Lecture 1: Laplace Transforms (1)

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Further Mathematics, Signals and Systems

Content

There are three topics:

- (Weeks 1-5) Laplace transforms
- (Weeks 5-8) Fourier series
- (Weeks 9-11) Matrix algebra

Lecture 1: Laplace Transforms

Today we shall cover the following topics on Laplace Transforms:

- Definition from first principles
- Using the tables
- Linearity
- Inversion of basic functions
- Discontinuous functions
- Heaviside/Unit Step function

Pierre-Simon Laplace (1749-1827)

"He brought the spirit of the *infinitely small* into the government."
Napoleon on why he fired Laplace as Minister of the Interior.

Definition from first principles

Let f be a function of time, t, that is zero for $t < 0$.
Then the Laplace transform of $f(t)$ is denoted by:

$$\mathcal{L}\{f(t)\} \qquad \text{or} \qquad \bar{f}(s)$$

It is defined for $t > 0$ and given by the integral:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

We have moved from the **time domain** (variable t) to the **complex frequency domain**, where the variable is s.

The condition on s for the transform to be valid can be found by looking for any occurences of $(s - \alpha)$ in the denominator of the transform, and then requiring $s > \alpha$.

Example using the Table of Laplace Transforms

In practice, we obtain Laplace transforms using a table of standard known transforms, rather than using integration.

Example: to determine the Laplace transform of:

$$f(t) = \sin(7t) \qquad \text{(for } t > 0, \text{ and zero otherwise)}$$

We look up the following entry in the table:

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}$$

Using this with $\omega = 7$:

$$\bar{f}(s) = \mathcal{L}\{\sin(7t)\} = \frac{7}{s^2 + 7^2} = \frac{7}{s^2 + 49}$$

valid for $s > 0$

Practice using the Table of Laplace Transforms

Using the table, find the Laplace Transform of a function that takes the following values for $t > 0$, and zero elsewhere:

(a)	3	(b)	t
(c)	t^2	(d)	$\sin(7t)$
(e)	$t\,\mathrm{e}^{-4t}$	(f)	$\sin(3t+\phi)$

Linearity

If $f(t)$ and $g(t)$ are time-dependent functions, and a and b are constants:

Linearity

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

This is called the “linearity” property.

Example (Using Linearity)

Using tables, determine the Laplace transform of

$$x(t) = \begin{cases} 0 & \text{for } t < 0; \\ 3t - 2t\,e^{-4t} & \text{for } t > 0. \end{cases}$$

$$\begin{aligned} \bar{x}(s) &= \mathcal{L}\{3t - 2t\,e^{-4t}\} \\ &= 3\mathcal{L}\{t\} - 2\mathcal{L}\{t\,e^{-4t}\} \\ &= 3\left(\frac{1}{s^2}\right) - 2\left(\frac{1}{(s+4)^2}\right) \\ &= \frac{3(s+4)^2 - 2s^2}{s^2(s+4)^2} = \frac{s^2 + 24s + 48}{s^2(s+4)^2} \end{aligned}$$

Valid for $s > 0$.

Linearity: A common mistake

Important!

Linearity does not mean that you can transform the product of two time-dependent functions by multiplying their individual transforms:

$$\mathcal{L}\{f(t) \times g(t)\} \neq \mathcal{L}\{f(t)\} \times \mathcal{L}\{g(t)\}$$

For example,

$$\mathcal{L}\left\{t\,e^{-\alpha t}\right\} = \frac{1}{(s+\alpha)^2}$$

which is **not** the same as

$$\mathcal{L}\{t\} \times \mathcal{L}\left\{e^{-\alpha t}\right\} = \frac{1}{s^2} \times \frac{1}{s+\alpha} = \frac{1}{s^2(s+\alpha)}$$

Inverse Transforms

If we are given the Laplace transform of a function, say $\bar{f}(s)$, and wish to obtain the corresponding function in the time-domain, $f(t)$, we use the inverse transform.

$$f(t) \xrightarrow{\mathcal{L}} \bar{f}(s)$$

$$f(t) \xleftarrow[\mathcal{L}^{-1}]{} \bar{f}(s)$$

For simple cases, look up the relevant entry in the tables.

We will also need to state the range of time for which the solution $f(t)$ is valid, which is $t > 0$ (for now!)

Example (inverse)

Invert the Laplace transform

$$\bar{v}(s) = \frac{1}{s+3}$$

The relevant entry is

$$\mathcal{L}^{-1}\left\{\frac{1}{s+\alpha}\right\} = \mathrm{e}^{-\alpha t}$$

with $\alpha = 3$.

Therefore,

$$v(t) = \begin{cases} 0 & \text{for } t < 0; \\ \mathrm{e}^{-3t} & \text{for } t > 0. \end{cases}$$

Practice inverse transforms

Using the table, find the inverse Laplace Transform of the following:

$$(a) \quad \bar{f}(s) = \frac{3\phi}{1+7s}$$

$$(b) \quad \bar{g}(s) = \frac{6E}{s^4}$$

where ϕ and E are constants.

Inverse Transforms

Often, $\bar{f}(s)$ will be a complicated fraction. Before taking the inverse transforms, we may need to simplify it using partial fractions or completing the square.

There will be plenty of questions to practice these techniques on this week's tutorial sheet.

Both techniques are described in Appendix 1, and the supplementary worksheet provides extra questions on partial fractions.

Example (inverse transform with partial fraction): Part 1

Invert the Laplace transform

$$\bar{f}(s) = \frac{1}{s(sCR + 1)}$$

where C, R are positive constants.

Example (inverse transform with partial fraction): Part 2

Begin with partial fractions:

$$\frac{1}{s(sCR+1)} = \frac{A}{s} + \frac{B}{sCR+1}$$

Cross-multiplying,

$$1 = A(sCR+1) + Bs$$

Choosing $s = 0$ reduces the equation to $A = 1$.
Choosing $s = -1/CR$, we obtain:

$$1 = \frac{-1}{CR}B \quad \Longrightarrow \quad B = -CR$$

Hence, the transformed function is equivalent to:

$$\bar{f}(s) = \frac{1}{s} - \frac{CR}{sCR+1}$$

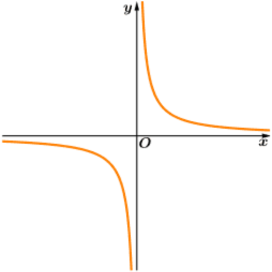
Example (inverse transform with partial fraction): Part 3

Then we can obtain the inverse Laplace transform of each term:

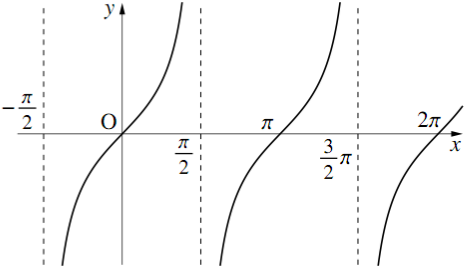
$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{CR}{sCR+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - CR\mathcal{L}^{-1}\left\{\frac{1}{sCR+1}\right\} \\ &= 1 - CR\frac{1}{CR}\,\mathrm{e}^{-t/CR} \\ &= 1 - \mathrm{e}^{-t/CR} \end{aligned}$$

This is the value of $f(t)$ for $t > 0$, and it is zero otherwise.

A function $f(x)$ is discontinuous at a point if there is a break in the graph of the function at that point.



(a) $y = \frac{1}{x}$



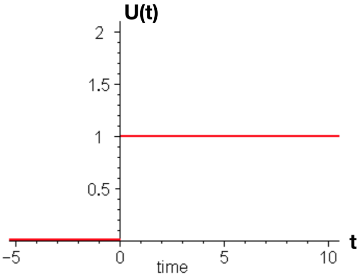
(b) $y = \tan(x)$

Heaviside Functions

The Unit Step Function, or Heaviside's unit function, $U(t)$, has a discontinuity at $t = 0$:

Unit Step Function

$$U(t) = \begin{cases} 0 & \text{if } t < 0; \\ 1 & \text{if } t > 0. \end{cases}$$



Heaviside Functions

We will use $U(t)$ instead of explicitly refering to the domain of t.

Instead of:

$$v(t) = \begin{cases} 0 & \text{if } t < 0; \\ e^{-3t}(2cos(5t) + \frac{1}{5}sin(5t)) & \text{if } t > 0. \end{cases}$$

write:

$$v(t) = e^{-3t}(2\cos(5t) + \frac{1}{5}\sin(5t))U(t)$$

to indicate that the function "switches on" at time $t = 0$.

Laplace transform and Heaviside Functions

Since the Laplace transform is only defined for functions that are zero for $t < 0$:

When transforming a function (e.g. $\sin(t)$), we usually assume that it only has this behaviour for $t > 0$.

Hence:

$$\mathcal{L}\{\sin(t)U(t)\} = \mathcal{L}\{\sin(t)\}$$

as the $U(t)$ merely formally states what we already assumed!

This is **only** true for $U(t)$, and **not** for $U(t-???)$, as we shall see next week!

Inverse Laplace transform and Heaviside Functions

Since the Laplace transform is only defined for functions that are zero for $t < 0$:

When obtaining an **inverse** Laplace transform we previously stated "valid for $t > 0$". From now on, multiply the final answer by $U(t)$ to indicate that it starts at $t = 0$.

Instead of:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = \begin{cases} 0 & \text{for } t < 0; \\ e^{-3t} & \text{for } t > 0. \end{cases}$$

We will write:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}\, U(t)$$

Summary

After today, you should . . .

- Be able to use the tables to take Laplace transforms of standard functions.
- Be able to use the tables to take *inverse* Laplace transforms of standard functions.
- Be able to use partial fractions to obtain the inverse transform of more complicated functions.
- Understand what a Heaviside or *step function* is. We will be using this a lot next week!

This Week

This week's lecture corresponds to Section 2.1-2.3 of the Course Notes, which contains many additional worked examples.

Before this week's tutorial:

- Attempt Questions 1-3 of Tutorial Sheet 1.

Next week's lecture will be on Laplace Transforms of functions that have a time delay (Section 2.4-2.5).

Extra Question - Inverting a function by completing the square

Invert the Laplace transform

$$\bar{x}(s) = \frac{7}{s^2 + 2s + 17}$$

The denominator $s^2 + 2s + 17$ doesn't have an integer factorisation, so completing the square:

$$\bar{x}(s) = \frac{7}{(s+1)^2 - 1^2 + 17} = \frac{7}{4}\left(\frac{4}{(s+1)^2 + 4^2}\right)$$

Then using the inverse transform $\mathcal{L}^{-1}\left\{\frac{\omega}{(s+\alpha)^2+\omega^2}\right\} = \mathrm{e}^{-\alpha t}\sin(\omega t)$ with $\alpha = 1$ and $\omega = 4$ we obtain:

$$x(t) = \frac{7}{4}\mathrm{e}^{-t}\sin(4t) \quad \text{for} \quad t > 0$$

Extra Question - Inversion with partial fractions: Part 1

Invert the Laplace transform:

$$\bar{g}(s) = \frac{E(sCR+1)}{s(sCR+2)^2}$$

Extra Question - Inversion with partial fractions: Part 2

Using partial fraction expansion:

$$\frac{E(sCR+1)}{s(sCR+2)^2} = \frac{A}{s} + \frac{B}{sCR+2} + \frac{D}{(sCR+2)^2}$$

$$E(sCR+1) = A(sCR+2)^2 + Bs(sCR+2) + Ds$$

- Choosing $s = 0$ yields $E = 4A$, and so $A = E/4$.
- Setting $s = -2/CR$ results in $D = ECR/2$.
- We could choose another value of s and solve for B, or equate the coefficients of s^2 to find $0 = C^2R^2A + CRB$ and thus obtain $B = \frac{-ECR}{4}$.

Extra Question - Inversion with partial fractions: Part 3

Hence,

$$\bar{g}(s) = \left(\frac{E}{4}\right)\frac{1}{s} + \left(\frac{-E}{4}\right)\frac{1}{s + (2/CR)} + \left(\frac{E}{2CR}\right)\frac{1}{(s + (2/CR))^2}$$

Then taking the inverse transform of each term:

$$\begin{aligned} g(t) &= \frac{E}{4} - \frac{E}{4}\,\mathrm{e}^{-2t/CR} + \frac{E}{2CR}\,t\,\mathrm{e}^{-2t/CR} \\ &= \frac{E}{4CR}\left\{ CR + (2t - CR)\,\mathrm{e}^{-2t/CR} \right\} \end{aligned}$$

for $t > 0$, and zero otherwise.

Extra Question - Inversion with partial fractions: Part 4

To state this using step functions instead:

$$g(t) = \frac{E}{4CR}\left\{ CR + (2t - CR)\,\mathrm{e}^{-2t/CR} \right\} U(t)$$