Lecture 2: Laplace Transforms (2)

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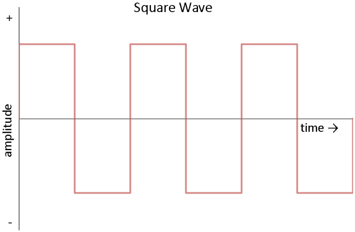
Further Mathematics, Signals and Systems

Lecture 2: Laplace Transforms

Today we shall cover:

- Constructing functions with time-delay using step functions.
- Delay form and the Delay Theorem.
- Laplace transforms of functions with time-delay.
- Inverse Laplace transforms of functions with time-delay.



Electrical signals often consist of multiple types of behaviour. We need to construct formula for such waves using delayed step functions, then obtain their Laplace transforms.

Heaviside Functions

Last time we defined the Heaviside or Unit Step function,

$$U(t) = \begin{cases} 0 & \text{if } t < 0; \\ 1 & \text{if } t > 0. \end{cases}$$

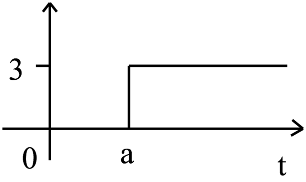
We can generalise this to step up or down at different times.

Heaviside Functions

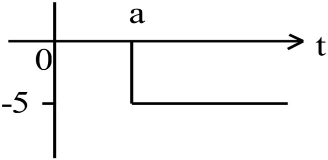
$$3U(t-a) = \begin{cases} 0 & \text{if } t < a; \\ 3 & \text{if } t > a. \end{cases}$$

is a step up by 3 at time $t = a$.

$-5U(t-a)$ is a step down of 5 at time $t = a$.



(b) $y = 3U(t-a)$



(c) $y = -5U(t-a)$

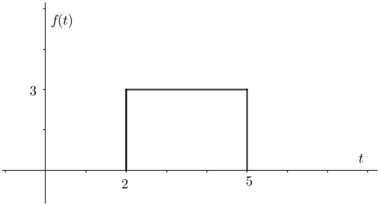
Heaviside Functions

Combine multiple step functions to switch signals on and off.

Example:

$$\begin{aligned} f(t) &= 3U(t-2) - 3U(t-5) \\ &= 3\Big(U(t-2) - U(t-5)\Big) \end{aligned}$$

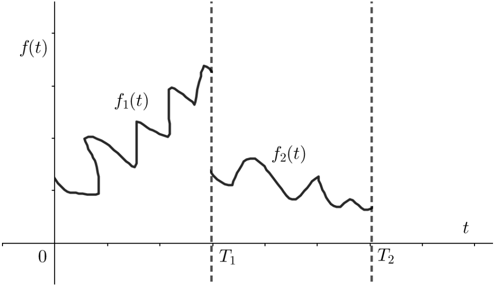
- Constant strength 3.
- Begins at time $t = 2$
- Ends at time $t = 5$



Heaviside Functions

Consider a function f that behaves like f_1 for the interval $[0, T_1]$, then changes to act like f_2 during the next interval $[T_1, T_2]$ before switching off.

$$f(t) = f_1(t)\Big(U(t) - U(t - T_1)\Big) + f_2(t)\Big(U(t - T_1) - U(t - T_2)\Big)$$

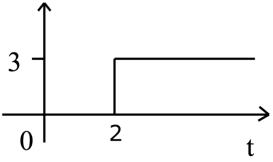


Laplace transforms of functions with time delay

Last week, we said that Laplace transforms are defined for functions that are zero for $t < 0$. How do we deal with a signal that starts (or changes) at a time *other than zero*?

Example: A constant signal that begins at time $t = 2$

$$f(t) = 3U(t-2)$$



This is the problem of how to take Laplace transforms of functions that include a time delay.

Delay Theorem

If our function contains a time-shifted step function $U(t-???)$, then we are taking the Laplace transform of a function with a delay.

We can use a special transform - the Heaviside or **delay theorem**:

Delay Theorem

$$\mathcal{L}\left\{g(t-T)U(t-T)\right\} = e^{-sT}\mathcal{L}\left\{g(t)\right\}$$

The constant T is the value of the time-delay.

Delay Form

The first stage of applying this theorem is to ensure that the function is in **delay form**. This means that all occurences of t are written explicitly as $t - T$, where T is the delay.

Example:

$$f(t) = tU(t-2) \qquad \textit{Not in delay form.}$$

$$= ((t-2)+2)U(t-2) \qquad \text{In delay form!}$$

One way to do this is simply replace all occurrences of t with $(t - T) + T$.

Exercise: Delay Form

Are the following functions in delay form?

$$(a) \qquad f(t) = 3(t-5)U(t-5)$$

$$(b) \qquad g(t) = tU(t-1)$$

$$(c) \qquad g(t) = \cos(3t)U(t)$$

$$(d) \qquad h(t) = 5(t-2)U(t-3)$$

$$(e) \qquad j(t) = \big(1+\sin(t)\big)U(t-4)$$

$$(f) \qquad k(t) = \mathrm{e}^{-\alpha(t-3)}\,U(t-3)$$

$$(g) \qquad v(t) = 3\cos(t)U(t-2T)$$

Procedure: Applying the Delay Theorem

Once we have ensured $f(t)$ is in delay form:

1. Name the part multiplied by the step function as $g(t - delay)$.
2. Replace $t - delay$ with t to obtain the function $g(t)$.
3. Take the Laplace transform of $g(t)$.
4. Multiply by $\mathrm{e}^{-s\times \mathrm{delay}}$

Look at the Delay Theorem again,

$$\mathcal{L}\{g(t-T)U(t-T)\} = e^{-sT}\mathcal{L}\{g(t)\}$$

Can you explain how this formula relates to the steps above?

Example 1: Applying the Delay Theorem

Find the Laplace transform of:

$$f(t) = \mathrm{e}^{-a(t-4T)}\, U(t-4T).$$

This is already in delay form, with delay of $4T$.

Declare $g(t-4T) = \mathrm{e}^{-a(t-4T)}$,

Therefore, $g(t) = \mathrm{e}^{-at}$

Using $\mathcal{L}\{\mathrm{e}^{-at}\} = \frac{1}{s+a}$ from the tables, apply the delay theorem:

$$\begin{aligned} \bar{f}(s) &= \mathcal{L}\{\mathrm{e}^{-a(t-4T)}\, U(t-4T)\} \\ &= \mathcal{L}\left\{\mathrm{e}^{-at}\right\} \times \mathrm{e}^{-s\times(4T)} = \frac{\mathrm{e}^{-4Ts}}{s+a} \end{aligned}$$

Example 2: Applying the Delay Theorem

Find the Laplace transform of:

$$f(t) = 3tU(t-2)$$

First, this must be rewritten in delay form:

$$f(t) = 3\big((t-2)+2\big)U(t-2)$$

Then declare $g(t-2) = 3\big((t-2)+2\big)$,

Therefore, $g(t) = 3(t+2) = 3t+6$

Applying the delay theorem:

$$\begin{aligned} \bar{f}(s) &= \mathcal{L}\{3tU(t-2)\} \\ &= \mathcal{L}\big\{3t+6\big\} \times \mathrm{e}^{-s\times 2} = \left(\frac{3}{s^2}+\frac{6}{s}\right)\mathrm{e}^{-2s} = \frac{3}{s^2}\big(1+2s\big)\,\mathrm{e}^{-2s} \end{aligned}$$

Exercise: Applying the Delay Theorem

Consider the function:

$$v(t) = (3t + 1)U(t - 2)$$

(a) What is the value of the time delay?

(b) Is this function written in delay form?

(c) Using the delay theorem, obtain the Laplace transform $\bar{v}(s)$.

Procedure: Applying the Inverse Delay Theorem

Inverse Delay Theorem

$$\mathcal{L}^{-1}\left\{e^{-sT}\bar{g}(s)\right\} = g(t-T)U(t-T).$$

1. Identify that the factor e^{-sT} means there will be a delay of T.
2. Identify $\bar{g}(s)$, where $\bar{f}(s) = \bar{g}(s)e^{-sT}$.
3. Invert this part to obtain $g(t) = \mathcal{L}^{-1}\{\bar{g}(s)\}$.
4. Change the variable from t to $t-T$, and so replace every occurrence of t in $g(t)$ with $t-T$, to get $g(t-T)$.
5. Multiply by the step function $U(t-T)$, to get:

$$f(t) = g(t-T)U(t-T)$$

Example: Applying the Inverse Delay Theorem (Part 1)

Find the inverse Laplace Transform of:

$$\bar{f}(s) = \frac{1}{s+2}\,\mathrm{e}^{-2sT}$$

First identify from the e^{-2sT} that the solution has a delay of $2T$.

Name the other part $\bar{g}(s)$:

$$\bar{g}(s) = \frac{1}{s+2}$$

Obtain the inverse transform of this:

$$g(t) = \mathcal{L}^{-1}\{\bar{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \mathrm{e}^{-2t}$$

Example: Applying the Inverse Delay Theorem (Part 2)

The inverse delay theorem says that we need $g(t-2T)$, so find this by replacing t with $t-2T$:

$$g(t) = \mathrm{e}^{-2t} \quad \Longrightarrow \quad g(t-2T) = \mathrm{e}^{-2(t-2T)}$$

To obtain the final answer, multiply by a step function with the same delay, $U(t-2T)$:

$$f(t) = \mathrm{e}^{-2(t-2T)}\, U(t-2T)$$

Exercise: Applying the Inverse Delay Theorem

Consider the function:

$$\bar{h}(s) = \frac{s}{s^2 + \omega^2}\,\mathrm{e}^{-3s}$$

- What is the value of the time delay?
- Using the inverse delay theorem, obtain the inverse Laplace transform $h(t)$.

Problems with multiple delays

Some functions have multiple step functions with different delays.

Example:

$$f(t) = 5t\big(U(t-2) - U(t-5)\big)$$

This signal turns on at time $t = 2$, and off at $t = 5$.

We would need to expand the brackets and treat each delay (of 2 and 5) separately - putting the bits multiplied by each step function into the corresponding delay form.

Summary

After today, you should be able to . . .

- Write a function with delay in delay form.
- Explain what the delay theorem says.
- Use the delay theorem to obtain the Laplace transform of a function with delay.
- Use the delay theorem to find the inverse transform of a delayed function.

This Week

This lecture corresponds to Section 2.4-2.5 of the Course Notes.

Before this week's tutorial:

- Work through some additional examples from this section of the Course Notes.
- Try Questions 1, 2(a)-(c) and 3(a)-(c) of Tutorial Sheet 2.

In next week's lecture we will begin using Laplace transforms to solve differential equations.

Extra Question - Multiple delays example: Part 1

Find the Laplace transform of:

$$f(t) = 5t\big(U(t-2) - U(t-5)\big)$$

Separate the differently-delayed parts and write each in delay form:

$$\begin{aligned} f(t) &= 5t\big(U(t-2) - U(t-5)\big) \\ &= 5tU(t-2) - 5tU(t-5) \\ &= 5\Big((t-2)+2\Big)U(t-2) - 5\Big((t-5)+5\Big)U(t-5) \end{aligned}$$

Now we are ready to apply the delay theorem twice.

Extra Question - Multiple delays example: Part 2

$$
\begin{aligned}
\bar{f}(s) &= \mathcal{L}\left\{ \big((t-2)+2\big)U(t-2) - 5\big((t-5)+5\big)U(t-5) \right\} \\
&= 5\mathcal{L}\left\{ \big((t-2)+2\big)U(t-2) \right\} - 5\mathcal{L}\left\{ \big((t-5)+5\big)U(t-5) \right\} \\
&= 5\mathcal{L}\left\{ t+2 \right\} \mathrm{e}^{-2s} - 5\mathcal{L}\left\{ t+5 \right\} \mathrm{e}^{-5s} \\
&= 5\left(\frac{1}{s^2} + \frac{2}{s} \right) \mathrm{e}^{-2s} - 5\left(\frac{1}{s^2} + \frac{5}{s} \right) \mathrm{e}^{-5s} \\
&= \frac{5}{s^2}\left((1+2s)\,\mathrm{e}^{-2s} - (1+5s)\,\mathrm{e}^{-5s} \right)
\end{aligned}
$$

Extra Question - Inverse with Multiple Delays: Part 1

Determine the inverse Laplace transform of:

$$\bar{f}(s) = \frac{3\,e^{-2s} + 2s\,e^{-3s}}{s^2}$$

Expanding this fraction,

$$\begin{aligned}\bar{f}(s) &= \frac{3}{s^2}\,e^{-2s} + \frac{2s}{s^2}\,e^{-3s} \\ &= 3\frac{1}{s^2}\,e^{-2s} + 2\frac{1}{s}\,e^{-3s}\end{aligned}$$

The two time-delay exponentials are multiplied by fundamentally different functions: $\frac{1}{s^2}$ and $\frac{1}{s}$. Therefore treat the two delays separately, and apply the inverse delay theorem twice.

Extra Question - Inverse with Multiple Delays: Part 2

Name what is multiplied by the first exponential $\bar{g}_1(s)$, and what is multiplied by the second exponential $\bar{g}_2(s)$:

$$\bar{g}_1(s) = \frac{3}{s^2}, \qquad \bar{g}_2(s) = \frac{2}{s}$$

Taking the inverse Laplace transform of each of these:

$$\begin{aligned} g_1(t) &= \mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 3t \\ g_2(t) &= \mathcal{L}^{-1}\left\{\frac{2}{s}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 2 \times 1 = 2 \end{aligned}$$

Finally we replace t with $t-2$ or $t-3$ and multiply by the corresponding step function:

$$\begin{aligned} f(t) &= g_1(t-2)U(t-2) + g_2(t-3)U(t-3) \\ &= 3(t-2)U(t-2) + 2U(t-3) \end{aligned}$$