians and degrees:

$$360^\circ = 2\pi^c,$$

$$180^\circ = \pi^c,$$

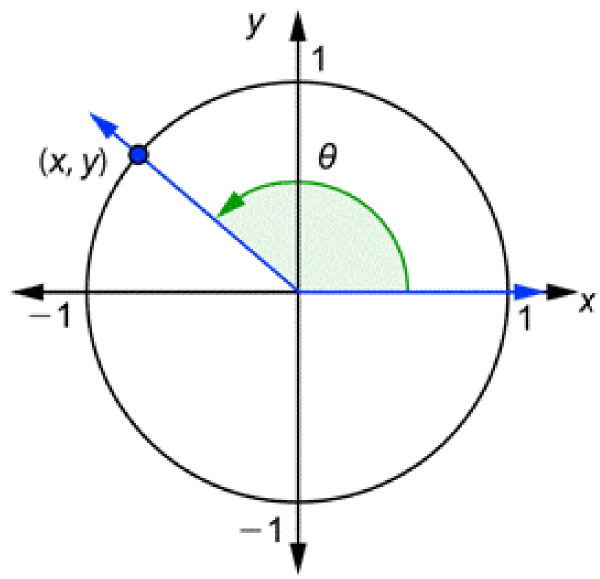
$$1^\circ = \frac{\pi}{180}^c,$$

$$30^\circ = 30 \times \frac{\pi}{180} = \frac{\pi}{6} = 0.5236^c,$$

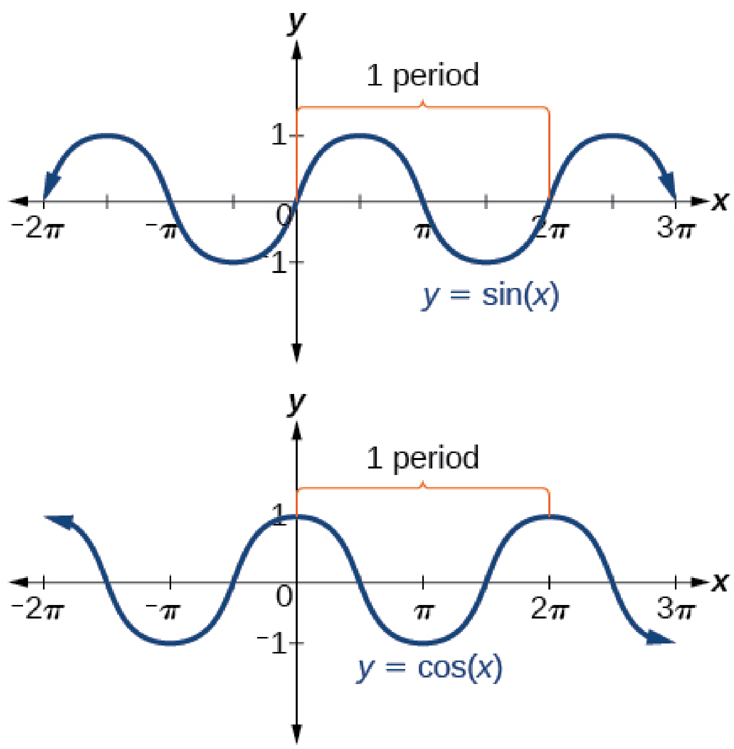
$$\frac{\pi}{4}^c = \frac{\pi}{4} \times \frac{180}{\pi} = 45^\circ$$

1.1.2 Trigonometric functions

- Trigonometric functions involve sin, cos and tan.
- We will study these functions in greater detail later in the course, but as we shall refer to them frequently we need to be familiar with their shape and special properties. The better you memorise their graphs, the easier it will be to work with them.
- Imagine a point on a unit circle centred at the origin. It starts at the position $(1, 0)$, and travels around the circumference of the circle in an anticlockwise direction. The angle θ in radians measures the angle that it has rotated from it's initial position on the positive horizontal axis.



- If we plot how the vertical and horizontal position of the point is going to vary as a function of this angle θ that it has turned through, this gives us the two main trigonometric functions: $\sin(\theta)$ and $\cos(\theta)$. Plotting the graphs of **sine** and **cosine** using radians:



- Both vary between -1 and +1.

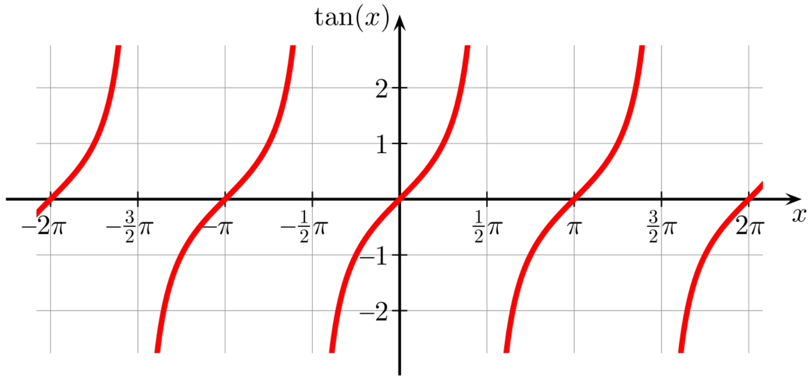
- At every integer multiple of π, the sine function crosses the horizontal axis so that:

$$\sin(n\pi) = 0$$

For any $n \in \mathbb{Z}$

- The **tangent** function is defined as the ratio of the other two functions:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$



- The tan function has asymptotic behaviour at regular intervals.
- **Periodicity:** sine and cosine have period 2π, while tan has period π.
- There are many special connections between sine and cosine.
  - The cosine curve is just the sine curve shifted left by $\pi/2$ radians.
  - Other connections exist, such as the following trigonometric identities (given in the Formula Booklet):

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\sin(2\theta) = 2\sin(A)\cos(\theta)$$

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

These relationships are true for any value of θ, which makes them *identities* which are stronger than equations.

1.1.3 Exponential and Logarithmic functions

- The **exponential** function is defined by the limit:

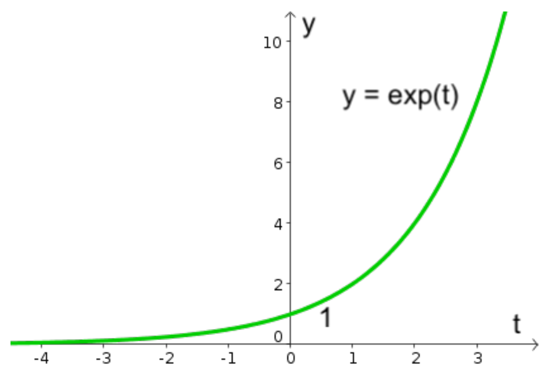
$$y = \mathrm{e}^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

  It may be written as $y = \mathrm{e}^x$ or $y = exp(x)$.

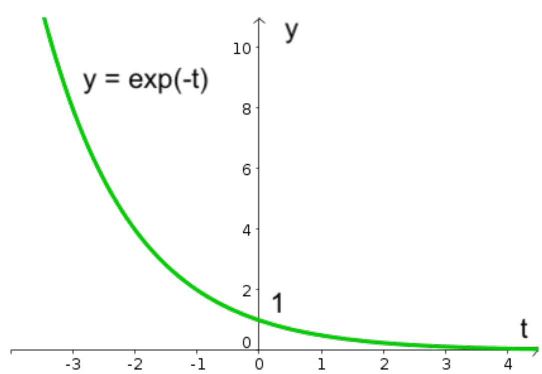
- The constant e, called Euler's number (after Swiss mathematician Leonhard Euler, 1707-1783), can be found by setting $x = 1$ to be $2.71828\ldots$ It is one of the fundamental mathematical constants (like π).

- This function was discovered by Swiss mathematician Jacob Bernoulli (1655-1705) while studying compound interest: the definition comes from taking the limit as compound interest is applied instantaneously to your money.

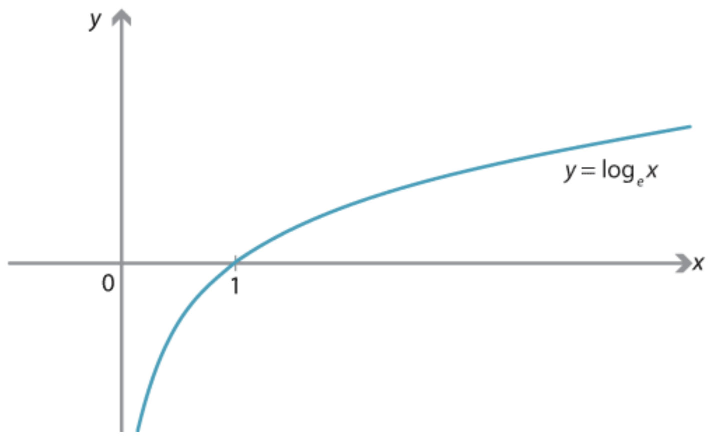
- An exponential function of time $y = \mathrm{e}^t$ is used to model **exponential growth**, capturing the behaviour of variables that grow with increasing rate such as compound interest, infectious disease spread, or population growth in ideal conditions.



- Change the sign of the independent variable in the index reverses the plot, and this behaviour (**exponential decay**) is used to model the decay of the mass of a radioactive substance.



- The **natural log** function is $y = \log_e(x)$ or $y = \ln(x)$. It is only valid for $x > 0$.
- It is the inverse of the exponential function, so if $y = e^x$ then $x = \ln(y)$. In other words, logarithms are useful for telling us what the power or index to a given base the input number would be.
- Because the natural log is the inverse of the exponential function, its plot is equivalent to that of the exponential function with the horizontal and vertical axes switched around:

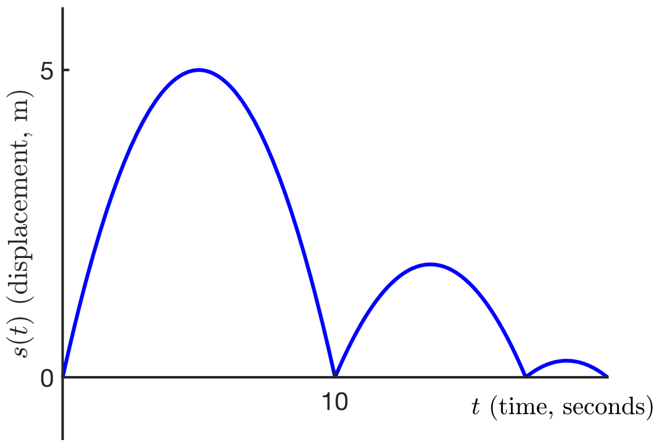


It is not defined for negative values of x, or at $x = 0$ where there is an asymptote.

1.2 Lecture 2: Introduction to Differentiation

- Differentiation of a function yields the derivative, which represents the gradient/slope of a function. This is equivalent to its rate of change.
- For example, given a function $y(x)$ which depends on x, the derivative (which we write as $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ describes the rate at which y is changing with respect to x.
- **Example**

  The height of a ball thrown vertically upward:

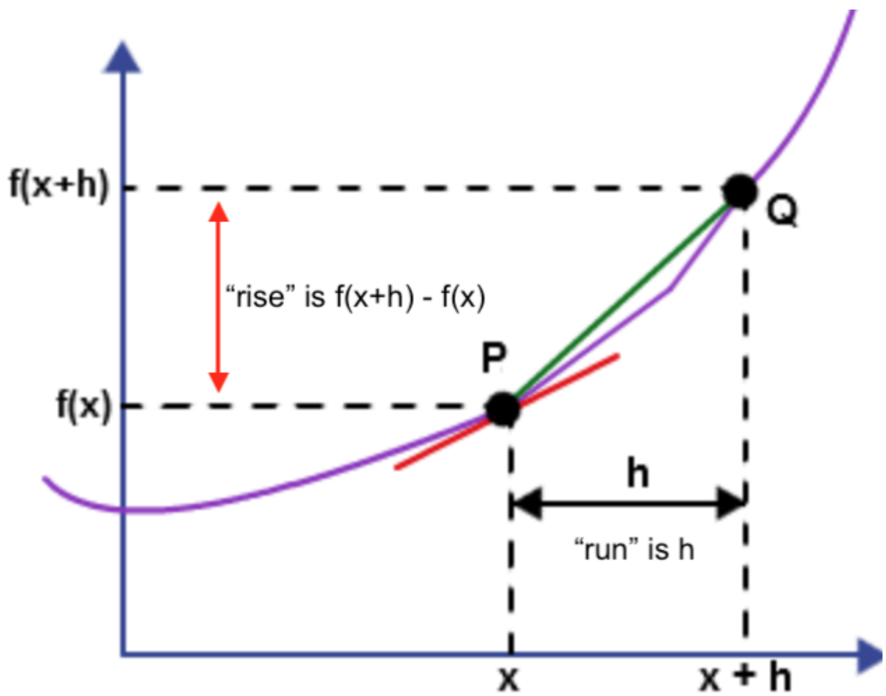
  Initially there is positive gradient as the height is increasing as the ball rises, then the ball slows and the gradient decreases (but is still positive as the ball is still rising), reaching a gradient of zero at the highest point, then a negative gradient as it falls. Differentiation is a process by which we can find exactly this information.

  Thus, if we had a formula for the height of this ball, for example a quadratic function in terms of time $s(t) = 0.2t(10 - t)$, then differentiating this function will give us another function that describes not the height, but this rate at which the height is changing (in other words, the velocity of the ball!) at any particular time t.

- Uses of differentiation:
  - Rates of change as applied to motion, so determining the velocity and acceleration of an object.
  - Maxima and minima of a function.
  - Modelling real-life systems using differential equations. Often scientists and engineers know about how things change, such as infectious diseases, population growth, chemical processes or mechanical processes, and these can be represented using equations involving derivatives.

1.2.1 Differentiation from first principles

- How is the derivative of a function determined? Consider a general curve $y = f(x)$.
  - The gradient of a curve is defined as the gradient of a tangent to the curve at that point.
  - Tangent: a straight line that just touches the curve at the point we are interested in.
  - This is calculated from rise/run of the chord between the points $(x, f(x))$ and $(x+h, f(x+h))$ on a curve in the limit as $h \to 0$ (so that the chord approximates the tangent at $(x, f(x))$).



- **Summary**

  Given a function $f(x)$, the the derivative is the instantaneous rate of change of f with respect to x. This is also the gradient of the graph $y = f(x)$, and so (by the above) it can be defined as:

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **Example (dependent on time - this is not required)**

  The displacement s of an object in t is given by $s(t) = t^2$.

  Find the velocity using differentiation by first principles:

$$
\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}t} &= \lim_{h\to 0}\frac{s(t+h)-s(t)}{h} \\
&= \lim_{h\to 0}\frac{(t+h)^2-t^2}{h} \\
&= \lim_{h\to 0}\frac{t^2+h^2+2th-t^2}{h} \\
&= \lim_{h\to 0}\frac{h^2+2th}{h} \\
&= \lim_{h\to 0}\frac{h(h+2t)}{h} \\
&= \lim_{h\to 0} h+2t \\
&= 2t
\end{aligned}
$$

  (So the rate of change of a quadratic function is described by a linear function.)

1.2.2 Elementary differentiation using standard rules

- We will employ rules (contained in the module Formula Sheet) that have been proven using differentiation from first principles.

  For constants a and n:

$$y = ax^n \implies \frac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}$$

$$y = ax \implies \frac{\mathrm{d}y}{\mathrm{d}x} = a$$

$$y = a \implies \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$y = \mathrm{e}^x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$$

$$y = \sin(x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x)$$

$$y = \cos(x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\sin(x)$$

$$y = \ln(x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

  To differentiate simple standard functions, we can try to match them to these formats and apply the rule.

- **Examples**

  Differentiate:

$$y = 5x^2, \qquad y = -3x^{10}, \qquad y = -\frac{4}{x^3}$$

$$y = 9x, \qquad y = 2\sqrt{x}, \qquad y = 100x^{0.5}$$

- You may need to revise the laws of powers, covered in Foundation Maths 1, in order to re-arrange some of these into a format compatible with the rules for differentiation.

2 Week 2

2.1 Lecture 3: Linearity and the gradient of a curve at a point

- Summary of what we know so far, and the notation for differentiation:

  If y is a variable whose value is dependent on x, then we say that y is a function of x and may also write it as $y(x)$.

  Here it makes sense to ask how much y will change if x changes by a small amount, which is the *derivative* of y with respect to x, or the *rate of change* of y as x changes.

  This is written as either $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, or as $y'(x)$ for short.

  However, if function g depends on a different variable, such as $g(t) = 3t^2 + 1$ then the notation $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ does not make sense, and it would be appropriate to use $\dfrac{\mathrm{d}g}{\mathrm{d}t}$ instead.

- **Linearity**
  Using our standard rules, we can easily differentiate standard expressions that are added together or subtracted, or are multiplied by a constant value.

  $$y = f(x) \pm g(x) \qquad \Longrightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = y' = \frac{\mathrm{d}f}{\mathrm{d}x} \pm \frac{\mathrm{d}g}{\mathrm{d}x} = f'(x) \pm g'(x)$$

  If a is a constant, then:

  $$y = af(x) \qquad \Longrightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = a\frac{\mathrm{d}f}{\mathrm{d}x} = af'(x)$$

  Thus, if $y = ax^k$, where a, k are constants, then:

  $$\frac{\mathrm{d}y}{\mathrm{d}x} = kax^{k-1}$$

- **Examples**

  Differentiate the following w.r.t. x:

  $$y = 10\sqrt{t}, \qquad\qquad y = 4x^3 + 5x + \frac{2}{x^3}$$

  $$y = (5x+2)(2x-1), \qquad\qquad y = -\frac{1}{\sqrt{2x}} - x^{3/2}$$

2.1.1 Gradient of a curve at a point

- Differentiating the formula for a curve results in a formula for the gradient of that curve at any point. In any case that is not a straight line, this gradient is dependent on x. Therefore, to find the gradient of a curve at a particular point, differentiate first and then substitute in the particular value of x to the formula obtained for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

  Example notation: given a function $y(x)$, the gradient of y at the point $x = 2$ may be indicated by $y'(2)$ or $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2}$

- **Examples**

  1. Find the gradient of the curve $y = x^2 + 4x - 7$ at the point $(2, 5)$.

  2. Find the gradient of the curve $y = 2x - \frac{9}{(3x)^{3/2}}$ at the point $(3, 17/3)$.

     Solution:

     First, we need to use the rules of indices to write the second term in a suitable form:

$$
\begin{aligned}
y &= 2x - \frac{9}{3^{3/2}x^{3/2}} \\
&= 2x - \frac{9}{3^{3/2}}x^{-3/2} \\
&= 2x - \frac{3^2}{3^{3/2}}x^{-3/2} \\
&= 2x - 3^{2-\frac{3}{2}}x^{-3/2} \\
&= 2x - 3^{\frac{4}{2}-\frac{3}{2}}x^{-3/2} \\
&= 2x - 3^{1/2}x^{-3/2}
\end{aligned}
$$

     Now all terms are in a suitable form, and differentiating with respect to x:

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\big(2x\big) - \frac{\mathrm{d}}{\mathrm{d}x}\big(3^{1/2}x^{-3/2}\big) \\
&= 2 - 3^{1/2}\big(-\frac{3}{2}\big)x^{-\frac{3}{2}-1} \\
&= 2 + \frac{1}{2}\cdot 3^{1/2}\cdot 3x^{-\frac{3}{2}-\frac{1}{2}} \\
&= 2 + \frac{1}{2}\cdot 3^{\frac{1}{2}+1}x^{-5/2} \\
&= 2 + \frac{1}{2}\cdot 3^{3/2}x^{-5/2}
\end{aligned}
$$

Finally, we evaluate this formula at $x = 3$ as required:

$$
\begin{aligned}
\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=3} &= 2 + \frac{1}{2}\cdot 3^{3/2}(3)^{-5/2} \\
&= 2 + \frac{1}{2}\cdot 3^{\frac{3}{2}-\frac{5}{2}} \\
&= 2 + \frac{1}{2}\cdot 3^{-2/2} \\
&= 2 + \frac{1}{2}\cdot 3^{-1} \\
&= 2 + \frac{1}{2}\cdot\frac{1}{3} \\
&= 2 + \frac{1}{6} \\
&= 2\frac{1}{6} \quad \text{or} \quad \frac{13}{6}
\end{aligned}
$$

- Using rules of indices, we may need to re-write expressions before they can be differentiated. Generalising the previous example:

  If a and k are constants, then the following may be useful:

  $$\frac{1}{(ax)^k} = \frac{1}{a^k x^k} = \frac{1}{a^k}x^{-k}$$

2.1.2 Extra Questions

- What are the co-ordinates of the points where this cubic has a gradient of 5?

  $$y = \frac{2}{3}x^3 - \frac{7}{2}x^2 + x - 14$$

  Solution:

  $$\big(4, -23\frac{1}{3}\big) \text{ and } \big(-\frac{1}{2}, -\frac{371}{24}\big) \text{ or } \big(-0.5, -15.48333\ldots\big)$$

- What is the gradient of this quadratic equation at the points where $y = 3$?

  $$y = -3x^2 + 11x - 5$$

  Can you see how these gradients make sense given the shape of the parabola?

  Solution:

  $$\frac{\mathrm{d}y}{\mathrm{d}x} = 5 \text{ at } x = 1, \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -5 \text{ at } x = 2\frac{2}{3}$$

2.2 Lecture 4: The Product and Quotient Rules

- Suppose $f(x)$ is the product of two functions (i.e. the result of multiplying them together):

$$f(x) = u(x) \cdot v(x)$$

Then the derivative is given by the Product Rule, which can be written as:

$$\begin{aligned}
\frac{\mathrm{d}f(x)}{\mathrm{d}x} &= v(x) \cdot \frac{\mathrm{d}u(x)}{\mathrm{d}x} + u(x) \cdot \frac{\mathrm{d}v(x)}{\mathrm{d}x} \\
(uv)' &= vu' + uv' \\
\frac{\mathrm{d}}{\mathrm{d}x}(uv) &= vu' + uv' \\
\frac{\mathrm{d}}{\mathrm{d}x}(uv) &= u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}
\end{aligned}$$

- **Examples**

  1. Differentiate $y = x^2 \sin(x)$.
  2. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \mathrm{e}^x \cos(x)$.
  3. Differentiate $y = (x^2 + 2)\ln(x)$.

- Suppose instead that our function is given by one function divided by another ("Quotient" is another word for a fraction):

$$y = f(x) = \frac{u(x)}{v(x)}$$

Then the derivative is given by the Quotient Rule, which can be written as:

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

or

$$f'(x) = \frac{vu' - uv'}{v^2}$$

- **Examples**

  1. Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin(x)}{x^3}\right)$.
  2. If $f(x) = \tan(x)$, then find $\frac{\mathrm{d}f}{\mathrm{d}x}$ using the Quotient Rule.

- **Examples**

  Ask the students which rule to use to differentiate:

$$y = x^2 \sin(x)$$

$$y = \ln(x) \tan(x)$$

$$y = x^7 \, \mathrm{e}^x$$

$$y = \frac{4x^9 + 2}{\tan(x)}$$

$$y = \frac{\ln(x)}{2x^2 + x + 1}$$

Consistency of the rules

- To see how the product rule and the quotient rule are consistent, differentiate:

$$h(t) = \frac{3\,\mathrm{e}^t}{t^2}$$

  Show that we can calculate the answer using either

  (a) the quotient rule

  (b) the product rule by first writing it as $h(t) = 3\,\mathrm{e}^t\, t^{-2}$.

  Will then need to break up the quotient rule solution into two terms to show that the answers are the same.

- To see how the product rule is consistent with the regular methods of differentiation:

$$y = (x^2 - 7)x^4$$

  Show

  (a) How to calculate it using the product rule

  (b) how to determine the answer by first multiplying out the brackets.

  Show that both methods are consistent and give rise to the same answer.

3 Week 3

3.1 Lecture 5: Chain rule

- **Recap**

  Revise the Product Rule and the Quotient Rule with one example of each:

  $$h(x) = \left(\frac{1}{x} - x + 2\right) \ln(x)$$

  $$y = \frac{\cos(x)}{\sqrt{x}}$$

- A composite function is defined as a "function of a function". For example, $h(x)$ is a composite function when:

  $$h(x) = g(f(x))$$

  where f and g are functions. In this case, to calculate the output of h, x is the input to function f and the output is then taken as the input for function g.

- **Examples of composite functions**

  $$\sin(3x+1), \qquad e^{x^2+2}, \qquad (2x-5)^4, \qquad \cos(\sin(x))$$

- To differentiate composite functions, we use the Chain Rule.

  If $y = g(f(x))$, then we write $u = f(x)$ and so $y = g(u)$.

  Then,

  $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

  Give your final answer in terms of the original variables - in this case, x and not u.

- **Examples using the Chain Rule:**

  1. If $y = \sin(3t^2 + 5)$, find y'
  2. Differentiate

     $$y = \tan(4t)$$

  3. Find $\frac{dy}{dx}$ when:

     $$y = e^{3x^2+3x-1}$$

- **Examples**

  1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \mathrm{e}^{3x^2+3x-1}$
  2. Differentiate $y = \sin(2x)$
  3. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = (x^5 + 2x + 4)^5$
  4. Find y' when $y = \cos(4x)$
  5. Differentiate $y = \sin\left(5t + \frac{1}{2}\right)$
  6. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \ln\left(\sqrt{x}\right)$

- From Examples 2 and 4 (and Example 2 in the previous lecture), the pattern suggests an additional set of rules that we can use as shortcuts:
  If a is a constant,

$$y = \sin(ax) \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = a\cos(ax)$$

$$y = \cos(ax) \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -a\sin(ax)$$

$$y = \tan(ax) \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = a\sec^2(ax)$$

$$y = \mathrm{e}^{ax} \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = a\,\mathrm{e}^{ax}$$

3.2 Lecture 6: Higher-order derivatives and applications

3.2.1 Second-order Derivatives

- Differentiating a function $y = f(x)$ with respect to x gives us the "first derivative" of y, denoted by y', $f'(x)$, or $\frac{\mathrm{d}y}{\mathrm{d}x}$. This tells us about the gradient or rate of change of y.

  Differentiating the result (with respect to x) yields the "second derivative" of y, denoted by:

  $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \quad \text{or} \quad y'' \quad \text{or} \quad f''(x)$$

  This tells us the rate at which the gradient of y changes.

- We can keep on differentiating many times. The n^{th} derivative is denoted by:

  $$\frac{\mathrm{d}^ny}{\mathrm{d}x^n} \quad \text{or} \quad y^{(n)} \quad \text{or} \quad f^{(n)}(x)$$

- **Examples**

  $$(1) \quad f(x) = 5x^3 \quad \Longrightarrow \quad f'(x) = 15x^2 \quad \Longrightarrow \quad f''(x) = 30x$$

  $$(2) \quad f(x) = 6\,\mathrm{e}^x \quad \Longrightarrow \quad f'(x) = 6\,\mathrm{e}^x \quad \Longrightarrow \quad f''(x) = 6\,\mathrm{e}^x$$

- If the independent variable is specifically time (usually denoted as t), then we may also use the notation $\dot{y}(t)$ and $\ddot{y}(t)$ for the first and second time-derivatives. This is applicable to the next topic!

3.2.2 Physical applications of differentiation as “rate of change”:

- Suppose an object moves in a straight line with its position along the line $x(t)$.
  - Velocity is the rate of change of position, so:

    $$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t}$$

  - Acceleration is the rate of change of velocity, so:

    $$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t}$$

    Therefore we also note that

    $$a(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

- **Example**

  A car moves in a straight line from A to B. At any time t (in seconds), the displacement of the car from A is given by:

  $$x(t) = t^3 + 2t^2 \quad \text{metres}$$

  1. What is the velocity of the car at $t = 3$s?
  2. What is the acceleration of the car after 4s?

  Solution:

  1. $$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(t^3 + 2t^2) = 3t^2 + 4t$$

     $$\therefore v(t = 3) = 3(3)^2 + 4(3) = 27 + 12 = 39\,\mathrm{ms}^{-1}$$

  2. $$a(t) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}(3t^2 + 4t) = 6t + 4$$

     $$\therefore a(t = 4) = 6(4) + 4 = 28\,\mathrm{ms}^{-2}$$

3.2.3 Combining the rules of differentiation

- We may encounter functions that are more complicated combinations of other functions - perhaps a function of a function that is then multiplied by another function - or a function of a function of a function.
- Complicated problems may require multiple uses of any combination of the rules. Break them down "from the outside/superstructure to the inside". Careful layout and presentation of solutions is key to avoid getting lost!
- **Example 1:**

  Differentiate

  $$y = \sin(5x)\sqrt{\cos(x)}$$

  The "outermost" structure here is $\sin(5x)$ multiplied by $\sqrt{\cos(x)}$, so this requires first an application of the product rule.

  Let,

  $$u = \sin(5x) \quad \text{and} \quad v = \sqrt{\cos(x)}$$

  Then to differentiate each of these inner component parts requires the chain rule:

  Let $w = 5x$, then $u = \sin(w)$, and so:

  $$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}w} \cdot \frac{\mathrm{d}w}{\mathrm{d}x} = \cos(w) \cdot 5 = 5\cos(5x)$$

  Let $z = \cos(x)$, then $v = \sqrt{z} = z^{1/2}$, and so:

  $$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}z} \cdot \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{2}z^{-1/2} \cdot (-\sin(x)) = -\frac{\sin(x)}{2\sqrt{\cos(x)}}$$

  Finally we combine these terms in the "big picture" product rule again:

  $$\begin{aligned}
  \frac{\mathrm{d}y}{\mathrm{d}x} &= u \cdot \frac{\mathrm{d}v}{\mathrm{d}x} + v \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \\
  &= \sin(5x) \cdot \left(-\frac{\sin(x)}{2\sqrt{\cos(x)}} \right) + \sqrt{\cos(x)} \cdot 5\cos(5x) \\
  &= 5\cos(5x)\sqrt{\cos(x)} - \frac{\sin(x)\sin(5x)}{2\sqrt{\cos(x)}}
  \end{aligned}$$

- Now for two examples of problems that involve "function of a function of a function".

- **Example 2:**

$$y = \ln\big(\sin(2x)\big)$$

This can be solved by using the chain rule twice.

If $y = f(g(h(x)))$, let the innermost function be $u = h(x)$, and the middle function $v = g(u)$, so that $y = f(v)$.

Then the Chain Rule states:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \cdot \frac{\mathrm{d}v}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

So for this example,

$$\text{Let} \quad u = 2x, \qquad \text{and} \quad v = \sin(u). \qquad \text{Then} \quad y = \ln(v)$$

Then

$$\frac{\mathrm{d}y}{\mathrm{d}v} = \frac{1}{v}, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \cos(u), \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \cdot \frac{\mathrm{d}v}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2\cos(u)}{v} = \frac{2\cos(2x)}{\sin(2x)} = \frac{2}{\tan(2x)}$$

- **Example 3:**

$$y = \cos\left(\mathrm{e}^{4x}\right)$$

Again this is a function (*cosine*), of a function (*exponential*), of a function (multiply by 4).

$$\text{Let} \quad u = 4x, \qquad \text{and} \quad v = \mathrm{e}^{u}. \qquad \text{Then} \quad y = \cos(v)$$

Then

$$\frac{\mathrm{d}y}{\mathrm{d}v} = -\sin(v), \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \mathrm{e}^{u}, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 4$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \cdot \frac{\mathrm{d}v}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = -4\,\mathrm{e}^{u}\sin(v) = -4\,\mathrm{e}^{4x}\sin(\mathrm{e}^{4x})$$

3.2.4 Extra Questions

1. Given that

$$g(t) = \ln(t^2 + 4)(3t^2 - 7)$$

What is $\frac{\mathrm{d}g}{\mathrm{d}t}$?

Solution:

$$g'(t) = 6t\ln(t^2 + 4) + \frac{2t(3t^2 - 7)}{t^2 + 4}$$

2. Determine the second derivative with respect to x of:

$$f(x) = \mathrm{e}^{x^2}$$

Solution:

$$\therefore f'(x) = 2x\,\mathrm{e}^{x^2}$$
$$\therefore f''(x) = 2\,\mathrm{e}^{x^2} + 4x^2\,\mathrm{e}^{x^2} = 2\,\mathrm{e}^{x^2}\left(1 + 2x\right)$$

3. Determine the second derivative with respect to x of:

$$y = (x + 1)\sin(4x - 1)$$

What is the value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x = \frac{1}{4}$?

Solution:

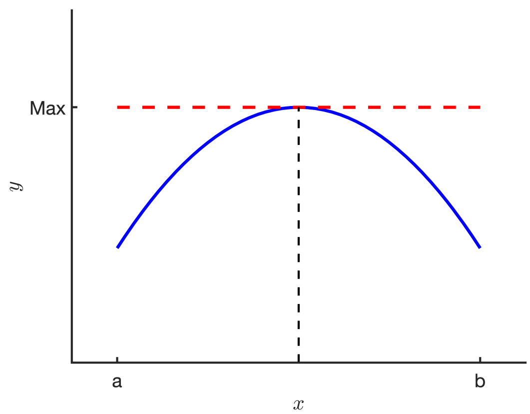
$$\begin{aligned}
\therefore y'(x) &= \sin(4x - 1) + 4(x + 1)\cos(4x - 1) \\
\therefore y''(x) &= 8\cos(4x - 1) - 16(x + 1)\sin(4x - 1) \\
\therefore y''\left(\frac{1}{4}\right) &= 8\cos(0) - 20\sin(0) = 8
\end{aligned}$$

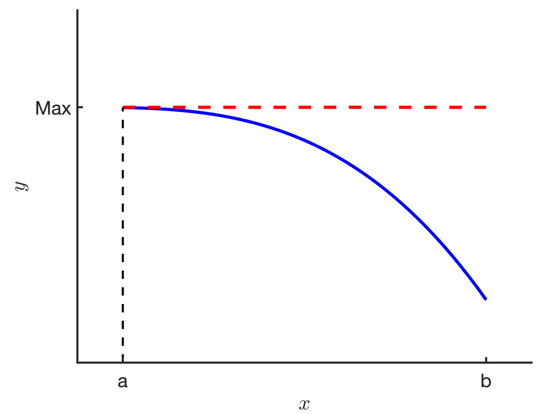
4 Week 4

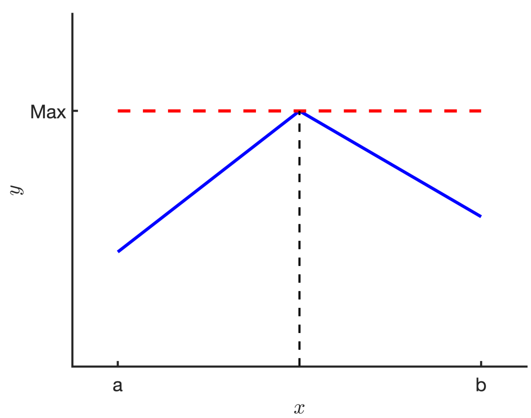
4.1 Lecture 7: Stationary Points and Extrema

We often want to maximise or minimise a function within a given range of x.

Suppose we want to maximise (or minimise) a function f in the range $a < x < b$. There are three ways a maximum can occur:







(a) The extreme value occurs at a point in (a, b) where:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

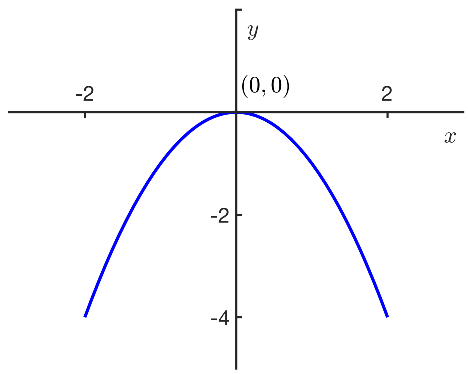
(b) The extreme value occurs at an endpoint.

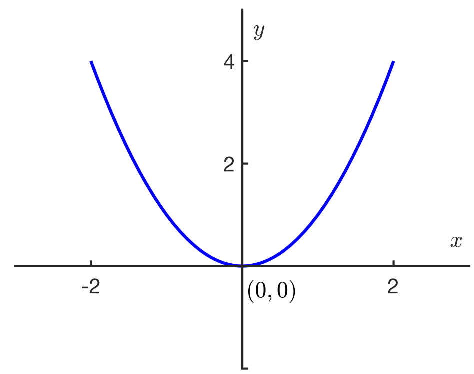
(c) The extreme value occurs at a point in (a, b) but the gradient $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined as the function is not "smooth".

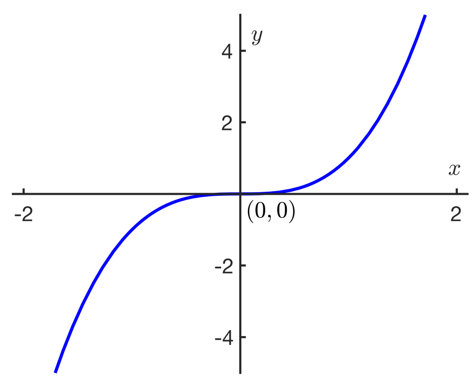
Similar possibilities exist for where minimum values occur. Of these, (a) is the case we are mainly interested in, but we need to be aware of the other possible situations that extreme values appear.

4.1.1 Stationary Points

If $f'(x) = 0$ when $x = a$ for some value a, then $\big(a, f(a)\big)$ is called a stationary point or turning point of the function f. There are three main types:







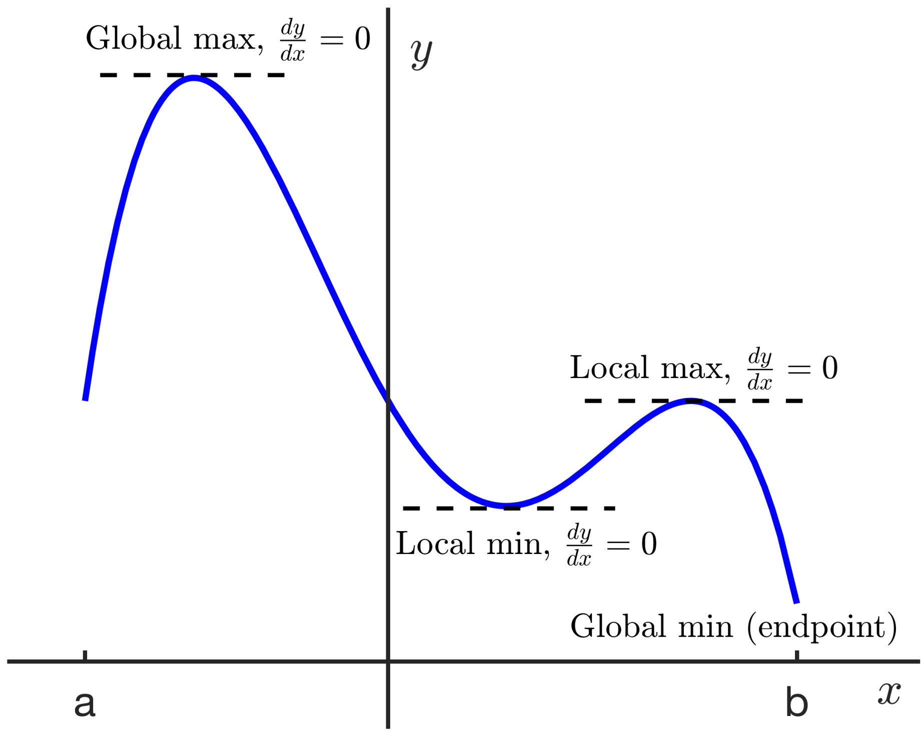
(a) Local maximum. e.g. the point $(0, 0)$ of $y = -x^2$.

(b) Local minimum. e.g. the point $(0, 0)$ of $y = x^2$.

(c) Point of inflection. e.g. the point $(0, 0)$ of $y = x^3$.

We use the terms "local" max/min, to distinguish from the global max/min which is the overall extreme value for the given range.

Example of local and global extrema:



4.1.2 Locating stationary points, and classifying them using the second derivative test

Therefore, to find the maximum and minimum points of a curve $y = f(x)$, we follow the following procedure:

(1) Determine the first derivative $\frac{dy}{dx}$.

(2) Solve the equation $\frac{dy}{dx} = 0$ for x. This tells us the location of the points where the gradient is zero - that is, the stationary points.

(3) Calculate the second derivative $\frac{d^2y}{dx^2}$.

(4) Determine the sign of $\frac{d^2y}{dx^2}$ at each stationary point, and apply the following test:

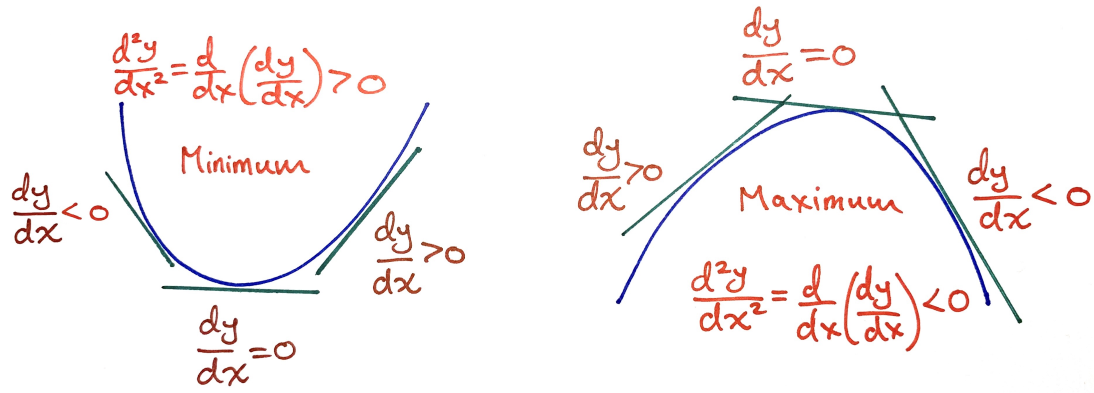
Second Derivative Test:

$$\left.\frac{d^2y}{dx^2}\right|_{x=a} > 0 \quad \Longrightarrow \quad \text{local minimum}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=a} < 0 \quad \Longrightarrow \quad \text{local maximum}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=a} = 0 \quad \Longrightarrow \quad \text{no conclusion - use the first derivative test instead.}$$

This test comes from considering the rate of change of the gradient around a maximum or minimum stationary point:



e.g. Around a local maximum, the gradient smoothly changes from positive to zero to negative. Thus it is decreasing around the stationary point, and so has a negative rate of change. Thus the second derivative of the function is negative at the stationary point.

(5) If we are only concerned with the behaviour in a limited interval $a < x < b$ of x, we should also evaluate the function at the endpoints, determining $f(a)$ and $f(b)$ to check if they are actually the global extrema in the range supplied.

4.1.3 Example I

Find and classify the stationary points of

$$y = x^2 - 4x + 5$$

Solution:

Observe that this is a quadratic function with a positive coefficient of x^2, so we would expect the graph to be a $\cup$-shaped parabola, and thus there should be a local minimum at the only stationary point!

Differentiating:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 4$$

Set this first derivative equal to zero and solve for x:

$$2x - 4 = 0$$

$$\therefore \quad 2x = 4$$

$$\therefore \quad x = 4/2 = 2$$

So there is a stationary point when $x = 2$. At this point, evaluating the original formula gives us the y-coordinate:

$$y(x = 2) = (2)^2 - 4(2) + 5 = 4 - 8 + 5 = 1$$

So the stationary point is located at $(2, 1)$. Is this a maximum or a minimum?

Determining the second derivative:

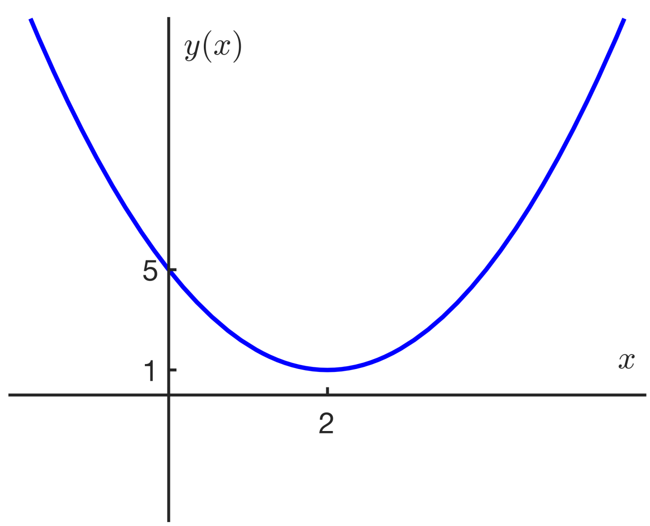
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}(2x - 4) = 2$$

In this case, it is a constant, so at the stationary point when $x = 2$ we still have:

$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=2} = 2 > 0$$

Hence there is a local minimum at $(2, 1)$.

This, of course, makes sense given the expected shape of the parabola:



4.1.4 Example II

Find and classify the stationary points of:

$$y = x^3 + x^2$$

Solution:

The first derivative is:

$$y' = 3x^2 + 2x$$

Setting this equal to zero and solving the resulting quadratic equation for x to locate the stationary points:

$$y' = 0$$

$$3x^2 + 2x = 0$$

$$x(3x + 2) = 0$$

$$\therefore x = 0 \quad \text{or} \quad 3x + 2 = 0$$

$$\therefore x = 0 \quad \text{or} \quad x = -\frac{2}{3}$$

Then determining the y-coordinate in each case:

$$y(x = 0) = 0^3 + 0^2 = 0$$

and

$$\begin{aligned} y\left(x = -\frac{2}{3}\right) &= \left(-\frac{2}{3}\right)^3 + \left(-\frac{2}{3}\right)^2 \\ &= -\frac{8}{27} + \frac{4}{9} \\ &= \frac{12 - 8}{27} \\ &= \frac{4}{27} \end{aligned}$$

So the stationary points are $(0, 0)$ and $\left(-\frac{2}{3}, \frac{4}{27}\right)$.

Then we differentiate again to obtain the second derivative:

$$y'' = 6x + 2$$

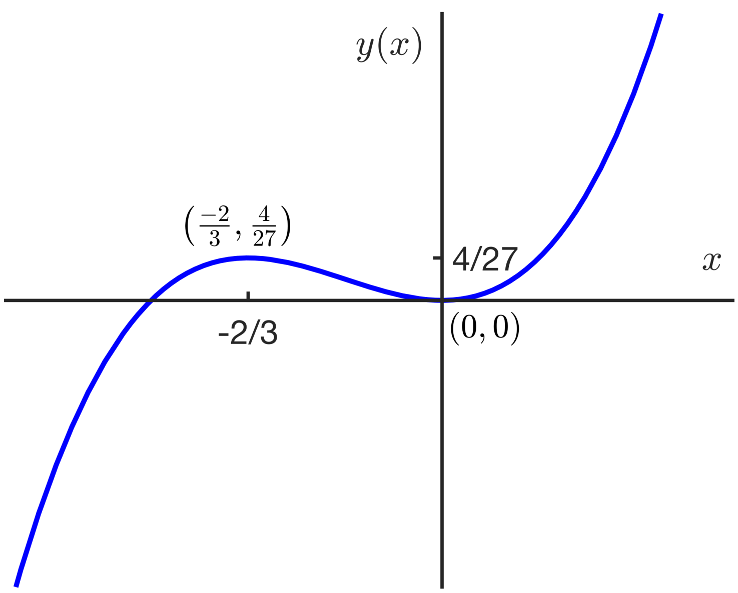
and evaluating this at the stationary points to apply the second derivative test in each case:

$$y''(x = 0) = 6(0) + 2 = 2 > 0$$

So $(0, 0)$ is a local minimum.

$$y''\left(x = -\frac{2}{3}\right) = 6\left(-\frac{2}{3}\right) + 2 = -4 + 2 = -2 < 0$$

So $\left(-\frac{2}{3}, \frac{4}{27}\right)$ is a local maximum.



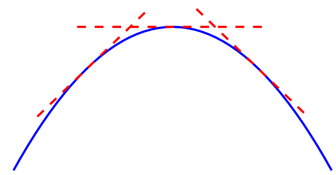
4.2 Lecture 8: First derivative test

If the Second Derivative Test is unsuccessful, we can use the First Derivative Test:

If $x = a$ is a stationary point, calculate the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at very close values of x on either side of a, called a^- and a^+. Then there are three possible scenarios:

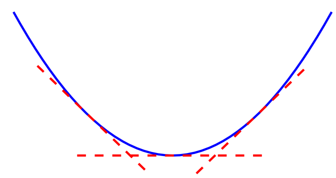
	a^-	a	a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	> 0 $/$	0 $-$	< 0 $\backslash$

Local maximum:



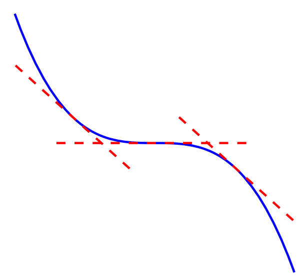
	a^-	a	a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	< 0 $\backslash$	0 $-$	> 0 $/$

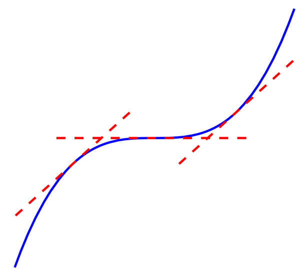
Local minimum:



	a^-	a	a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	< 0 $\backslash$	0 $-$	< 0 $\backslash$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	> 0 $/$	0 $-$	> 0 $/$

Point of inflection:





4.2.1 Example

Find and classify the stationary points of

$$f(x) = x^4 + 2x^3$$

Solution:

The first derivative is:

$$f'(x) = 4x^3 + 6x^2$$

Setting this equal to zero and solving for x to locate the stationary points:

$$\begin{aligned}
&f'(x) = 0\\
&4x^3 + 6x^2 = 0\\
&2x^3 + 3x^2 = 0\\
&x^2(2x+3) = 0\\
&\therefore x^2 = 0 \quad \text{or} \quad 2x+3 = 0\\
&\therefore x = 0 \quad \text{or} \quad x = -\frac{3}{2}
\end{aligned}$$

Then determining the y-coordinate in each case:

$$f(x=0) = 0^4 + 2(0)^3 = 0$$

and

$$\begin{aligned}
f\Big(x = -\frac{3}{2}\Big) &= \Big(-\frac{3}{2}\Big)^4 + 2\Big(-\frac{3}{2}\Big)^3\\
&= \frac{81}{16} - \frac{27}{4}\\
&= -\frac{27}{16}
\end{aligned}$$

So the stationary points are $(0,0)$ and $\big(-\frac{3}{2}, -\frac{27}{16}\big)$.

Then we differentiate again to obtain the second derivative:

$$f''(x) = 12x^2 + 12x$$

and evaluating this at the stationary points to apply the second derivative test in each case:

$$f''(x = 0) = 12(0)^2 + 12(0) = 0$$

So the second derivative test fails in this case.

$$\begin{aligned} f''\left(x = -\frac{3}{2}\right) &= 12\left(-\frac{3}{2}\right)^2 + 12\left(-\frac{3}{2}\right) \\ &= \frac{12 \times 9}{4} - \frac{36}{2} \\ &= 27 - 18 \\ &= 9 > 0 \end{aligned}$$

So $\left(-\frac{3}{2}, -\frac{27}{16}\right)$ is a local minimum.

Now, we can apply the first derivative test to classify $(0, 0)$ by evaluating $f'(x)$ around $x = 0$:

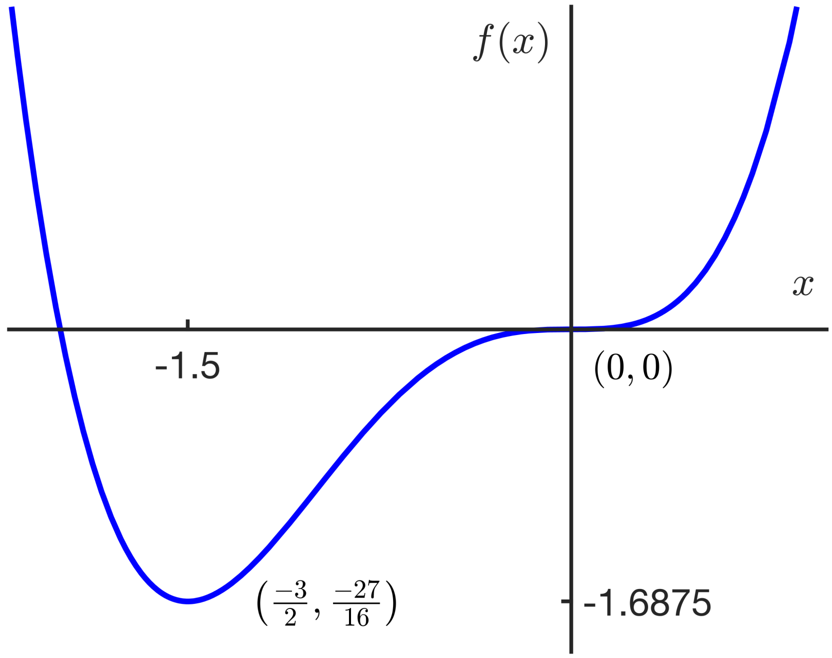
- Try $x = -0.0001$:

$$\begin{aligned} f'(x = -0.0001) &= 4(-0.0001)^3 + 6(-0.0001)^2 \\ &= 5.9996 \times 10^{-8} > 0 \end{aligned}$$

- Try $x = +0.0001$:

$$\begin{aligned} f'(x = -0.0001) &= 4(0.0001)^3 + 6(0.0001)^2 \\ &= 6.0004 \times 10^{-8} > 0 \end{aligned}$$

Hence we have a positive point of inflection at $(0, 0)$.



4.2.2 Summary

- Given a function $y = f(x)$, the local maximum or minimum value of y occurs at stationary points, where:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

- Given that $x = a$ is a stationary point of $y = f(x)$, we can determine whether there is a maximum or minimum located at $\big(a, f(a)\big)$ using the Second Derivative Test:

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=a} \begin{cases} > 0 & \implies \text{Local Minimum.} \\ < 0 & \implies \text{Local Maximum.} \\ = 0 & \implies \text{Use the First Derivative Test.} \end{cases}$$

5 Week 5

5.1 Lecture 9: Optimisation Problems

5.1.1 Introduction

- **Aim**

  Learn how to use interpret written physical problems, and use differentiation techniques to obtain solutions that maximise or minimise a quantity.

- Revise the principles of using differentiation to find and classify stationary points:

  Do this as a multiple-choice oral quiz.

  - Given a function $y = f(x)$, the local maximum or minimum value of y occurs at stationary points, where:

    $$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

  - Given that $x = a$ is a stationary point of $y = f(x)$, we can determine whether it is a maximum or minimum using the Second Derivative Test:

    $$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=a} \begin{cases} > 0 & \Longrightarrow \text{ Local Minimum.} \\ < 0 & \Longrightarrow \text{ Local Maximum.} \\ = 0 & \Longrightarrow \text{ Use the First Derivative Test.} \end{cases}$$

5.1.2 Motivation

Next, we shall learn how to use interpret written physical problems, and use the techniques of differential calculus to obtain solutions that maximise or minimise a quantity.

- We have learned that differentiation can be used to find maximum or minimum values of functions.
- Motivation: maximising profits in a company, or the surface area of a solar panel, or minimising the amount of material used in production of an item.
- We will be looking for the largest or smallest value of a function subject to some kind of constraint.
- The constraint will be some condition (usually described by an equation) that must absolutely be true no matter what our solution is.
- For example, we might need to minimise the material used in making an oil drum, but it must have a particular capacity no matter what dimensions we choose.

5.1.3 General principles

- Begin by reading the question, twice!
- We need to clearly understand what are (a) the quantity to be optimised, and (b) the constraint that must be satisfied.
- It often helps to draw a diagram of the situation.
- Assign some variable names to the unknowns, and turn our constraint and the optimised quantity into equations.
- We will want to use the constraint to obtain a formula in one variable for the quantity to be optimised.
- Then we can find the choice for which it is maximised or minimised by differentiating this formula and setting the derivative equal to zero.
- Finally, use the first or second derivative test to confirm that the answer is specifically a maximum or minimum as desired.

5.1.4 Example 1

Find two positive numbers whose sum is 300 and whose product is a maximum.

1. Label our unknowns: the two numbers are x and y.

2. The constraint: the sum must be 300, so $x + y = 300$.

3. The quantity to be optimised is the product: $P = xy$

4. Use the constraint to obtain the product in one variable:

$$y = 300 - x \quad \Longrightarrow \quad P = x(300 - x) = 300x - x^2$$

5. Differentiate the formula for the product:

$$\frac{\mathrm{d}P}{\mathrm{d}x} = 300 - 2x$$

6. Find the value of x that gives an extreme value by setting $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$:

$$300 - 2x = 0 \quad \Longrightarrow \quad 2x = 300 \quad \Longrightarrow \quad x = 150$$

7. Find the other number using the constraint:

$$y = 300 - x = 300 - 150 = 150$$

8. Confirm that this choice $(x = 150, y = 150)$ maximises the product using the Second Derivative Test:

$$\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = -2 \quad \Longrightarrow \quad \left.\frac{\mathrm{d}^2P}{\mathrm{d}x^2}\right|_{x=150} = -2 < 0 \quad \therefore \text{ Local Maximum.}$$

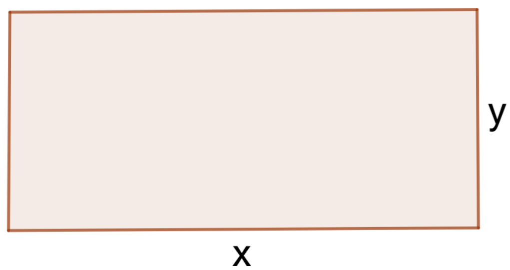
5.1.5 Example 2

A farmer wants to erect a rectangular pen in his field. He has 40m of wire fencing. What dimensions should he use to make the pen as large as possible?

1. Ask yourself: What is the quantity to be optimised? What is the constraint?

   The optimisation is that we need to *maximise* the area of the pen. The constraint is that the *perimeter* must equal the fencing length of 40m.

2. Draw a rectangle to visualise the problem.



3. Name the unknowns: the length x and breadth y of the pen.

4. Ask yourself: Using these symbols, how can the constraint and the optimised quantity be expressed as equations?

5. Use the information provided to create a formula for the constraint: the perimeter $P = 40$ of the pen.

$$P = x + y + x + y = 2x + 2y = 40 \quad \Longrightarrow \quad x + y = 20$$

6. Write a formula for what we are trying to optimise: the area A of the pen.

$$A = xy$$

7. Use the constraint equation to eliminate a variable from the area formula:

$$y = 20 - x \quad \Longrightarrow \quad A = x\big(20 - x\big) = 20x - x^2$$

8. Now that the area is stated in terms of one variable, differentiate:

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 20 - 2x$$

9. Set the gradient equal to zero and solve to find the value of x that maximises the area A:

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \quad \implies \quad 20 - 2x = 0 \quad \implies \quad 2x = 20 \quad \implies \quad x = 10m$$

10. Use the constraint to determine the other variable, so that the full dimensions are known:

$$y = 20 - x = 20 - 10 = 10m$$

11. Confirm that this choice ($x = 10m, y = 10m$) maximises the product using the Second Derivative Test:

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -2 \quad \implies \quad \left.\frac{\mathrm{d}^2A}{\mathrm{d}x^2}\right|_{x=10} = -2 < 0$$

Hence this is a local maximum area.

12. What is the largest pen he can make? Substitute both values back into the formula for area to determine what this maximum area actually is:

$$x = 10, y = 10 \quad \implies \quad A = xy = 10 \times 10 = 100m^2$$

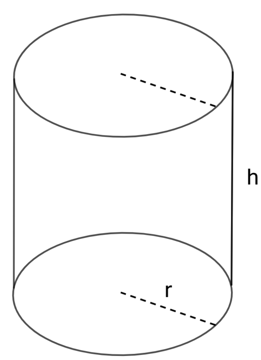
5.1.6 Example 3

An oil drum is to be manufactured with a cylindrical shape and a capacity of $1000cm^3$ of oil. The drum is made of gold, so it would be best to use as little as possible. What dimensions will minimise the amount of metal used to construct the drum? You may assume that the thickness of the drum is negligible.

1. Ask yourself: What is the quantity to be optimised? What is the constraint?

   The "optimisation" is that we want to *minimise* surface area, as this will be roughly equivalent to the material used given the negligible thickness. The constraint is that we must have a capacity (i.e. a *volume*) equal to $1000cm^3$

2. Draw a diagram, and label the dimensions: r is the radius, h is the height, V the volume and S the surface area of the drum.



3. Ask yourself: Using these symbols, how can the constraint and the optimised quantity be expressed as equations?

4. Constraint: Volume of a cylinder is given by $\pi r^2 h$ and equal to 1000. Rearranging this gives a formula for h in terms of r:

$$\pi r^2 h = 1000 \quad \Longrightarrow \quad h = \frac{1000}{\pi r^2}$$

5. Quantity to optimise: Surface area S given by the sum of the areas of the circular lid, circular base and cylindrical sides:

$$S = \pi r^2 + \pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r h$$

6. Then substitute in the volume restriction to eliminate h:

$$S = 2\pi r^2 + 2000 r^{-1}$$

7. Differentiate this function:

$$\frac{\mathrm{d}S}{\mathrm{d}r} = 4\pi r - 2000r^{-2}$$

8. Set $\frac{\mathrm{d}S}{\mathrm{d}r} = 0$ and solve for r to determine the stationary point of the surface area function:

$$4\pi r - 2000r^{-2} = 0 \quad \Longrightarrow \quad 4\pi r^3 - 2000 = 0$$

Hence:

$$\therefore r^3 = \frac{2000}{4\pi} \quad \Longrightarrow \quad r = \sqrt[3]{\frac{2000}{4\pi}} = 5.419cm$$

9. Use the value of r to calculate h and thus obtain the full dimensions of the drum:

$$h = \frac{1000}{\pi(5.419)^2} = 10.84cm$$

10. Use the Second Derivative Test to confirm that this is a minimum:

$$\frac{\mathrm{d}^2S}{\mathrm{d}r^2} = 4\pi + 4000r^{-3} \quad \Longrightarrow \quad \left.\frac{\mathrm{d}^2S}{\mathrm{d}r^2}\right|_{r=5.419} = 4\pi + 4000(5.419)^{-3} = 37.70 > 0$$

Hence this is a local minimum surface area of:

$$\begin{aligned} A &= 2\pi(5.419)^2 + \frac{2000}{5.419} \\ &= 553.58 \approx 554\mathrm{cm}^2 \end{aligned}$$

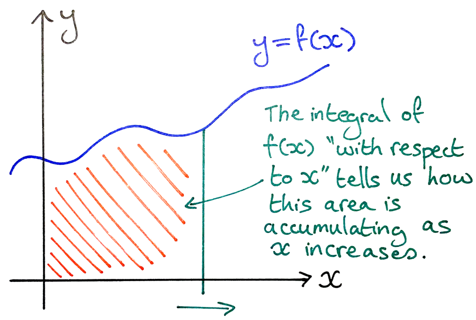
5.1.7 Extra Questions

- Show that a rectangle with a fixed area and minimum perimeter is a square.
- Show that a rectangle with a fixed perimeter and a maximum area is a square.
- Repeat Example 2, but the farmer is able to use a river as one of the four sides of the pen. How much larger can the pen be in this case, with the same length of fencing?

5.2 Lecture 10: Introduction to Integration

- Integration is the inverse process to differentiation: given a formula for the gradient $\frac{\mathrm{d}y}{\mathrm{d}x}$, can you find the equation of the curve $y = f(x)$?

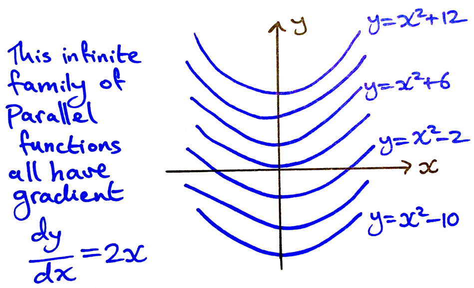
  It is also equivalent to obtaining an equation for the area enclosed between the curve and the x-axis.



- **Example**

  Suppose $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$.

  To integrate this, we need to find the function that differentiates to give $2x$.



  This could be $y = x^2$, but because a constant term has derivative zero, there are infinitely many other solutions such as
  $y = x^2 + 1$,
  $y = x^2 + 45$,
  $y = x^2 - 2000$,
  etc.

  We represent all of these by including the constant of integration $y = x^2 + c$. This is called "indefinite integration".

- Notation (for indefinite integrals):

$\int$	$f(x)$	$\mathrm{d}x$
Integral sign	Integrand	w.r.t. x

- Hence, writing the above result using proper notation:

$$\int 2x \, \mathrm{d}x = x^2 + c$$

- This is an example of a simple integral that can be obtained by reversing our differentiation rules. For a more general polynomial function ax^n, to differentiate it we "multiply by the power, then reduce the power by one". Doing the opposite of each step, and in reverse order, we would "increase the power by one, and divide by the new power":

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \quad \Longrightarrow \quad y = x^3 + c$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 \quad \Longrightarrow \quad y = \frac{1}{4}x^4 + c$$

- Generalising this to a rule:

$$\int ax^n \,\mathrm{d}x = \frac{a}{n+1}x^{n+1} + c \qquad (n \neq -1)$$

- **Examples**

  Evaluate

$$\int 4x^7 \,\mathrm{d}x \qquad \int t^{100} \,\mathrm{d}t$$

  Integrate the following with respect to x:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^3}$$

- **Linearity Rules for integration:**

$$\int af(x) \,\mathrm{d}x = a\int f(x) \,\mathrm{d}x \qquad \text{(Multiplicative Law)}$$

$$\int f(x) \pm g(x) \,\mathrm{d}x = \int f(x) \,\mathrm{d}x \pm \int g(x) \,\mathrm{d}x \qquad \text{(Additive Law)}$$

- **Examples**

  Using these rules (checking the answers by differentiation):

  1. Evaluate:
  $$\int 3x^2 + 8 \, \mathrm{d}x$$
  2. Find y, if:
  $$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 4x + 5$$

- Note that the integral of a constant is simply given by:
  $$\int a \, \mathrm{d}x = ax + c$$
  For example:
  $$\int 4.79 \, \mathrm{d}x = 4.79x + c, \qquad \int \pi \, \mathrm{d}x = \pi x + c$$

- Rules for integration of some standard functions:

  (Note that these are the reversal of the standard derivatives. They are in the formula booklet. Try not to confuse integration and differentiation.) For any constant a:

  $$\int a\cos(x) \, \mathrm{d}x = a\sin(x) + c$$
  $$\int a\sin(x) \, \mathrm{d}x = -a\cos(x) + c$$
  $$\int a\sec^2(x) \, \mathrm{d}x = a\tan(x) + c$$
  $$\int a\,\mathrm{e}^x \, \mathrm{d}x = a\,\mathrm{e}^x + c$$
  $$\int \frac{a}{x} \, \mathrm{d}x = a\ln|x| + c$$

- **Examples**

1. Evaluate:

$$\int \frac{-1}{2}\cos(t)\,\mathrm{d}t$$

Solution:

$$\int \frac{-1}{2}\cos(t)\,\mathrm{d}t = \frac{-1}{2}\sin(t) + c$$

2. Evaluate:

$$\int 6\,\mathrm{e}^x - 3x^5 + \frac{19}{x} - \frac{1}{\sqrt{x}}\,\mathrm{d}x$$

Solution:

Before evaluating any integrals, re-write the final term using laws of indices:

$$\begin{aligned}
\int 6\,\mathrm{e}^x - 3x^5 + \frac{19}{x} - \frac{1}{\sqrt{x}}\,\mathrm{d}x &= \int 6\,\mathrm{e}^x - 3x^5 + \frac{19}{x} - x^{-1/2}\,\mathrm{d}x \\
&= 6\,\mathrm{e}^x - \frac{3}{6}x^6 + 19\ln|x| - \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \\
&= 6\,\mathrm{e}^x - \frac{1}{2}x^6 + 19\ln|x| - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
&= 6\,\mathrm{e}^x - \frac{1}{2}x^6 + 19\ln|x| - 2\sqrt{x} + c
\end{aligned}$$

where the final fraction *must* be simplified, using:

$$\frac{1}{\frac{1}{2}} = 1 \div \frac{1}{2} = \frac{1}{1} \div \frac{1}{2} = \frac{1}{1} \times \frac{2}{1} = \frac{2}{1} = 2$$

We should never have nested fractions in our final solutions as they can always be simplified in this fashion.

3. A curve passes through the point $(1, 7)$ and the gradient of the curve at the point (x, y) is given by $2x^2(2x + 1)$. Find the equation of the curve.

Solution:

Begin by integrating the gradient:

$$\begin{aligned}
y &= \int 2x^2(2x+1)\,\mathrm{d}x \\
&= \int 4x^3 + 2x^2\,\mathrm{d}x \\
&= \frac{4x^4}{4} + \frac{2x^3}{3} + c \\
&= x^4 + \frac{2}{3}x^3 + c
\end{aligned}$$

Substituting in the co-ordinates $y = 7$ when $x = 1$, determine the value of c:

$$\begin{aligned}
& 7 = (1)^4 + \frac{2}{3}(1)^3 + c \\
\therefore\quad & 7 = 1 + \frac{2}{3} + c \\
\therefore\quad & 7 = \frac{5}{3} + c \\
\therefore\quad & c = \frac{21}{3} - \frac{5}{3} = \frac{16}{3}
\end{aligned}$$

Hence the fully-defined equation of the curve is:

$$y = x^4 + \frac{2}{3}x^3 + \frac{16}{3}$$

6 Week 6

6.1 Lecture 11: Integration by Substitution

- Opening problem:

  $$(a)\quad \text{Find a formula for } \; y \; \text{ if:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 6x^5 + \frac{3}{x} - \sqrt{x}$$

  $$(b)\quad \text{Evaluate } \int 4\cos(x) - \frac{2}{3}x^{-4}\,\mathrm{d}x$$

  Go through this and revise the basics of integration.

- Now, how might we evaluate:

  $$\int -2\sin(2t)\,\mathrm{d}t$$

  We notice that the integrand in this case looks like the result of a derivative using the chain rule.

- Integration by substitution is the reverse of the chain rule for differentiation. We can use it to integrate more complicated "functions of functions".

- **General method**

  1. Generally, let u be the inner part of the most complicated term in the integrand.
  2. Obtain $\frac{\mathrm{d}u}{\mathrm{d}x}$ by differentiating this formula for $u(x)$, then rearrange this to obtain $\mathrm{d}x$ in terms of $\mathrm{d}u$.
  3. Substitute everything in - replacing both u and $\mathrm{d}x$.
  4. We should now have the problem stated entirely in terms of u, and it should hopefully be a soluble integral. Execute it!
  5. Substitute $u(x)$ back in, so that we obtain the final answer in terms of the original variable x.
  6. Check our answer by differentiation. Can we recover the original integrand?

- **Examples**

(a) Evaluate:

$$\int (x+4)^5 \, \mathrm{d}x$$

Solution:

Let $u = x + 4$

Then,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \qquad \Longrightarrow \qquad \mathrm{d}x = \mathrm{d}u$$

Substituting both parts in:

$$\begin{aligned} \int (x+4)^5 \, \mathrm{d}x &= \int u^5 \, \mathrm{d}u \\ &= \frac{1}{6} u^6 + c \\ &= \frac{1}{6} (x+4)^6 + c \end{aligned}$$

(b) $$\int x^2 (2x^3+3)^5 \, \mathrm{d}x \quad = \quad \frac{1}{36} (2x^3+3)^6 + c$$

(c) $$\int (2x-4)^3 \, \mathrm{d}x \quad = \quad \frac{1}{8} (2x-4)^4 + c$$

(d) $$\int (6x^2+4x) \sin\left(x^3+x^2\right) \mathrm{d}x \quad = \quad -2\cos\left(x^3+x^2\right) + c$$

(e) $$\int x\sqrt{3x^2-2} \, \mathrm{d}x \quad = \quad \frac{1}{9} \left(3x^2-2\right)^{\frac{3}{2}} + c$$

(f) $$\int 2x \, \mathrm{e}^{x^2-\frac{1}{2}} \, \mathrm{d}x \quad = \quad \mathrm{e}^{x^2-\frac{1}{2}} + c$$

- Note that (similar to the analogous cases in differentiation) we can use this process when the inner function is linear to effectively derive some more generalised rules of integration:

  Given that both a and n are constant:

$$\int a\sin(nx)\,\mathrm{d}x = \frac{-a}{n}\cos(nx) + c$$

$$\int a\cos(nx)\,\mathrm{d}x = \frac{a}{n}\sin(nx) + c$$

$$\int a\,\mathrm{e}^{nx}\ \mathrm{d}x = \frac{a}{n}\,\mathrm{e}^{nx} + c$$

6.2 Lecture 12: Definite Integration

- We have been working with indefinite integrals, which feature an arbitrary constant c which is unknown without further information:

$$\int f(x)\,\mathrm{d}x = F(x) + c$$

- The definite integral:

$$\int_a^b f(x)\,\mathrm{d}x$$

has the known limits of integration $x = a$ and $x = b$. In this case,

$$\int_a^b f(x)\,\mathrm{d}x = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

There is no arbitrary constant c when the limits are known. Instead of adding $+c$ after the integration is carried out, we substitute in the upper and lower limits b and a, and find the difference.
We could imagine that there exists $+c$ for both the upper and lower limits but they cancel each other.

In physical terms, this definite integral is the area under the curve $f(x)$ in the interval $a < x < b$. It is a *numerical value*, rather than a function.

- **Standard examples**

(1) $$\int_1^4 2x - 4\,\mathrm{d}x = 3$$

(2) $$\int_0^1 \sqrt{x} + 3x - 2\,\mathrm{d}x = \frac{1}{6}$$

(3) $$\int_1^5 \frac{9}{t} - 2t + t^2\,\mathrm{d}t = 31.81\ldots$$

(4) $$\int_0^2 3x - x^2\,\mathrm{d}x = \frac{10}{3}$$

(5) $$\int_{-\pi}^{\pi} 4\sin(x) + 2\cos(x)\,\mathrm{d}x = 0$$

(6) $$\int_2^3 \frac{1}{t^2} - \sin(t) + 0.5\,\mathrm{d}t = 0.0928$$

- **Substitution**

  If we are calculating a definite integral using substitution, then when the new variable u is declared and $\mathrm{d}x$ replaced with $\mathrm{d}u$, we also need to replace the limits $x = a$ and $x = b$ with the corresponding limits in terms of $u = u(a)$ and $u = u(b)$ in order to fully convert the integral to the substituted variable.

  As the result will be a numerical value, it will no longer be necessary to substitute the original variable back into the final solution.

- **Examples of definite integration by substitution:**

(1) $$I = \int_0^1 3\sqrt{x}\big(4x^{3/2} + 1\big)^2 \,\mathrm{d}x = \frac{62}{3}$$

(2) Find the integral of $(2x - 4)^3$ between $x = 2$ and $x = 3$. (Answer $= 2$)

(3) $$I = \int_{-1}^1 \left(6x^2 + 4\right) \sin\left(x^3 + 2x\right) \,\mathrm{d}x = 0$$

6.2.1 Extra Questions

(1) $$\int_{-3}^1 6x^2 - 5x + 2 \,\mathrm{d}x = 84$$

(2) $$\int_{-\pi}^{\pi} \sin(x) \,\mathrm{d}x = 0$$

(3) $$\int_0^1 -12\,\mathrm{e}^t \,\mathrm{d}t = -12\,\mathrm{e} + 12 = -20.6\ldots$$

7 Week 7

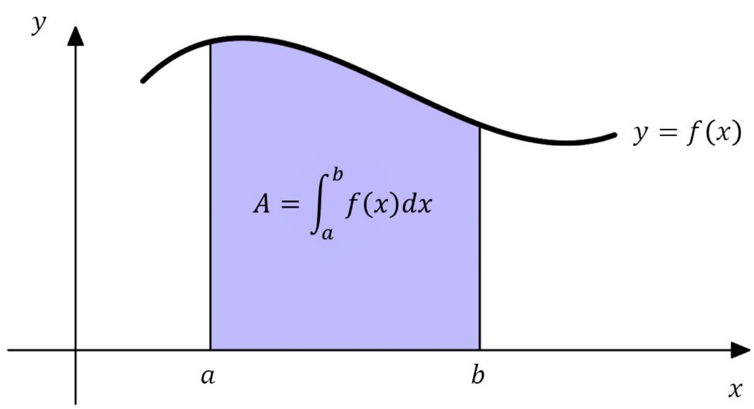
7.1 Lecture 13: Area under a Curve

- Opening problem:

  (a) Calculate $y = \displaystyle\int -2x(4-x^2)^7 \,\mathrm{d}x \quad = \quad \frac{1}{8}(4-x^2)^8 + c$

  (b) Calculate $y = \displaystyle\int_1^5 3t^2 - \frac{5}{t} \,\mathrm{d}t \quad = \quad 124 - 5\ln(5) = 115.95\ldots$

- Consider a curve $y = f(x)$ which is above the x-axis in the region $a < x < b$. Suppose A is the area bounded by the curve $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$:

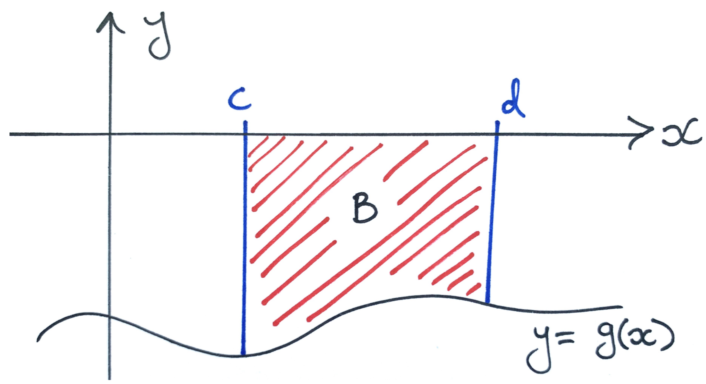


  Then A is called the area under the curve between $x = a$ and $x = b$.

  The definite integral of the function in a region where the curve is above the x-axis yields a positive value, which is exactly this area. Hence:

  $$A = \int_a^b f(x)\,\mathrm{d}x$$

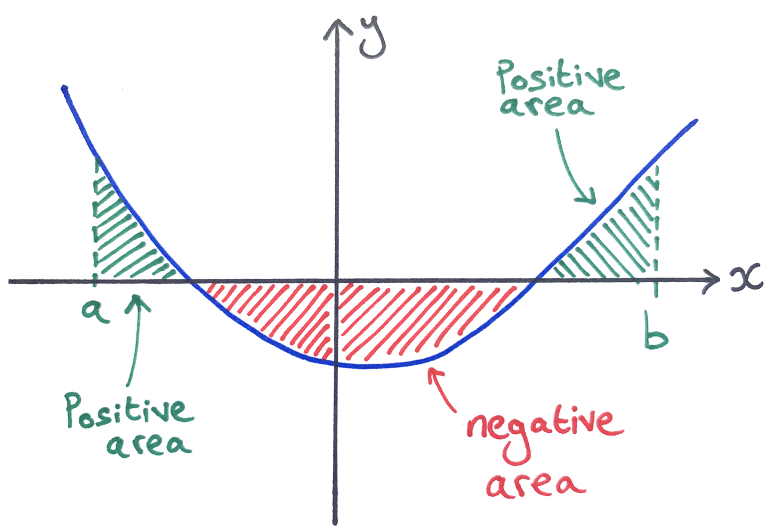
- However, integrating the function of a curve over a range of x where it is *below* the x-axis gives a negative value, which is precisely $-1\times$ the area between the curve and the x-axis.



Thus, we have:

$$-B = \int_c^d g(x)\,\mathrm{d}x$$

- So what if we wish to calculate the area enclosed by a curve that is both above and below the x-axis in different regions?

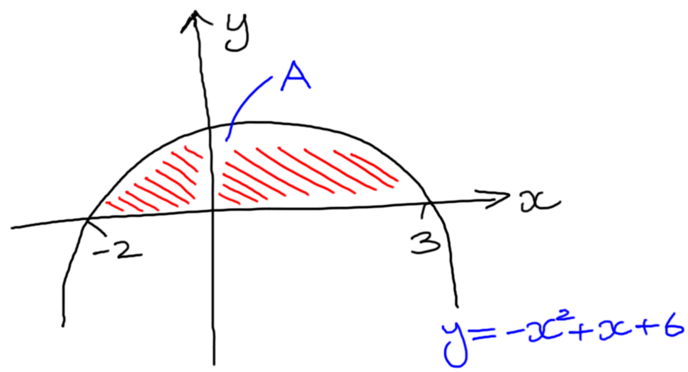


In this case, when asked to calculate the total *area* we must determine this positive value by separately calculating the integrals for regions where the curve is above and below the x-axis, and then summing their magnitudes/absolute values.

Therefore, we must begin by solving an equation to find where the curve crosses the x-axis.

- **Examples**

(1) Find the area between the curve $y = -x^2 + x + 6$ and the x-axis.



Solution

1. Either by using the quadratic equation, or factorising to:

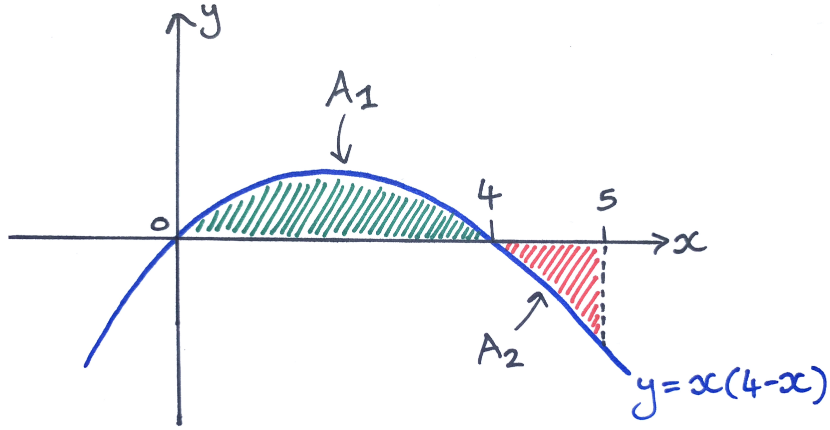
$$y = -(x^2 - x + 6) = -(x-3)(x+2)$$

   locate the roots at $x = -2$ and $x = 3$.
2. Draw a graph.
3. Formulate the definite integral:

$$\begin{aligned}
A &= \int_{-2}^{3} y(x)\,\mathrm{d}x \\
&= \int_{-2}^{3} -x^2 + x + 6\,\mathrm{d}x \\
&= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 6x \right]_{-2}^{3} \\
&= \left(-\frac{(3)^3}{3} + \frac{(3)^2}{2} + 6(3) \right) - \left(-\frac{(-2)^3}{3} + \frac{(-2)^2}{2} + 6(-2) \right) \\
&= \big(-9 + \frac{9}{2} + 18 \big) - \big(\frac{8}{3} + 2 - 12 \big) \\
&= 9 + \frac{9}{2} - \frac{8}{3} + 10 = \frac{19 \times 6}{6} + \frac{27}{6} - \frac{16}{6}
\end{aligned}$$

4. Answer $= 125/6$ square units.

(2) Find the area between the curve $y = 4x - x^2$ and the x-axis from $x = 0$ to $x = 5$.



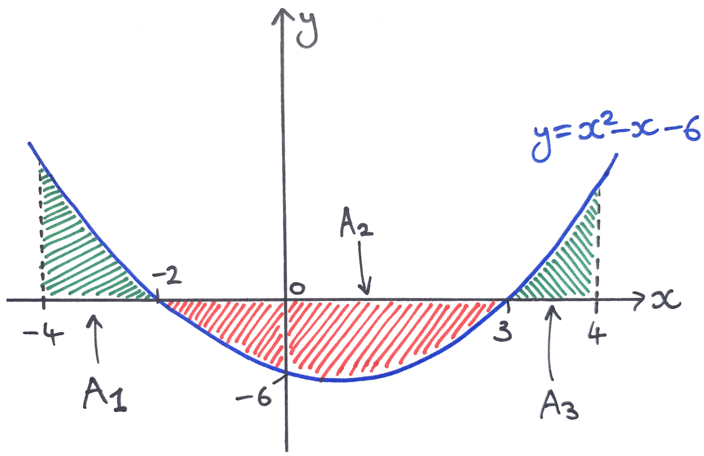
Solution

1. Either by using the quadratic equation, or factorising to $y = x(4 - x)$, deduce roots at $x = 0$ and $x = 4$.
2. Draw the graph.
3. Since a root occurs in the range, the total area is split in two parts: above the x-axis in $0 < x < 4$, and below the x-axis in $4 < x < 5$. We must formulate these two integrals separately, then add their magnitudes:

$$
\begin{aligned}
A &= A_1 + A_2 \\
&= \left| \int_0^4 4x - x^2 \, \mathrm{d}x \right| + \left| \int_4^5 4x - x^2 \, \mathrm{d}x \right| \\
&= \left| \left[2x^2 - \frac{1}{3}x^3 \right]_0^4 \right| + \left| \left[2x^2 - \frac{1}{3}x^3 \right]_4^5 \right| \\
&= \left| \frac{32}{3} \right| + \left| \frac{-7}{3} \right| \\
&= \frac{32}{3} + \frac{7}{3} \\
&= 13 \text{ square units}
\end{aligned}
$$

(3) Find the area between the curve $y = x^2 - x - 6$ and the x-axis between $x = -4$ and $x = 4$, and compare this with the integral:

$$\int_{-4}^{4} x^2 - x - 6 \, \mathrm{d}x$$



Solution

1. Either by using the quadratic equation, or factorising to $y = (x-3)(x+2)$, deduce roots at $x = -2$ and $x = 3$.
2. Draw the graph.
3. This time two roots occur in the range, and so the graph shows three regions. The integrals over $-4 < x < -2$ and $3 < x < 4$ will give positive results, while the integral over $-2 < x < 3$ will be negative.
4. To find the total area we add the magnitudes of the three areas:

$$\begin{aligned}
A &= A_1 + A_2 + A_3 \\
&= \left| \int_{-4}^{-2} x^2 - x - 6 \, \mathrm{d}x \right| + \left| \int_{-2}^{3} x^2 - x - 6 \, \mathrm{d}x \right| + \left| \int_{3}^{4} x^2 - x - 6 \, \mathrm{d}x \right| \\
&= \left| \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x \right]_{-4}^{-2} \right| + \left| \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x \right]_{-2}^{3} \right| + \left| \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x \right]_{3}^{4} \right| \\
&= \left| \frac{38}{3} \right| + \left| \frac{-125}{6} \right| + \left| \frac{17}{6} \right| \\
&= \frac{38}{3} + \frac{125}{6} + \frac{17}{6} \\
&= \frac{109}{3} \text{ square units}
\end{aligned}$$

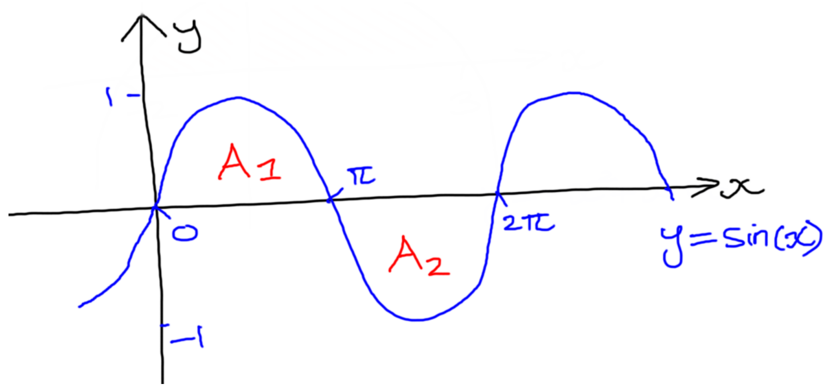
5. By comparison, the single integral (where the middle region adds a negative area) gives a smaller and negative result:

$$\frac{38}{3} - \frac{125}{6} + \frac{17}{6} = -\frac{16}{3}$$

This is the *net* area under the curve.

(4) What do you expect the answer of the following integral to be?

$$\int_0^{2\pi} \sin(x)\mathrm{d}x$$



Drawing a sketch, by symmetry we see that the area between $0 < x < \pi$ and the area between $\pi < x < 2\pi$ will cancel each other out.

We confirm by calculation that the answer is zero:

$$\int_0^{2\pi} \sin(x)\,\mathrm{d}x \quad = \quad \Big[-\cos(x)\Big]_0^{2\pi} = (-1) - (-1) = 0$$

7.2 Lecture 14: Integration by Parts

- We have seen how to integrate functions using integration by substitution:

$$\int \cos(3x+1)\,\mathrm{d}x, \qquad \int x\sin\left(1-x^2\right)\mathrm{d}x, \qquad \int 3t\,\mathrm{e}^{-3t^2}\ \mathrm{d}t$$

  In each case, the integrand involves a composite function.

- However, what if we want to integrate the following:

$$\int t^2\ln(t)\,\mathrm{d}t, \qquad \int (2x+1)\,\mathrm{e}^{3x}\ \mathrm{d}x, \qquad \int_0^{\pi}(4x-1)\sin(2x)\,\mathrm{d}x$$

  In these cases, the integrand does not usually involve a significant composite function, but does consist of the product of two functions.

- To integrate the product of two functions, we may attempt a technique called "integration by parts":

  1. Choose one part to be u, and the other to be $\frac{\mathrm{d}v}{\mathrm{d}x}$ (or $\frac{\mathrm{d}v}{\mathrm{d}t}$ etc. as appropriate).
  2. Differentiate u to obtain $\frac{\mathrm{d}u}{\mathrm{d}x}$, and integrate $\frac{\mathrm{d}v}{\mathrm{d}x}$ to obtain v (don't bother with "+c" at this stage).
  3. Substitute these four pieces (u, v, $\frac{\mathrm{d}u}{\mathrm{d}x}$, $\frac{\mathrm{d}v}{\mathrm{d}x}$) into the following formula and evaluate the second integral (which should be easier than the original):

$$\boxed{\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x}$$

     If this second integral is still not possible, then we have made the wrong choice of u or may have to try integration by substitution instead.

- As a guide for deciding which part we should choose to be u (the part that we will differentiate to result in an easier second integral): L-A-T-E

  1. **L**og: always choose $u = \ln(x)$ if possible.
  2. **A**lgebra: a polynomial (x, x^2, t^3).
  3. **T**rigonometric functions ($\sin(x)$, $\cos(2t)$).
  4. **E**xponential functions are least preferable (e^{2x}, e^{-3t}).

- **Examples:**

$$(1) \qquad \int t^2 \ln(t)\ \mathrm{d}t$$

Solution:

Let $u = \ln(t)$ and let $\frac{\mathrm{d}v}{\mathrm{d}t} = t^2$

Differentiating u:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{t}$$

And integrating to obtain v:

$$v = \int \frac{\mathrm{d}v}{\mathrm{d}t}\ \mathrm{d}t \quad = \quad \int t^2\ \mathrm{d}t \quad = \quad \frac{1}{3}t^3$$

Then using the integration by parts formula:

$$\begin{aligned}
\int t^2 \ln(t)\ \mathrm{d}t &= uv - \int v\frac{\mathrm{d}u}{\mathrm{d}t}\ \mathrm{d}t \\
&= \big(\ln(t)\big)\left(\frac{1}{3}t^3\right) - \int \left(\frac{1}{3}t^3\right)\left(\frac{1}{t}\right)\ \mathrm{d}t \\
&= \frac{1}{3}t^3 \ln(t) - \frac{1}{3}\int t^2\ \mathrm{d}t \\
&= \frac{1}{3}t^3 \ln(t) - \frac{1}{9}t^3 + c \\
&= \frac{1}{9}t^3\big(3\ln(t) - 1\big) + c
\end{aligned}$$

$$\begin{aligned}
(2) \qquad \int (2x+1)\,\mathrm{e}^{3x}\ \mathrm{d}x &= \frac{1}{3}(2x+1)\,\mathrm{e}^{3x} - \frac{2}{9}\,\mathrm{e}^{3x} + c \\
&= \frac{1}{9}\,\mathrm{e}^{3x}\,\big(6x+1\big) + c
\end{aligned}$$

$$(3) \qquad \int 3x\,\mathrm{e}^{2x+1}\ \mathrm{d}x = \frac{3}{2}x\,\mathrm{e}^{2x+1} - \frac{3}{4}\,\mathrm{e}^{2x+1} + c$$

Notes: We may need to employ integration by substitution within these examples.

- For definite integrals using integration by parts, the formula becomes:

$$\boxed{\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \big[uv\big]_a^b - \int_a^b v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x}$$

- **Example:**

(4) $\quad I = \displaystyle\int_0^{\pi} (4x-1)\sin(2x)\,\mathrm{d}x$

Solution:

Let $u = 4x-1$ and $v' = \sin(2x)$. Then $u' = 4$ and $v = -\dfrac{1}{2}\cos(2x)$

Substituting into the "by parts" formula for definite integrals:

$$\begin{aligned}
I &= \left[(4x-1)\left(-\frac{1}{2}\cos(2x)\right)\right]_0^{\pi} - \int_0^{\pi}\left(-\frac{1}{2}\cos(2x)\right)(4)\,\mathrm{d}x \\
&= \left[-\frac{1}{2}(4x-1)\cos(2x)\right]_0^{\pi} + 2\int_0^{\pi}\cos(2x)\,\mathrm{d}x \\
&= \left[-\frac{1}{2}(4x-1)\cos(2x)\right]_0^{\pi} + 2\left[\frac{1}{2}\sin(2x)\right]_0^{\pi} \\
&= \left[-\frac{1}{2}(4x-1)\cos(2x)\right]_0^{\pi} + \Big[\sin(2x)\Big]_0^{\pi} \\
&= \left[-\frac{1}{2}(4x-1)\cos(2x) + \sin(2x)\right]_0^{\pi} \\
&= \left(-\frac{1}{2}(4\pi-1)\cos(2\pi) + \sin(2\pi)\right) - \left(-\frac{1}{2}(-1)\cos(2\cdot 0) + \sin(2\cdot 0)\right) \\
&= -\frac{1}{2}(4\pi-1) - \frac{1}{2} \\
&= -2\pi
\end{aligned}$$

- More complicated questions require multiple applications of integration by parts. This is especially common where one part is a trigonometric or exponential term, and the other is a polynomial function of order two or higher (that is, a quadratic or cubic function etc.).

  In this example, we will need to use it twice. The original integral (I_1) involves the product of a quadratic function and a trigonometric function. The first use of integration by parts yields an integral (I_2) that is *simpler* - the product of a linear function and a trigonometric function - but still not simple enough to evaluate immediately. A second application of integration by parts reduces this to a simple integral of a trigonometric term:

$$\begin{aligned} I_1 &= \int (2x^2 + 3x + 1) \cos(4x - 7) \, \mathrm{d}x \\ &= (2x^2 + 3x + 1) \frac{1}{4} \sin(4x - 7) - I_2 \end{aligned}$$

  where

$$\begin{aligned} I_2 &= \int \frac{1}{4}(4x + 3) \sin(4x - 7) \, \mathrm{d}x \\ &= -\frac{1}{16}(4x + 3) \cos(4x - 7) + \frac{1}{16} \sin(4x - 7) + c \end{aligned}$$

  Don't forget to add in the constant of integration at the very end of the solution!

- **Advanced examples**

Certain problems (usually involving both a trigonometric part and an exponential part) appear to go in circles, so that performing two applications of integration by parts takes you back to some multiple of the original integral.

These can be solved by treating the original integral as an algebraic variable and transposing the equation to solve for it.

Example:

$$I_1 = \int \sin(x)\,\mathrm{e}^{x-1}\ \mathrm{d}x$$

Performing integration by parts once here results in:

$$I_1 = \sin(x)\,\mathrm{e}^{x-1} - I_2$$

where

$$I_2 = \int \mathrm{e}^{x-1}\cos(x)\ \mathrm{d}x \quad = \quad \cos(x)\,\mathrm{e}^{x-1} + \int \sin(x)\,\mathrm{e}^{x-1}\ \mathrm{d}x$$

by a second use of integration by parts. Hence,

$$\begin{aligned} I_1 &= \sin(x)\,\mathrm{e}^{x-1} - (\cos(x)\,\mathrm{e}^{x-1} + I_1) \\ &= \sin(x)\,\mathrm{e}^{x-1} - \cos(x)\,\mathrm{e}^{x-1} - I_1 \end{aligned}$$

But we can transpose this equation and solve for I_1 as follows:

$$I_1 = \frac{\mathrm{e}^{x-1}}{2}\big(\sin(x) - \cos(x)\big)$$

8 Week 8

8.1 Lecture 15: Matrices

- **Definition:** A matrix is a rectangular array of variables, enclosed by either square or round brackets. They are denoted by a capital letter. A one-dimensional matrix is also known as a vector. The individual components of the matrix are called elements.

- **Order of a matrix:** the size and shape, described by the number of rows and then the number of columns.

- **Examples:**

$$\begin{array}{ccccc} \begin{pmatrix} 1 \\ 3 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} 5 & 3 & -6 & 0 & -1 \end{pmatrix} & \begin{pmatrix} 7 & -3 \\ 1 & 4 \\ 9 & -2 \end{pmatrix} & \begin{pmatrix} 4 & 8 & 1 \\ 0 & 6 & -5 \\ 2 & 1 & -1 \end{pmatrix} \\ \\ 2 \times 1 & 2 \times 2 & 1 \times 5 & 3 \times 2 & 3 \times 3 \end{array}$$

- **Square matrix:** these have order $n \times n$ (where n is an integer). They have special properties.

8.1.1 Basic matrix operations

- **Addition and Subtraction:**
  We can add or subtract two matrices only if they have precisely the same order (same number of rows and same number of columns). In this case we add or subtract each of their corresponding elements.

  In general, for a pair of 2×2 matrices $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$ and $B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$:

$$A \pm B = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \pm \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} a_{1,1} \pm b_{1,1} & a_{1,2} \pm b_{1,2} \\ a_{2,1} \pm b_{2,1} & a_{2,2} \pm b_{2,2} \end{pmatrix}$$

- **Scalar multiplication:**
  A scalar is a real or complex *number*, in contrast to a vector or matrix. To multiply a matrix by a scalar, we simply multiply ("scale") each element of the matrix by that scalar. Scalar multiplication is always admissible for any scalar and any matrix.

  In general, for a scalar α and a matrix $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$:

$$\alpha \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} \alpha a_{1,1} & \alpha a_{1,2} \\ \alpha a_{2,1} & \alpha a_{2,2} \end{pmatrix}$$

- **Examples:**

$$A = \begin{pmatrix} 7 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

Find each of the following, if they exist:

(i) $2A$ $\quad$ (ii) $-5B$ $\quad$ (iii) $B + C$

(iv) $B - A$ $\quad$ (v) $2B + C$ $\quad$ (vi) $C - 3B$

Solution:

(i) $$2A = 2\begin{pmatrix} 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 2(7) \\ 2(0) \end{pmatrix} = \begin{pmatrix} 14 \\ 0 \end{pmatrix}$$

(ii) $$-5B = -5\begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 5(-1) & 5(3) \\ 5(4) & 5(0) \end{pmatrix} = \begin{pmatrix} 5 & -15 \\ -20 & 0 \end{pmatrix}$$

(iii) $$B + C = \begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1+2 & 3+1 \\ 4+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$$

(iv) $$B - A = \begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix} - \begin{pmatrix} 7 \\ 0 \end{pmatrix} \qquad \text{Invalid.}$$

As B is a 2×2 matrix while A is a 2×1 matrix, their orders differ and they cannot be subtracted.

(v) $$\begin{aligned} 2B + C &= 2\begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2(-1) & 2(3) \\ 2(4) & 2(0) \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 6 \\ 8 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2+2 & 6+1 \\ 8+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 0 & 7 \\ 8 & 1 \end{pmatrix} \end{aligned}$$

(vi) $$\begin{aligned} C - 3B &= \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} - 3\begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3(-1) & 3(3) \\ 3(4) & 3(0) \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -3 & 9 \\ 12 & 0 \end{pmatrix} = \begin{pmatrix} 2-(-3) & 1-9 \\ 0-12 & 1-0 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ -12 & 1 \end{pmatrix} \end{aligned}$$

8.1.2 Matrix multiplication

- Two matrices can be multiplied only if the number of columns in the first matrix matches the number of rows in the second matrix.
- If this is satisfied, the order of the resulting matrix is given by the number of rows in the first matrix and the number of columns in the second matrix.
- Each element in the result is given by matching the corresponding row of the first matrix with the corresponding column from the second matrix and taking the sum of the product of each pair.
- Matrix multiplication is a **non-commutative** operation, meaning that the order of multiplying the matrices is important and *can not be changed*. For two matrices A and B, in general we find that $AB \neq BA$. One of these may not even exist while the other does, or they may both exist but give different results.

- **Examples:**
  Let

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$$

Calculate (i) BC, (ii) CB, (iii) AC and (iv) CA:

Solutions:

$$\text{(i)} \qquad BC = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$$

B has order 2×1 while C has order 2×2, so the number of columns of B (1) does not equal the number of rows of C (2). Thus, this is not a valid operation and BC does not exist.

$$\text{(ii)} \qquad CB = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

The orders are 2×2 and 2×1. The inner two numbers match, so this is a valid multiplication. From the outer two numbers, the result will be a 2×1 matrix:

$$\begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} (-1)(2) + (2)(-1) \\ (4)(2) + (5)(-1) \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

(iii) $\qquad AC = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$

The orders are both 2×2, so the inner two numbers match and from the outer two numbers, the result will be another 2×2 matrix:

$$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} (1)(-1) + (0)(4) & (1)(2) + (0)(5) \\ (2)(-1) + (-1)(4) & (2)(2) + (-1)(5) \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -6 & -1 \end{pmatrix}$$

(iv) $\qquad CA = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$

Again the orders are both 2×2, so the inner two numbers match and from the outer two numbers, the result will be another 2×2 matrix:

$$\begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} (-1)(1) + (2)(2) & (-1)(0) + (2)(-1) \\ (4)(1) + (5)(2) & (4)(0) + (5)(-1) \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 14 & -5 \end{pmatrix}$$

These results demonstrate non-commutativity: BC does not exist while CB does, and while both AC and CA do exist they are not equal to each other.

8.1.3 Special matrices

- **Identity Matrix:**
  These are square matrices with 1 on the leading diagonal elements, and 0 everywhere else. They act like the number 1 when it comes to matrix multiplication, being the only matrices that satisfy:

  $$AI = A = IA \quad \text{for any matrix } A \text{ of suitable order.}$$

  Consider the 2×2 and 3×3 identity matrices:

  $$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- **Zero Matrix:**
  This is a square matrix where every entry is zero. For example,

  $$\underline{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \underline{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  It acts like the number 0 in matrix addition and matrix multiplication, so:

  $$A + \underline{O} = A = \underline{O} + A$$

  and

  $$A\underline{O} = \underline{O} = \underline{O}A \quad \text{for any matrix } A \text{ of suitable order.}$$

8.1.4 Determinant

- Square matrices have a determinant, which is a scalar associated with the matrix. In matrix applications to geometry, it is associated with ideas of size and scaling factor.

  For a square matrix A, the determinant may be denoted $|A|$ or $\det(A)$ or by replacing the brackets with vertical lines.

  For a general 2×2 matrix

  $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  the determinant is found by subtracting the product of the second diagonal from the product of the lead diagonal. Hence:

  $$\boxed{|A| = ad - bc}$$

- **Example**

  Calculate the determinant of $C = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$:

  Solution:

  This is a square matrix, so the determinant does exist.

  $$|C| = \begin{vmatrix} -1 & 2 \\ 4 & 5 \end{vmatrix} = (-1)(5) - (2)(4) = -5 - 8 = -13$$

8.2 Lecture 16: Solving simultaneous equations with matrices

- **Independent opening practice working with matrices.**

  Given $\quad A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

  Find (if they exist):

  $A + B, \quad 3B, \quad AB, \quad BA, \quad \det(A)$

- **Inverse Matrix:**

  For a square matrix A, there *may* exist an inverse matrix A^{-1}, such that:

  $AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$

  For a general 2×2 square matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse matrix is calculated by:

$$\boxed{A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

  where $|A|$ is the determinant of A.

  If the determinant of a square matrix is equal to zero, then that matrix has no inverse.

- **Examples:**

  For the following square matrices, find the inverse matrix if it exists.

  (1) $\quad A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \qquad A^{-1} = \dfrac{1}{(1)(2) - (-1)(0)}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}$

  (2) $\quad B = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix} \qquad B^{-1} = \dfrac{1}{(1)(2) - (0)(-3)}\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3/2 & 1/2 \end{pmatrix}$

  (3) $\quad C = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \qquad$ (Zero determinant - Inverse does not exist.)

- **Simultaneous Equations:**

  We may be familiar with solving systems of two simultaneous linear equations using the elimination or substitution methods. However, they can also be solved using an alternative matrix method. Computer systems can easily apply the same technique to solving systems of hundreds of variables and equations.

  Consider a pair of simultaneous equations involving variables x and y:

  $$ax + by = p$$

  $$cx + dy = q$$

  where a, b, c, d, p, q are (known) constants, and we wish to solve for x and y.

  1. Ensure the equations are in the consistent form above, then we can represent this system by a single matrix equation:

     $$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

     and so:

     $$AX = B$$

     where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix of coefficients, the vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$ contains the unknown variables, and $B = \begin{pmatrix} p \\ q \end{pmatrix}$.

  2. Calculate the inverse matrix A^{-1} of the matrix of coefficients.

     Note: If A has determinant zero, and thus cannot be inverted, the method fails at this point. This indicates that the pair of equations either have *infinite* solutions, or *zero* solutions.

  3. Pre-multiply both sides by the inverse matrix to solve for the vector X:

     $$A^{-1}AX = A^{-1}B \quad \Longrightarrow \quad X = A^{-1}B$$

  4. From the entries in X, read off the values of x and y.
  5. Verify solutions by substituting the values of x and y back into the original equations.

- **Example 1**

Solve for x and y:

$$5x + 2y = 10$$

$$4x - 3y = 14$$

Re-writing this as a matrix equation:

$$\begin{pmatrix} 5 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

so we have $AX = B$, where

$$A = \begin{pmatrix} 5 & 2 \\ 4 & -3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

Then,

$$A^{-1} = \frac{1}{(5)(-3) - (2)(4)} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix} = \frac{-1}{23} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix}$$

and so

$$X = A^{-1}B = \frac{-1}{23} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 10 \\ 14 \end{pmatrix} = \frac{-1}{23} \begin{pmatrix} -58 \\ 30 \end{pmatrix} = \begin{pmatrix} 58/23 \\ -30/23 \end{pmatrix}$$

Thus we find:

$$x = \frac{58}{23} \approx 2.52 \quad \text{and} \quad y = -\frac{30}{23} \approx -1.30$$

Verifying:

$$5x + 2y = 5\left(\frac{58}{23}\right) + 2\left(-\frac{30}{23}\right) = \frac{290}{23} - \frac{60}{23} = \frac{230}{23} = 10$$

and

$$4x - 3y = 4\left(\frac{58}{23}\right) - 3\left(-\frac{30}{23}\right) = \frac{232}{23} + \frac{90}{23} = \frac{322}{23} = 14$$

as required.

- **Example 2**

  Solve for x and y:

$$3x - 5y = 7$$

$$2x = 20 - 4y$$

  First we must transpose the second equation so that both are in a consistent format:

$$3x - 5y = 7$$

$$2x + 4y = 20$$

  Re-writing this as a matrix equation:

$$\begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 20 \end{pmatrix}$$

  so we have $AX = B$, where

$$A = \begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad B = \begin{pmatrix} 7 \\ 20 \end{pmatrix}$$

  Then,

$$A^{-1} = \frac{1}{(3)(4) - (-5)(2)} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix}$$

  and so

$$X = A^{-1}B = \frac{1}{22} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ 20 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 128 \\ 46 \end{pmatrix} = \begin{pmatrix} 64/11 \\ 23/11 \end{pmatrix}$$

  Thus we find:

$$x = \frac{64}{11} \approx 5.82 \quad \text{and} \quad y = \frac{23}{11} \approx 2.09$$

  Verifying:

$$3x - 5y = 3\left(\frac{64}{11}\right) - 5\left(\frac{23}{11}\right) = \frac{192}{11} - \frac{115}{11} = \frac{77}{11} = 7$$

  and

$$2x + 4y = 2\left(\frac{64}{11}\right) + 4\left(\frac{23}{11}\right) = \frac{128}{11} + \frac{92}{11} = \frac{220}{11} = 20$$

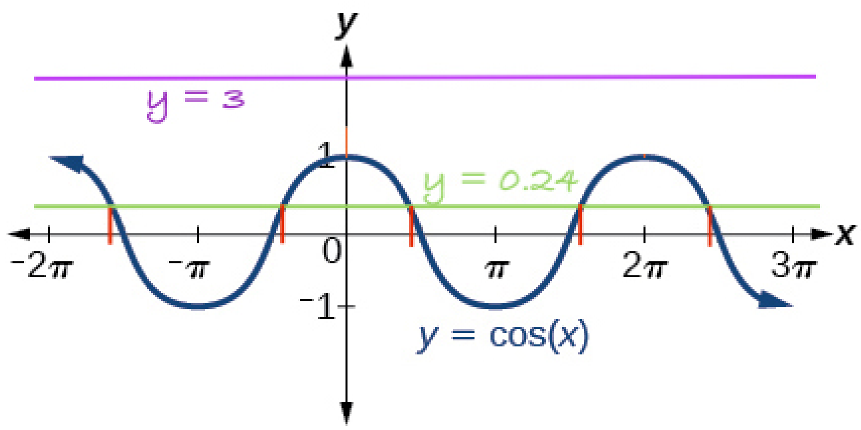
  as required.

9 Week 9

9.1 Lecture 17: Solving Trigonometric Equations

- Recall the graphs of sine, cosine and tangent and key features.
- Remember to use radians!
- **Example 1**

  Use this to illustrate the problem of multiple solutions due to periodicity.

  How many values of x are there such that:

  (a) $\cos(x) = 0.24$?

  Answer: 1.328 and 4.955, but there are infinitely many values!

  (b) $\cos(x) = 3$?

  Answer: None - outside the possible range of values.

- A solution to $\cos(x) = 0.24$ is a value of x on the graph where the straight line $y = 0.24$ intersects with the curve $y = \cos(x)$.
- Clearly, if there are going to be any solutions to a trigonometric equation, there will actually be *infinitely many* unless we restrict our interest to a specific range of x.

- **Method: Solving trigonometric equations of the form** $\sin(x) = a$

  1. Use the inverse trigonometric function on your calculator to obtain the **principal value**:

     $$x_0 = \sin^{-1}(a)$$

     This is the first solution, and will be the value closest to the y-axis.

  2. For sine and cosine, use the symmetry present in the graph (or a CAST diagram) to locate the other solution that occurs within the first cycle. Often, *but not always*, this takes the form $x_1 = \pi - x_0$ for sine, and $x_1 = -x_0$ for cosine.

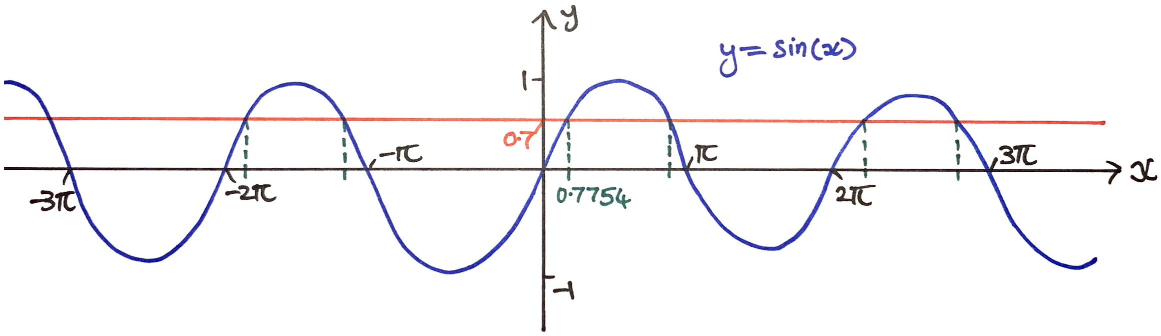
  3. To find all of the other solutions, then:
     - For sine and cosine, add and subtract integer multiples of 2π to **both** x_0 and x_1 until we are outside of the stated range.
     - For tangent, add and subtract multiplies of π to the principal value x_0 until we are outside of the stated range.

     Note: To avoid confusion it may help to use a specific algorithm such as alternately adding 2π to x_0 and x_1 until both are beyond the upper limit.

  4. Substitute each final answer back into the original formula $\sin(x) = a$ to verify them.

It is important to use a high degree of precision when calculating each solution, as errors may compound as we use x_0 to determine x_1 and then use that determine subsequent solutions. In the following solutions, we use 4 d.p. in all cases as a sufficient level of precision.

- **Example 2**



Solve $\sin(x) = 0.7$ for $-2\pi \leq x \leq 3\pi$:

First, note that the range approximates to $-6.2832 \leq x \leq 9.4248$.

Obtain the principal value:

$$x_0 = \sin^{-1}(0.7) = 0.7754$$

From the symmetry of the first peak on the graph, the distance between the origin and x_0 is the same as the distance back from $x = \pi$ to the other solution. Thus:

$$x_1 = \pi - x_0 = \pi - 0.7754 = 2.3662$$

Then add and subtract multiples of the period 2π until we have exhausted the range:

$$x_2 = x_0 + 2\pi = 0.7754 + 2\pi = 7.0586$$

$$x_3 = x_1 + 2\pi = 2.3662 + 2\pi = 8.6494$$

$$x_4 = x_0 + 4\pi = 7.0586 + 2\pi = 13.3418 \quad \text{(Too large)}$$

$$x_5 = x_1 - 2\pi = 2.3662 - 2\pi = -3.9170$$

$$x_6 = x_0 - 2\pi = 0.7754 - 2\pi = -5.5078$$

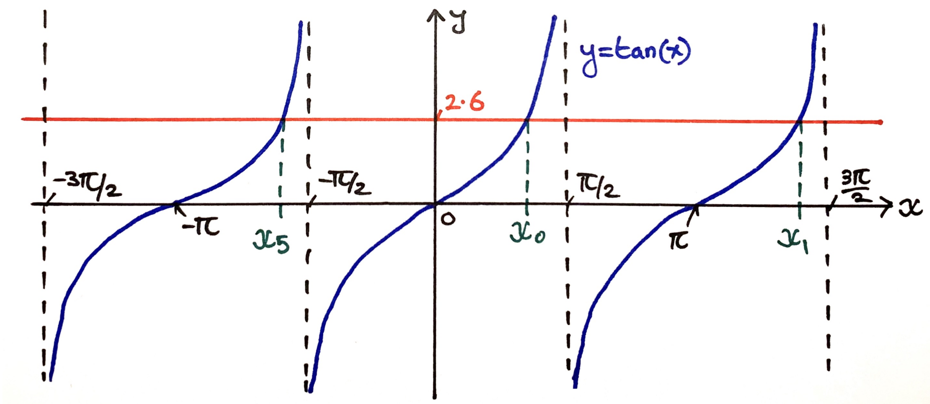
$$x_7 = x_1 - 4\pi = -3.9170 - 2\pi = -10.2002 \quad \text{(Too small)}$$

Hence, we obtain six solutions for x in the range:

$$0.7754, \quad 2.3662, \quad 7.00586, \quad 8.6494, \quad -3.9170, \quad -5.5078$$

Confirm by substituting the values back in, and by looking at the graph, that our results are sensible.

- **Example 3**



Find all values of x in the range $-2\pi \leq x \leq 4\pi$ for which $\tan(x) = 2.6$:

First, note that the range approximates to $-6.2832 \leq x \leq 12.5664$ and obtain the principal value:

$$x_0 = \tan^{-1}(2.6) = 1.2036$$

There is only one solution in a single period of tan (and so we expect six solutions in the range from -2π to 4π as this covers six periods). Then since it has period π (instead of 2π like sin and cos), add and subtract multiples of π from this initial solution:

$$x_1 = x_0 + \pi = 1.2036 + \pi = 4.3452$$

$$x_2 = x_0 + 2\pi = 1.2036 + 2\pi = 7.4868$$

$$x_3 = x_0 + 3\pi = 1.2036 + 3\pi = 10.6284$$

$$x_4 = x_0 + 4\pi = 1.2036 + 4\pi = 13.7700 \quad \text{(Too large)}$$

$$x_5 = x_0 - \pi = 1.2036 - \pi = -1.9380$$

$$x_6 = x_0 - 2\pi = 1.2036 - 2\pi = -5.0796$$

$$x_7 = x_0 - 3\pi = 1.2036 - 3\pi = -8.2212 \quad \text{(Too small)}$$

Hence, we obtain six solutions for x in the range:

$$1.2036, \quad 4.3452, \quad 7.4868, \quad 10.6284, \quad -1.9380, \quad -5.0796$$

Confirm by substituting the values back in to $\tan(x)$ that our results are sensible.

9.1.1 Solving trigonometric equations of the form $\sin(ax + b) = c$

- We have learned how to find solutions to problems of the form $\sin(x) = a$ in a given range of x.

  But what if we want to obtain the solutions to the more general equation:

  $$\sin(ax + b) = c$$

  where a, b, c are constants.

  1. Define a new variable $u = ax + b$ to transform this into a problem that we already know how to solve: $\sin(u) = c$.
  2. Calculate the new limits in terms of u, by substituting the limits of x into this formula.
  3. Determine the set of solutions for u.
  4. Convert solutions for u back to the corresponding solutions for x using:

     $$x = \frac{x - b}{a}$$

  5. Verify the final solutions by substituting back into $\sin(ax + b)$ and evaluating.

- **Example 4**

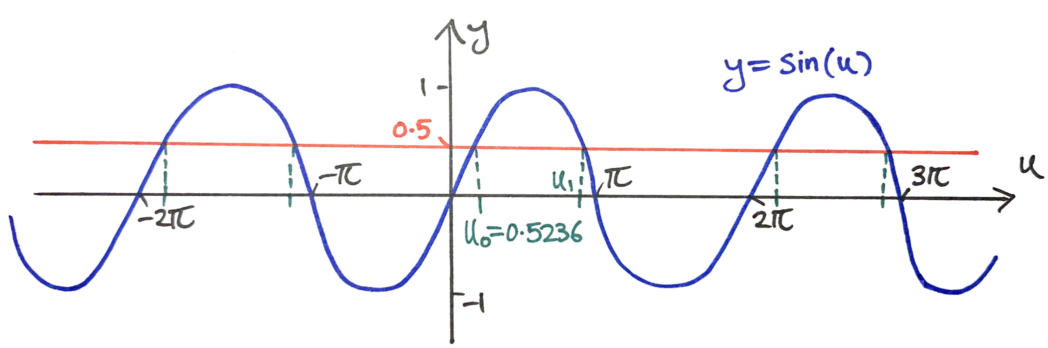
  Solve $\sin(3x + 0.2) = 0.5$ for $-\pi \leq x \leq \pi$.

  1. Let $u = 3x + 0.2$, then the range becomes $-3\pi + 0.2 \leq u \leq 3\pi + 0.2$, or

     $$-9.2248 \leq u \leq 9.6248$$

  2. Now the problem has been converted to:

     "Solve $\sin(u) = 0.5$ for $-9.2248 \leq u \leq 9.6248$."



3. Obtain the principal value:

$$u_0 = \sin^{-1}(0.5) = \frac{\pi}{6} = 0.5236$$

4. From the symmetry of the graph, the other solution in the first period is:

$$u_1 = \pi - 0.5236 = 2.6180$$

5. Adding and subtracting multiples of 2π, we find six solutions for u in the acceptable range. Then convert these back to solutions for x using $x = \frac{u-0.2}{3}$:

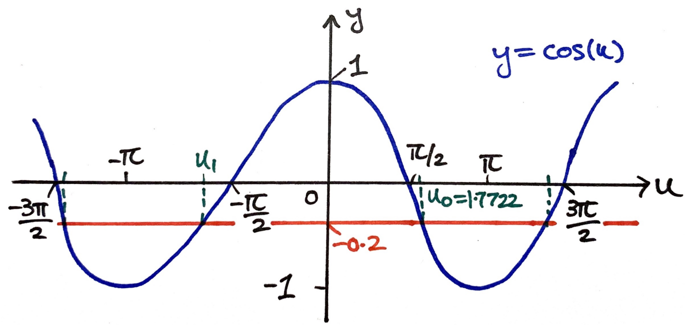
u	In Range?	$x = \dfrac{u-0.2}{3}$
$u_0 = 0.5236$	Yes	$(0.5236 - 0.2)/3 = 0.1079$
$u_0 + 2\pi = 0.5236 + 2\pi = 6.8058$	Yes	2.2023
$u_0 + 4\pi = 0.5236 + 4\pi = 13.090$	No	-
$u_0 - 2\pi = 0.5236 - 2\pi = -5.7596$	Yes	-1.9865
$u_0 - 2\pi = 0.5236 - 4\pi = -12.043$	No	-
$u_1 = 2.6180$	Yes	0.806
$u_1 + 2\pi = 2.6180 + 2\pi = 8.9012$	Yes	2.9004
$u_1 - 2\pi = 2.6180 - 2\pi = -3.6652$	Yes	-1.2884
$u_1 - 4\pi = 2.6180 - 4\pi = -9.9484$	No	-

9.2 Lecture 18: General Trigonometric functions

- **Example 5**

Find all solutions for $\cos(3x+1) = -0.2$ in the range $-\pi \le x \le \pi$.

1. Let $u = 3x+1$, then the problem becomes solving $\cos(u) = -0.2$ in the range $-3\pi + 1 \le u \le 3\pi + 1$, or $-8.425 \le u \le 10.425$.



2. Obtain the principal value:

$$u_0 = \cos^{-1}(-0.2) = 1.7722$$

3. Then from symmetry obtain the other solution in the first period:

$$u_1 = -u_0 = -1.7722$$

4. Adding and subtracting multiples of 2π, locate all solutions for u in the range.

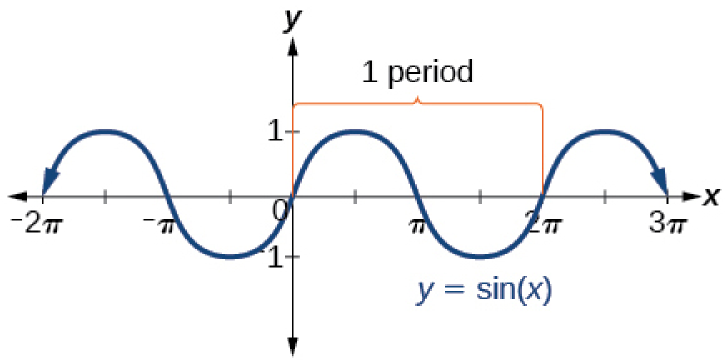
u	In Range?	$x = \dfrac{u-1}{3}$
$u_0 = 1.7722$	Yes	$(1.7722 - 1)/3 = 0.2564$
$u_0 + 2\pi = 1.7722 + 2\pi = 8.0554$	Yes	2.3518
$u_0 + 4\pi = 1.7722 + 4\pi = 14.339$	No	-
$u_0 - 2\pi = 1.7722 - 2\pi = -4.51106$	Yes	-1.8370
$u_0 - 4\pi = 1.7722 - 4\pi = -10.7942$	No	-
$u_1 = -1.7722$	Yes	0.9241
$u_1 + 2\pi = -1.7722 + 2\pi = 4.5110$	Yes	1.1703
$u_1 + 4\pi = -1.7722 + 4\pi = 10.7942$	No	-
$u_1 - 2\pi = -1.7722 - 2\pi = -8.0554$	Yes	-3.0185

Of course, $u_1 < u_0$, so $u_1 - 4\pi < u_0 - 4\pi$ and thus we know $u_1 - 4\pi$ will definitely be too small without needing to calculate it.

9.2.1 Graphs of the form $A\sin(ax+b)$

- **Standard sine wave**

  Begin with the graph of $y = \sin(x)$:



  It crosses the x-axis at:

  $$x = \ldots,\ -3\pi,\ -2\pi,\ -\pi,\ 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots$$

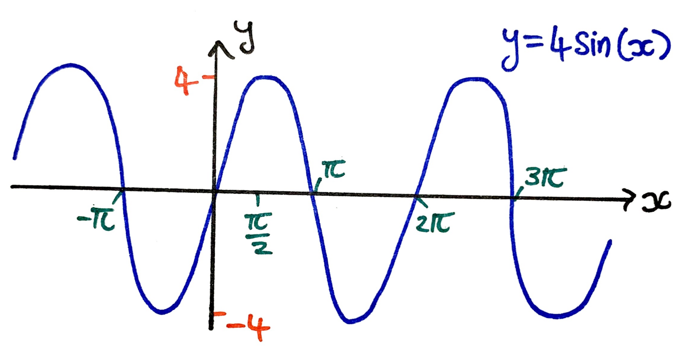
  The max. and min. values of $+1$ and -1 occur at:

  $$x = \ldots,\ -\frac{5}{2}\pi,\ -\frac{3}{2}\pi,\ -\frac{1}{2}\pi,\ \frac{1}{2}\pi,\ \frac{3}{2}\pi,\ \frac{5}{2}\pi,\ \ldots$$

  We consider how modifying the parameters in a generalised sinusoidal function of the form $y = A\sin(ax+b)+c$ (where A, a, b, c are constants) impacts these properties.

- **Amplitude:**

  Consider $y = 4\sin(x)$. All special values occur at the same points, but the maximum and minimum values become $+4$ and -4 respectively.



  Amplitude A in $y = A\sin(x)$ stretches the graph by a factor of A along the y-axis.

- **Frequency:**

  Consider $y = \sin(3x)$.

  It crosses the x-axis at:

  $$3x = \ldots,\ -3\pi,\ -2\pi,\ -\pi,\ 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots$$

  which is when

  $$x = \ldots,\ -\pi,\ -\frac{2}{3}\pi,\ -\frac{1}{3}\pi,\ 0,\ \frac{1}{3}\pi,\ \frac{2}{3}\pi,\ \pi,\ \ldots$$

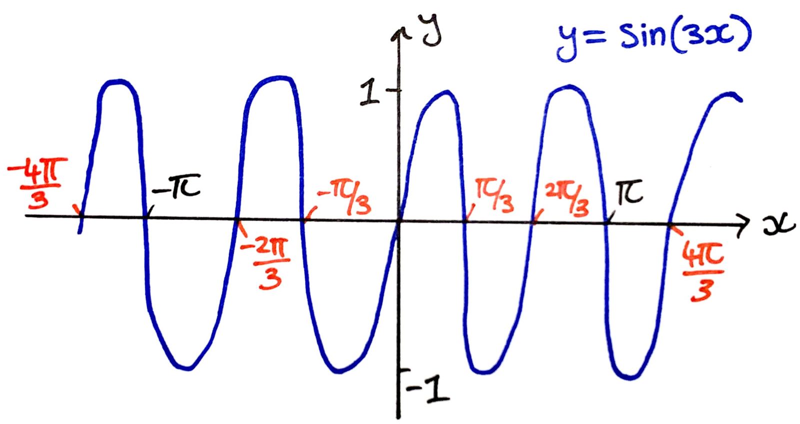
  The max. and min. values of $+1$ and -1 occur at:

  $$3x = \ldots,\ -\frac{5}{2}\pi,\ -\frac{3}{2}\pi,\ -\frac{1}{2}\pi,\ \frac{1}{2}\pi,\ \frac{3}{2}\pi,\ \frac{5}{2}\pi,\ \ldots$$

  which is when

  $$x = \ldots,\ -\frac{5}{6}\pi,\ -\frac{3}{6}\pi,\ -\frac{1}{6}\pi,\ \frac{1}{6}\pi,\ \frac{3}{6}\pi,\ \frac{5}{6}\pi,\ \ldots$$

  So points in $\sin(3x)$ occur three times as often as in $\sin(x)$, and the graph is "squashed" by factor of 3 along the x-axis.

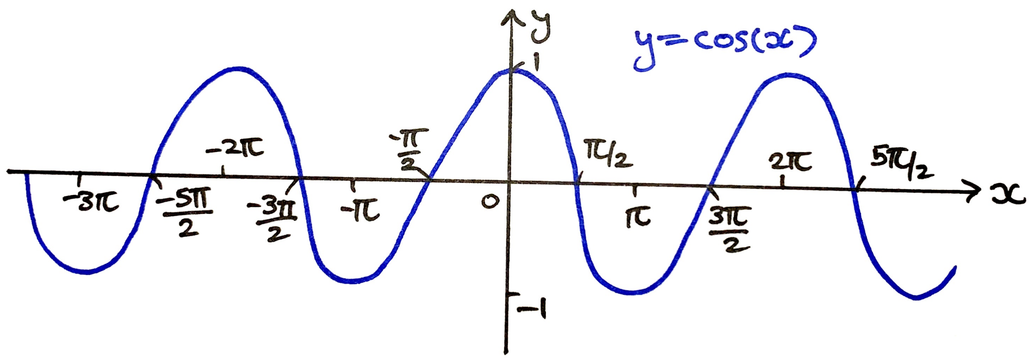


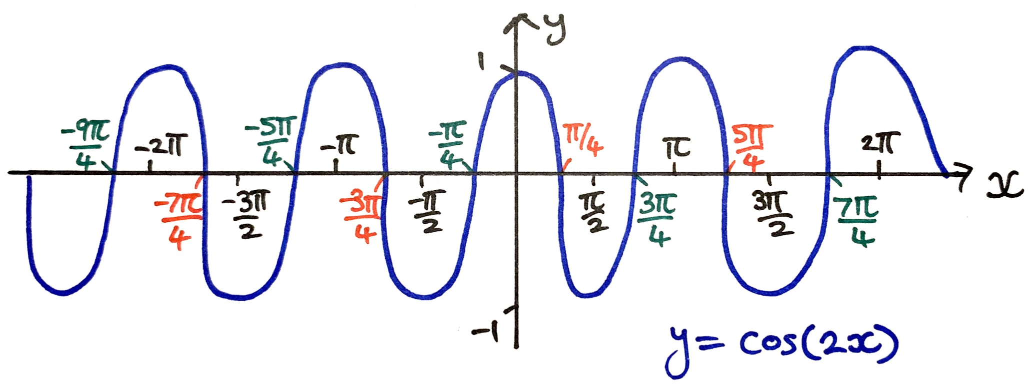
  This factor a in $\sin(ax)$ is called the **frequency**.

  If the frequency is less than 1, then the graph is "stretched" in the x-axis.

- **Example (Frequency):**

  Draw the graphs of $y = \cos(x)$ and $y = \cos(2x)$.





- **Phase:**

  Consider $y = \sin(x + \frac{\pi}{6})$.

  It crosses the x-axis at:

  $$x + \frac{\pi}{6} = \ldots,\ \ -3\pi,\ \ -2\pi,\ \ -\pi,\ \ 0,\ \ \pi,\ \ 2\pi,\ \ 3\pi,\ \ \ldots$$

  which is when

  $$x = \ldots,\ \ -3\pi - \frac{\pi}{6},\ \ -2\pi - \frac{\pi}{6},\ \ -\pi - \frac{\pi}{6},\ \ -\frac{\pi}{6},\ \ \pi - \frac{\pi}{6},\ \ 2\pi - \frac{\pi}{6},\ \ 3\pi - \frac{\pi}{6},\ \ \ldots$$

  and so when

  $$x = \ldots,\ \ -\frac{19}{6}\pi,\ \ -\frac{13}{6}\pi,\ \ -\frac{7}{6}\pi,\ \ -\frac{\pi}{6},\ \ \frac{5}{6}\pi,\ \ \frac{11}{6}\pi,\ \ \frac{17}{6}\pi,\ \ \ldots$$

  The max. and min. values of $+1$ and -1 occur at:

  $$x + \frac{\pi}{6} = \ldots,\ \ -\frac{5}{2}\pi,\ \ -\frac{3}{2}\pi,\ \ -\frac{1}{2}\pi,\ \ \frac{1}{2}\pi,\ \ \frac{3}{2}\pi,\ \ \frac{5}{2}\pi,\ \ \ldots$$

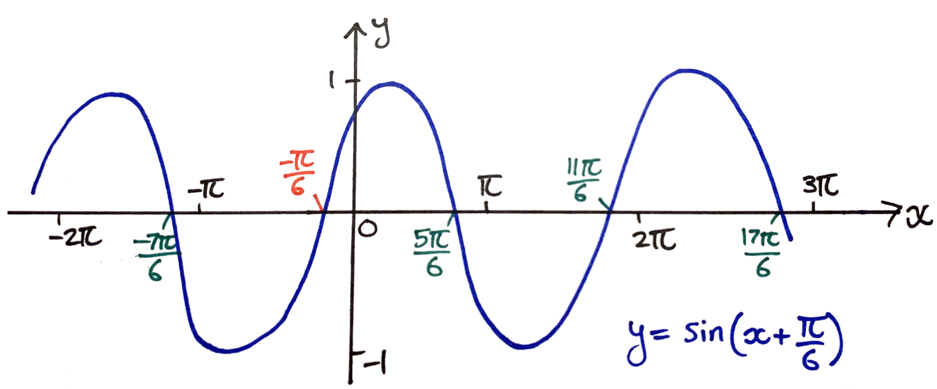
  which is when

  $$x = \ldots,\ \ -\frac{5}{2}\pi - \frac{\pi}{6},\ \ -\frac{3}{2}\pi - \frac{\pi}{6},\ \ -\frac{1}{2}\pi - \frac{\pi}{6},\ \ \frac{1}{2}\pi - \frac{\pi}{6},\ \ \frac{3}{2}\pi - \frac{\pi}{6},\ \ \frac{5}{2}\pi - \frac{\pi}{6},\ \ \ldots$$

  and so when

  $$x = \ldots,\ \ -\frac{5}{3}\pi,\ \ -\frac{2}{3}\pi,\ \ \frac{1}{3}\pi,\ \ \frac{4}{3}\pi,\ \ \ldots$$

  So $y = \sin(x + \frac{\pi}{6})$ looks like $y = \sin(x)$ shifted left along the x-axis by $\frac{\pi}{6}$.



  This value b in $\sin(x + b)$ is called the phase.

- **Note:** If phase is *negative* (e.g. $y = \sin(x - \frac{\pi}{6})$), the graph is shifted to the *right*.
- **Interaction between frequency and phase:**

  If plotting a function of the form $\sin(ax+b)$, first factorise out the frequency:

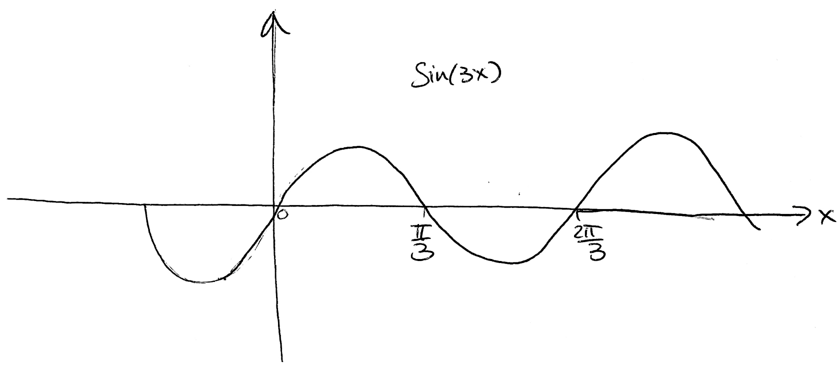
  $$\sin(ax+b) = \sin\left(a\big(x + \frac{b}{a}\big)\right)$$

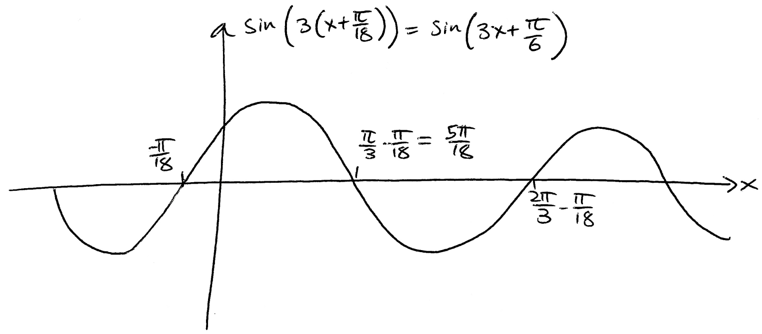
  This means (i) first squash the sine graph along the x-axis by a factor of a, and *then* (ii) shift the resulting graph by $\frac{b}{a}$ in the negative direction (left).

  Example:

  $$\sin\left(3x + \frac{\pi}{6}\right) = \sin\left(3\big(x + \frac{\pi}{18}\big)\right)$$

  First, draw the graph of $\sin(3x)$. Then shift the graph of this function left by $\frac{b}{a} = \frac{\pi}{18}$:





Check: $\sin\left(3x+\frac{\pi}{6}\right)\Big|_{x=\frac{5\pi}{18}} = \sin\left(3\left(\frac{5\pi}{18}\right)+\frac{\pi}{6}\right)$

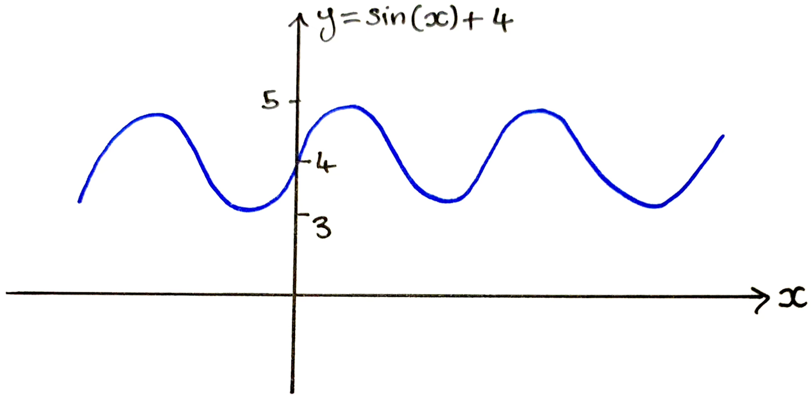
$= \sin\left(\frac{15\pi}{18}+\frac{3\pi}{18}\right) = \sin\left(\frac{18\pi}{18}\right)$

$= \sin(\pi)$

- **Vertical shift**

  Adding a value $+c$ to the trigonometric function simply shifts the graph upwards, and does not affect the locations of any important values or the shape of the curve.

  For example, consider $y = \sin(x) + 4$:



  The maximum and minimum values are now $1 + 4 = 5$ and $-1 + 4 = 3$ respectively.

- **Summary:**

  For $\;y = A\sin(ax + b) + c$

  - A is the amplitude. The maximum and minimum values are $\pm A$. The graph is stretched by A along the y-axis.
  - a is the frequency. This squashes the graph by a factor of a along the x-axis.
  - b is the phase. This shifts the graph leftward by $\frac{b}{a}$ along the x-axis.
  - c is the vertical shift. This shifts the graph upward by c along the y-axis.

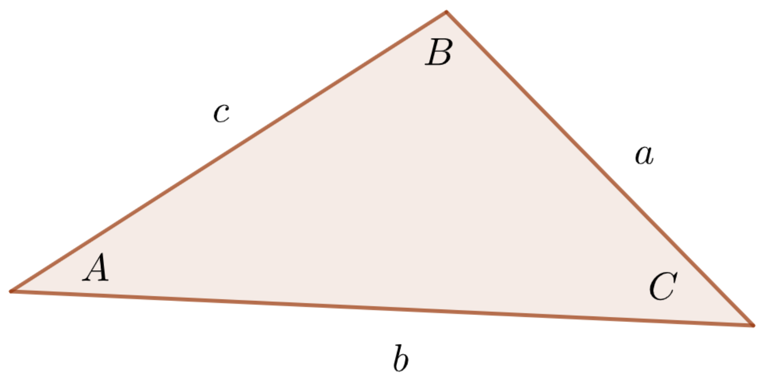
10 Week 10

10.1 Lecture 19: Trigonometric Rules and Identities

We have already learned some trigonometric rules (SOH-CAH-TOA) that apply specifically to right-angled triangles and help us calculate unknown sides and angles.

There exist more general rules that apply to *any* triangle.

Note: when working with triangles, use **degrees** and not radians as the unit of angle measurement.



In general, when working with triangles, we think of the sides and angles as existing in pairs - given an angle (A, B, C) the "corresponding" or "opposite" side (a, b, c) that pairs with it is the one side that the angle does not actually touch. We always use the same letter, with uppercase denoting the angle, and lowercase denoting the side.

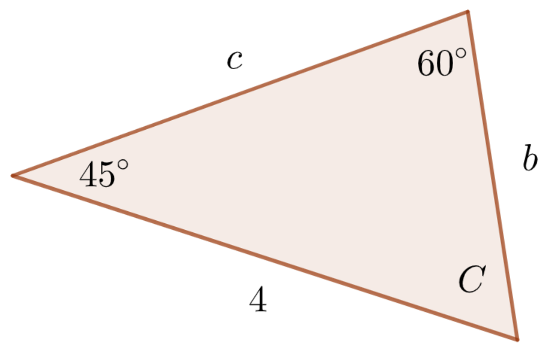
10.1.1 The Sine rule

We can use this rule to help us when we already have a side-angle pair that we know:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \qquad \text{or} \qquad \frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

10.1.2 Example

Find the missing sides b and c.



1. First, we know the angle (45°) opposite side b, and we also have a full side-angle pair (4 and 60°). Therefore, we can use the sine rule to find b:

$$\frac{b}{\sin(45)} = \frac{4}{\sin(60)}$$

Re-arranging to make b the subject:

$$b = \frac{4\sin(45)}{\sin(60)} = \frac{4\sqrt{6}}{3} = 3.266$$

2. Since we know the other two angles, we can easily find the third angle C using the fact that (for all triangles) all the angles must sum to 180°. Thus:

$$C = 180 - (60 + 45) = 75^\circ$$

3. Now that we know the angle C, we can employ the sine rule a second time to find it's opposing side c:

$$c = \frac{4\sin(75)}{\sin(60)} = 4.461$$

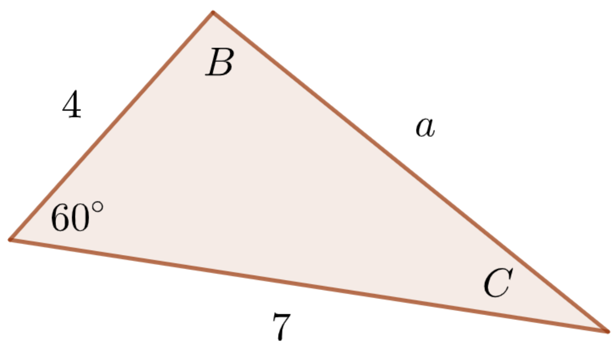
10.1.3 The Cosine Rule

This rule can be used when either we know all three sides and want to find an angle, or when we know two sides and the angle opposite the missing side and wish to find that side:

$$a^2 = b^2 + c^2 - 2bc\cos(A) \qquad \text{or} \qquad \cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

10.1.4 Example

Find the missing side a and angle B.



1. Use the cosine rule to find a:

$$\begin{aligned} a^2 &= 7^2 + 4^2 - 2 \times 7 \times 4 \times \cos(60) \\ &= 49 + 16 - 56 \times \frac{1}{2} \\ &= 37 \end{aligned}$$

   Taking the square root:

$$\therefore a = \sqrt{37} = 6.083$$

2. We can find B using the sine rule:

$$\sin B = \frac{7\sin(60)}{6.083}$$

   Then taking the inverse sine of both sides:

$$B = \sin^{-1}(0.9966) = 85.27^\circ$$

Alternatively, this could be achieved using the cosine rule, rewritten in the form:

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

Hence, substituting in the values of a, b, c:

$$49 = 37 + 16 - 48.664\cos(B)$$

which, when transposed for B, yields:

$$B = \cos^{-1}(0.0822) = 85.28^\circ$$

10.1.5 Proof of Pythagoras' theorem

Pythagoras' theorem is actually a special case of this more general cosine rule. If we apply it to a right-angled triangle, with the angle $A = 90^\circ$ being the right-angle, then the opposite side a is the hypotenuse (b and c can simply be the other two sides, it doesn't matter which is which). Then, since

$$\cos(90^\circ) = 0$$

The cosine rule yields:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cdot 0 \\ &= b^2 + c^2 \end{aligned}$$

And so we have derived Pythagoras' theorem!

10.1.6 Trigonometric Identities

10.1.7 Important identities

An *identity* is an equation that is true for ALL angles. There are several that relate the trigonometric functions, and the most important one is:

$$\sin^2(A) + \cos^2(A) = 1$$

Before we proceed, let's define some extra trigonometric functions:

$$\sec(A) = \frac{1}{\cos(A)}, \quad \operatorname{cosec}(A) = \frac{1}{\sin(A)}, \quad \cot(A) = \frac{1}{\tan A}$$

Then we can divide the identity $\sin^2(A)+\cos^2(A) = 1$ by $\cos^2(A)$, to obtain a new identity:

$$\tan^2(A) + 1 = \sec^2(A) \qquad \text{or} \qquad \sec^2(A) - \tan^2(A) = 1$$

We can use these identities, as well as other rules (such as the definition of tangent in terms of sine and cosine) to prove other relationships between the trigonometric functions.

10.1.8 Example

Prove that,

$$\sec^4(x) - \sec^2(x) = \tan^4(x) + \tan^2(x)$$

Solution:

To prove a relationship, we need to either start from the left-hand side (LHS) of the equation and show, step-by-step, that it is the same as the right-hand side (RHS) or vice-versa.

In general, it is often easier to start with the "more complicated"-looking side if there is one, and try to show that it is the same as the "simpler" side. In this particular example, both sides are about the same, so it doesn't matter which side we start with.

Start from the LHS:

$$\begin{aligned}
\sec^4(x) - \sec^2(x) &= (\sec^2(x))^2 - \sec^2(x) \\
&= (\tan^2(x) + 1)^2 - (\tan^2(x) + 1) \\
&= \tan^4(x) + 2\tan^2(x) + 1 - \tan^2(x) - 1 \\
&= \tan^4(x) + \tan^2(x)
\end{aligned}$$

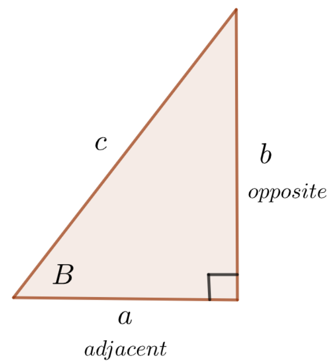
So by using an identity at the second step, expanding the bracket, and then simplifying, we have arrived at the RHS and thus proven this new relationship.

10.1.9 Deriving the identity

Using a right-angled triangle and Pythagoras' Theorem, we can prove the identity:

$$\sin^2(x) + \cos^2(x) = 1$$

Consider the following right-angled triangle:



Using SOH-CAH-TOA, we can obtain the following:

$$\sin(B) = \frac{b}{c} \qquad \text{and} \qquad \cos(B) = \frac{a}{c}$$

Rearranging these:

$$b = c\sin(B) \qquad \text{and} \qquad a = c\cos(B)$$

Then substituting both into Pythagoras' Theorem:

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (c\cos(B))^2 + (c\sin(B))^2 \\ &= c^2\big(\cos^2(B) + \sin^2(B)\big) \end{aligned}$$

Then dividing both sides by c^2:

$$1 = \cos^2(B) + \sin^2(B)$$

This is true for any angle B in radians or degrees.

10.2 Lecture 20: Trig. formulae for sums and differences

10.2.1 Multiple angle formulae

We will use six formulae that allow us to calculate the sine, cosine and tangent of the sum of two angles, or the difference between two angles A and B. These are true when using either radians or degrees.

For sine,

$$\sin(A + B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$

and

$$\sin(A - B) = \sin(A)\cos(B) - \sin(B)\cos(A)$$

Both of these identities can be written together as:

$$\boxed{\sin(A \pm B) = \sin(A)\cos(B) \pm \sin(B)\cos(A)}$$

For cosine,

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

and

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

Both of these identities can be written together as:

$$\boxed{\cos(A + B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)}$$

For tangent,

$$\boxed{\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}}$$

10.2.2 Example

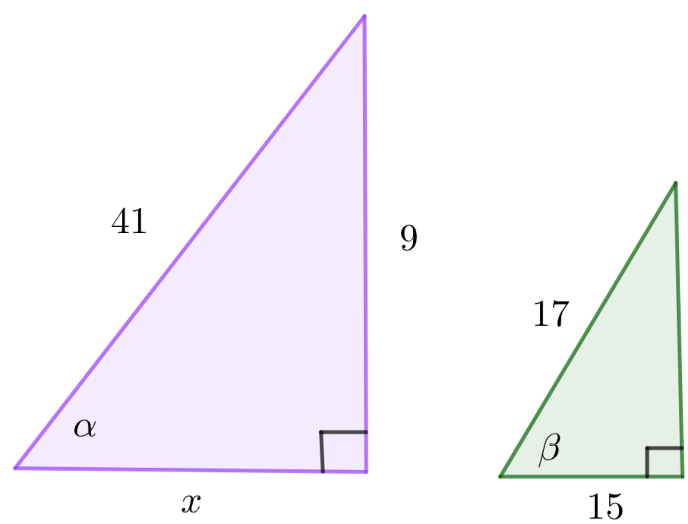
Suppose that α and β are acute angles (that is, $0 < \alpha, \beta < 90$) such that

$$\sin(\alpha) = \frac{9}{41} \quad \text{and} \quad \cos(\beta) = \frac{15}{17}$$

Find $\sin(\alpha + \beta)$ and $\cos(\alpha - \beta)$.

Solution:

Draw right-angled triangles, and use Pythagoras' Theorem and SOH-CAH-TOA to find the other two parts that we don't yet have. That is: $\cos(\alpha)$ and $\sin(\beta)$.



For the first triangle, Pythagoras' theorem gives:

$$x^2 + 9^2 = 41^2$$

and so

$$x = \sqrt{41^2 - 9^2} = 40$$

Hence:

$$\cos(\alpha) = \frac{adjacent}{hypotenuse} = \frac{40}{41}$$

Similarly, for the second triangle, the side opposite B is given by:

$$\sqrt{17^2 - 15^2} = 8$$

And thus we obtain

$$\sin(\beta) = \frac{opposite}{hypotenuse} = \frac{8}{17}$$

Then we have our four constituent parts of the two formulae that we are trying to find:

$$\begin{aligned}
\sin(\alpha + \beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\
&= \frac{9}{41} \cdot \frac{15}{17} + \frac{40}{41} \cdot \frac{8}{17} \\
&= \frac{455}{697} \\
&= 0.65
\end{aligned}$$

And:

$$\begin{aligned}
\cos(\alpha - \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\
&= \frac{40}{41} \cdot \frac{15}{17} + \frac{9}{41} \cdot \frac{8}{17} \\
&= \frac{672}{697} \\
&= 0.96
\end{aligned}$$

10.2.3 Special case: Double-angle Formulae

Setting $A = B$ in the sine formula, we obtain:

$$\boxed{\sin(2A) = 2\sin(A)\cos(A)}$$

and setting $A = B$ in the cosine formula gives:

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

Then substituting either of these terms using the identity $\sin^2(A) + \cos^2(A) = 1$, we can obtain:

$$\cos(2A) = 2\cos^2(A) - 1 \qquad \text{or} \qquad \cos(2A) = 1 - 2\sin^2(A)$$

And setting $A = B$ in the tangent formula gives:

$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

10.2.4 Special case: Half-angle Formulae

To find a formula for the cosine of half of an angle, take the double-angle formula $\cos(2A) = 2\cos^2(A) - 1$, set $X = 2A$ and rearrange to solve for $\cos(X/2)$:

$$\cos^2\left(\frac{X}{2}\right) = \frac{1}{2}\big(\cos(X) + 1\big) \qquad \text{or} \qquad \cos\left(\frac{X}{2}\right) = \sqrt{\frac{\cos(X) + 1}{2}}$$

Similarly, we can rearrange $\cos(X) = 1 - 2\sin^2(X/2)$ to obtain the half-angle formula for sine:

$$\sin^2\left(\frac{X}{2}\right) = \frac{1}{2}\big(1 - \cos(X)\big)$$

10.2.5 Expanding wave expressions of the form $R\sin(A + \phi)$

10.2.6 Example

Write

$$2\sin(A) + 3\sin(A + 35)$$

in the form $R\sin(A + \phi)$.

Solution:

We need to find the values of the constants R and ϕ that make this true for *all* values of A.

First, use the expansion formula for $\sin(A + B)$:

$$3\sin(A + 35) = 3\big(\sin(A)\cos(35) + \cos(A)\sin(35)\big)$$

so that we have:

$$2\sin(A) + 3\sin(A + 35) = \sin(A)\big(2 + 3\cos(35)\big) + \cos(A)\big(3\sin(35)\big)$$

Similarly expand $R\sin(A + \phi)$:

$$R\sin(A + \phi) = R\big(\sin(A)\cos(\phi) + \cos(A)\sin(\phi)\big)$$

Then equating these two equations:

$$\sin(A)\big(2 + 3\cos(35)\big) + \cos(A)\big(3\sin(35)\big) = \sin(A)\big(R\cos(\phi)\big) + \cos(A)\big(R\sin(\phi)\big)$$

Match the coefficients of $\sin(A)$ and $\cos(A)$:

$$R\cos(\phi) = 2 + 3\cos(35), \qquad R\sin(\phi) = 3\sin(35)$$

Divide one of these equations by the other to find the phase angle ϕ:

$$\tan(\phi) = \frac{R\sin(\phi)}{R\cos(\phi)} = \frac{3\sin(35)}{2 + 3\cos(35)} = 0.386$$

$$\therefore \phi = \tan^{-1}(0.386) = 21.11°$$

(Note: if this gave a negative value, we would add 360° (or 2π if we were using radians) to make ϕ positive and preserve the original sine functions)

To find the amplitude:

$$\begin{aligned}
\big(R\cos(\phi)\big)^2 + \big(R\sin(\phi)\big)^2 &= \big(2 + 3\cos(35)\big)^2 + \big(3\sin(35)\big)^2 \\
R^2\big(\cos^2(\phi) + \sin^2(\phi)\big) &= 22.83 \\
R^2 &= 22.83
\end{aligned}$$

Hence:

$$R = 4.78$$

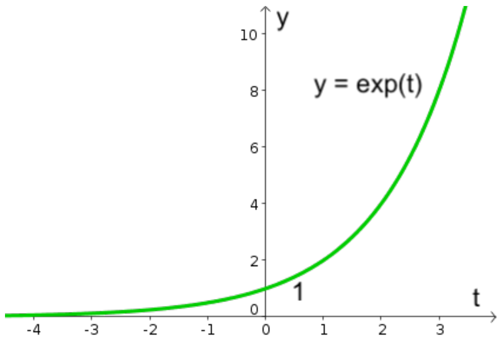
Thus:

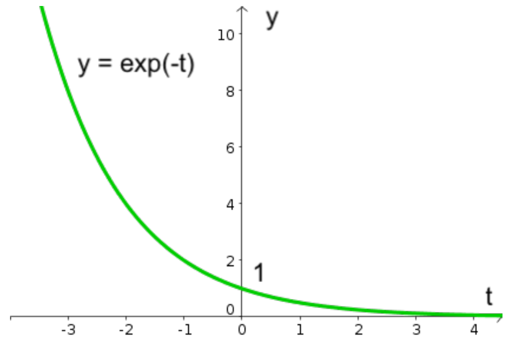
$$2\sin(A) + 3\sin(A + 35) = 4.78\sin(A + 21.11^\circ)$$

11 Week 11

11.1 Lecture 21: Exponential and Natural Log functions

- From Lecture 1, recall the exponential function: $y = \mathrm{e}^x$ or $y = exp(x)$.





The value is defined at each point by the limit:

$$y = \mathrm{e}^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

The constant e (Euler's number) can be found by setting $x = 1$ to be $2.71828\ldots$

- We often use the independent variable t when using the exponential function to decribe time-dependent functions of growth and decay.

- As t increases, e^t very quickly gets extremely large. We say that it tends to infinity:

$$\mathrm{e}^t \to \infty \quad \text{as} \quad t \to \infty$$

As t becomes large and negative, e^t becomes very small - but never actually reaches zero or becomes negative.

$$\mathrm{e}^t \to 0 \quad \text{as} \quad t \to -\infty$$

- The graph of $y = \mathrm{e}^{-t}$ looks like $y = \mathrm{e}^t$ reflected in the y-axis. Hence,

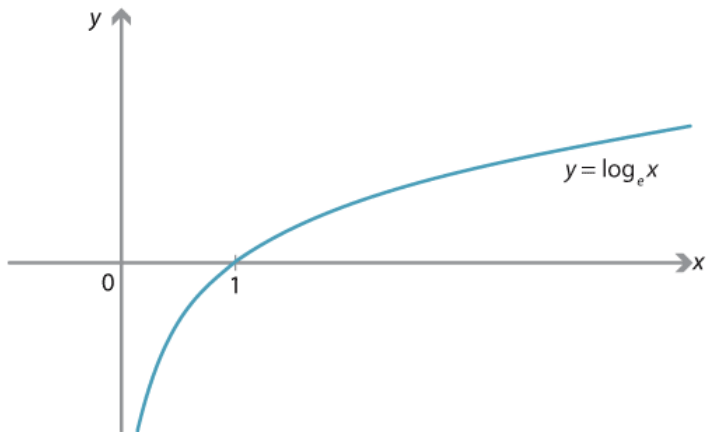
$$\mathrm{e}^{-t} \to 0 \quad \text{as} \quad t \to \infty, \quad \text{and } \mathrm{e}^{-t} \to \infty \quad \text{as} \quad t \to -\infty$$

- Both graphs have a y-intercept of $\mathrm{e}^0 = 1$.

- **The Natural Log function**

  This is written as $y = \log_e(x)$ ("log base e") or $y = \ln(x)$, where x is positive $(x > 0)$. It is the inverse of the exponential function $y = \mathrm{e}^x$. That is,

  $$\text{If} \quad y = \mathrm{e}^x \quad \text{then} \quad x = \ln(y)$$

  Visually, this means that the graph is the same as the exponential function with the axes switched.

- As x gets large and positive (i.e. as $x \to \infty$), then $y = \ln(x)$ increases indefinitely, so $y \to \infty$. However this does not happen very quickly - it is slower than exponential growth.

- $\ln(1) = 0$ so the x-intercept is at $x = 1$.

- As x approaches zero (from positive values) $y = \ln(x)$ gets very large and negative, so $y \to -\infty$ as $x \to 0$.

- $\ln(x)$ is not defined for $x = 0$ or for any negative values of x.

- **Laws of the Natural Log**

  For any real numbers x and y:

  $$\ln(xy) = \ln(x) + \ln(y)$$

  $$\ln(x/y) = \ln(x) - \ln(y)$$

  $$\ln\left(x^n\right) = n\ln(x)$$

  $$\ln(1) = 0, \quad \ln\left(\mathrm{e}^1\right) = 1, \quad \ln\left(\mathrm{e}^x\right) = x$$

- **Example 1**

  Show the following using the laws of logs:

$$(a) \qquad \ln(24) = \ln(3) + 3\ln(2)$$

$$(b) \qquad \ln(32) = 5\ln(2)$$

$$(c) \qquad \ln\left(\frac{5}{4}\right) = \ln(5) - 2\ln(2)$$

$$(d) \qquad \ln\left(\frac{3^2 7}{5}\right) = 2\ln(3) + \ln(7) - \ln(5)$$

$$(e) \qquad \ln\left(\frac{xy}{z^2}\right) = \ln(x) + \ln(y) - 2\ln(z)$$

- When solving equations with logs and exponential functions, the key is that log() and exp() are inverse operations, so they can be used to undo each other as required.

- **Example 2**

  Make x the subject of $y = A\,\mathrm{e}^{kx}$ where $A = 2$ and $k = 4$.

  Solution:

$$y = 2\,\mathrm{e}^{4x}$$

1. Divide by 2 to get rid of the 2:

$$\frac{y}{2} = \mathrm{e}^{4x}$$

2. Now undo the exponential by taking the ln of both sides and using the rule above that $\ln\left(\mathrm{e}^{4x}\right) = 4x$:

$$\ln\left(\frac{y}{2}\right) = \ln\left(\mathrm{e}^{4x}\right) = 4x$$

3. Divide both sides by 4:

$$x = \frac{1}{4}\ln\left(\frac{y}{2}\right)$$

- **Example 3**

  Solve the following equation for P:

  $$x = \frac{\ln(P/21)}{-0.00013}$$

  Solution:

  Begin by multiplying both sides by -0.00013:

  $$-0.00013x = \ln(P/21)$$

  Use the exponential function to "undo" the log:

  $$\mathrm{e}^{-0.00013x} = \mathrm{e}^{\ln(P/21)} = \frac{P}{21}$$

  Finally multiply both sides by 21 to make P the subject:

  $$P = 21\,\mathrm{e}^{-0.00013x}$$

  So P experiences exponential decay as x increases.

11.2 Lecture 22: Exponential and Logarithmic word problems

- Real-life growth and decay processes are often modelled using exponential functions. These can be manipulated and solved using logarithms.
- For each question, read the question carefully to determine: what is the information we are given, and what quantity is it that we are trying to find?

- **Example 1**

  The growth formula for the number of websites on the internet is given by

  $$W = 65000\,\mathrm{e}^{0.7t}$$

  where W is the number of websites, and t is the time in years from a fixed starting date. How many years will it take for there to be 60 million websites?

  Let $W = 60000000$, then the equation becomes:

  $$60000000 = 65000\,\mathrm{e}^{0.7t}$$

  We want to solve this for t:

  $$\frac{60000000}{65000} = \mathrm{e}^{0.7t}$$

  Simplifying,

  $$\frac{12000}{13} = \mathrm{e}^{0.7t}$$

  Then to "undo" the exponential, we can take the natural log of both sides:

  $$\ln\left(\frac{12000}{13}\right) = \ln\left(\mathrm{e}^{0.7t}\right) = 0.7t$$

  So, dividing both sides by 0.7, we obtain a solution for t:

  $$t = \frac{1}{0.7}\ln\left(\frac{12000}{13}\right) = 9.75 \text{ years from the start date}$$

- **Example 2**

Suppose the population P (in millions) of a particular country is modelled by the following formula:

$$P = A\,\mathrm{e}^{kt}$$

where t is the time (in years) since 1975, and A and k are unknown constants. The population was counted as 10 million in January 1975 and again as 15 million in January 2000.

(i) What was the forecasted population in 2010?

(ii) What year did the population exceed 20 million?

Solution:

For these questions, we begin by using the given sets of information to find the constants:

Using $P(t = 0) = 10$, we find:

$$10 = A\,\mathrm{e}^{k\times 0} = A \times 1 = A$$

Hence,

$$A = 10$$

Then using $A = 10$ and $P(t = 25) = 15$, we get:

$$15 = 10\,\mathrm{e}^{10k}$$

Rearranging to make k the subject:

$$k = \frac{1}{25}\ln(15/10) = 0.016219\ldots$$

Hence,

$$P = 10\,\mathrm{e}^{0.016219t}$$

(i) 2010 is 35 years after 1975, so set $t = 35$ and then solve for P:

$$\begin{aligned} P(t = 35) &= 10\,\mathrm{e}^{0.016219\times 35} \\ &= 17.641\ldots \end{aligned}$$

So there will be a population of 17.6 million in 2010.

(ii) We wish to solve for t when $P = 20$:

$$20 = 10\,\mathrm{e}^{0.016219t}$$

$$2 = \mathrm{e}^{0.016219t}$$

$$\ln(2) = \ln\left(\mathrm{e}^{0.016219t}\right) = 0.016219t$$

$$t = \frac{\ln(2)}{0.016219} = 42.74\ldots$$

So it is projected to occur 42.7 years after 1975, which is in Autumn 2017.

- **Example 3**

Assume that the atmospheric pressure (in kPa) at height h (in m) above sea level is given by the formula:

$$P = A\,\mathrm{e}^{-kh}$$

and you are provided with the following dataset:

P	h	
93.9	570	(1)
	895	
75.6	2250	(2)
32.5		

(a) Use the table to find A and k.

(b) Determine the missing entries in the table.

Solution:

(a) Using the two complete data sets ((1) and (2)), we can obtain two simultaneous equations for P and h:

Using (1) we obtain the equation:

$$93.9 = A\,\mathrm{e}^{-570k}$$

Using (2) we similarly obtain:

$$75.6 = A\,\mathrm{e}^{-2250k}$$

To solve for k first, we divide this first equation by the second to eliminate A:

$$\frac{93.9}{75.6} = \frac{A\,\mathrm{e}^{-570k}}{A\,\mathrm{e}^{-2250k}} = \mathrm{e}^{-570k-(-2250k)} = \mathrm{e}^{-570k+2250k} = \mathrm{e}^{1680k}$$

Then taking the natural log of both sides:

$$\ln\left(\frac{93.9}{75.6}\right) = \ln\left(\mathrm{e}^{1680k}\right) = 1680k$$

and rearranging:

$$k = \frac{1}{1680}\ln\left(\frac{93.9}{75.6}\right) = 0.000129$$

Finally, substitute this value for k back into either of our two equations and solve for A. Using equation (1):

$$93.9 = A\,\mathrm{e}^{-570\times 0.000129}$$

Transposing for A:

$$A = 93.9 \div \mathrm{e}^{-570\times 0.000129} = 101.06$$

Hence,

$$P = 101.06\,\mathrm{e}^{-0.000129h}$$

(b) First substitute in $h = 895$:

$$P = 101.06\,\mathrm{e}^{-0.000129\times 895} = 90.0\ kPa$$

Then, set $P = 32.5$ and solve for h:

$$32.5 = 101.06\,\mathrm{e}^{-0.000129h}$$

Thus,

$$h = \frac{1}{-0.000129}\ln\left(\frac{32.5}{101.06}\right) = 8794.4\ m$$

Remember to include units and check that these solutions seem sensible.