Lecture 3: Laplace Transforms (3)

Dr Gavin M Abernethy

Further Mathematics, Signals and Systems

Lecture 3: Laplace Transforms

Today we shall cover:

- How to obtain Laplace transforms of integrals and derivatives.
- Using Laplace transforms to solve a single integro-differential equation.

Next week we will extend these ideas to solving a system of several ODEs, and apply them to analysing electronic circuits.

Modelling Electronic Components

- Capacitor with capacitance C and current i passing through it:

$$\text{Potential difference} = \frac{1}{C}\int_0^t i(t)\,\mathrm{d}t + \text{ initial p.d.}$$

- Resistor with resistance R and current i passing through it:

$$\text{Potential difference} = iR.$$

- Inductor with inductance L and current i passing through it:

$$\text{Potential difference} = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$$

To solve equations modelling these components, we may need to obtain Laplace transforms of integrals and derivatives.

Laplace Transforms of Integrals and Derivatives

If $f(t)$ is some time-dependent function:

Laplace Transforms of Derivatives and Integrals

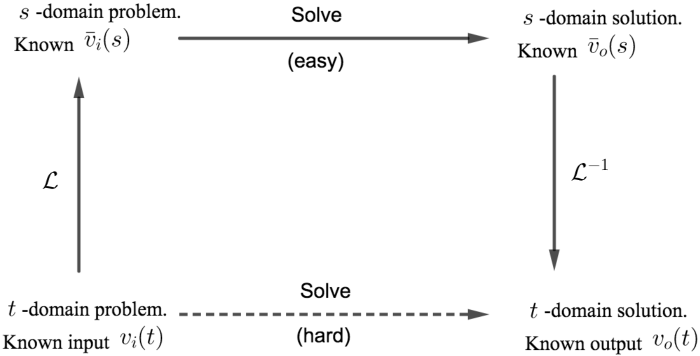
$$\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = s\bar{f}(s) - f(0)$$

$$\mathcal{L}\left\{\frac{\mathrm{d}^2f(t)}{\mathrm{d}t^2}\right\} = s^2\bar{f}(s) - sf(0) - \dot{f}(0)$$

$$\mathcal{L}\left\{\int_0^t f(t)\mathrm{d}t\right\} = \frac{\bar{f}(s)}{s}$$

General strategy: Solving ODEs using Laplace transforms

We will use what we have learned about using Laplace transforms to analyse electronic circuits. By moving our problem to the complex frequency domain they become easier to solve.



General strategy: Solving ODEs using Laplace transforms

Given a problem in the time-domain consisting of a set of equations with an input, output, and other time-dependent variables:

1. Take Laplace transforms of each equation to obtain the problem in the s-domain.
2. Rearrange and substitute the equations to obtain a formula for the transform of the output solely in terms of the transform of the input (all other s-dependent variables eliminated).
3. Incorporate a specific input function, if one is given.
4. Use the inverse Laplace transform to obtain the output in the time-domain.

Example 1 (Question)

Solve the following first-order ordinary differential equation for $x(t)$ using Laplace transforms:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 3x(t) = U(t)$$

with the initial condition $x(0) = 0$.

You have already encountered examples such as this in the Level 4 Engineering Mathematics module.

Example 1 (Solution)

Step 1 - Take Laplace transforms of both sides:

$$\mathcal{L}\left\{\frac{\mathrm{d}x}{\mathrm{d}t} + 3x\right\} = \mathcal{L}\{1U(t)\}$$

Remember: we can basically ignore $U(t)$ if no time delay.

$$\mathcal{L}\left\{\frac{\mathrm{d}x}{\mathrm{d}t}\right\} + 3\mathcal{L}\{x\} = \mathcal{L}\{1\}$$

$$s\bar{x}(s) - x(0) + 3\bar{x}(s) = \frac{1}{s}$$

$$s\bar{x}(s) - 0 + 3\bar{x}(s) = \frac{1}{s} \qquad \text{using the initial condition}$$

$$\bar{x}(s)\big(s+3\big) = \frac{1}{s}$$

Example 1 (Solution)

Step 2 - Rearrange to solve for $\bar{x}(s)$:

$$\bar{x}(s) = \frac{1}{s(s+3)}$$

Step 3 - Using partial fractions, invert to obtain $x(t)$:

$$\bar{x}(s) = \frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+3}\right)$$

$$\begin{aligned}
\therefore x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+3}\right)\right\} \\
&= \frac{1}{3}\left(\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}\right) \\
&= \frac{1}{3}\left(1 - \mathrm{e}^{-3t}\right)U(t)
\end{aligned}$$

Important practicalities

When taking Laplace transforms of equations, it is important to distinguish between two types of **unknowns**:

Constants and time-dependent variables.

For example, if the resistance R is a constant it can be factored out due to linearity:

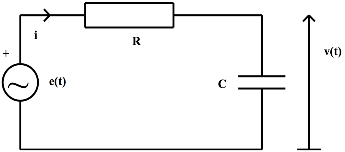
$$\mathcal{L}\{R\} = R\mathcal{L}\{1\}$$

But the current $i(t)$ depends on time. Assuming we don't already know exactly *how* it depends on time (i.e. we don't know if it is $t^2 + 1$, $5\sin(t)$, etc.), simply name its transform $\bar{i}(s)$:

$$\mathcal{L}\{i(t)\} = \bar{i}(s)$$

Example 2 (Question (a))

A series circuit consisting of an AC voltage source with voltage $e(t)$, a resistor of resistance R, and a capacitor with capacitance C, has current $i(t)$.



Consider the potential differences:

$$e(t) = Ri(t) + \frac{1}{C}\int_0^t i(t)\mathrm{d}t,$$

where $v(t) = \dfrac{1}{C}\displaystyle\int_0^t i(t)\mathrm{d}t$ is the p.d. across the capacitor.

We wish to find the output $v(t)$, given an input voltage $e(t)$.

Example 2 (Solution (a))

First take Laplace transforms of both equations:

$$\begin{aligned} e(t) &= Ri + \frac{1}{C}\int_0^t i dt \\ \therefore \bar{e}(s) &= R\bar{i}(s) + \frac{1}{Cs}\bar{i}(s) \\ &= \bar{i}(s)\left(R + \frac{1}{Cs}\right) \end{aligned}$$

and

$$\begin{aligned} v(t) &= \frac{1}{C}\int_0^t i dt \\ \therefore \bar{v}(s) &= \frac{1}{Cs}\bar{i}(s) \\ \therefore \bar{i}(s) &= sC\bar{v}(s) \end{aligned}$$

Example 2 (Solution (a))

Then we can make a substitution to eliminate $\bar{i}(s)$, and so obtain the output $\bar{v}(s)$ solely in terms of the input $\bar{e}(s)$:

$$\begin{aligned}
\bar{e}(s) &= \bar{i}(s)\left(R + \frac{1}{Cs}\right) \\
&= sC\,\bar{v}(s)\left(R + \frac{1}{Cs}\right) \\
&= \bar{v}(s)(sCR + 1)
\end{aligned}$$

$$\therefore \bar{v}(s) = \frac{1}{sCR + 1}\bar{e}(s)$$

Example 2 (Question (b))

Continuing with this circuit, determine the output voltage $v(t)$ for each of the following inputs:

$$\begin{array}{ll} i) & e(t) = EU(t) \\ ii) & e(t) = EU(t-T) \end{array}$$

where E and T are positive constants.

i.e. What is the p.d. across the capacitor when the input voltage is a constant starting at time zero, or a constant starting after a delay of time T?

Example 2 (Solution (b))

i) $e(t) = EU(t)$

No time-delay, so: $\bar{e}(s) = E\mathcal{L}\{1\} = \frac{E}{s}$

Hence the transform of the output is:

$$\bar{v}(s) = \frac{E}{s(1 + CRs)} = E\left(\frac{1/CR}{s(s + 1/CR)}\right)$$

Invert using $\mathcal{L}^{-1}\left\{\frac{\alpha}{s(s+\alpha)}\right\} = 1 - \mathrm{e}^{-\alpha t}$ from the transform tables:

$$v(t) = E\left(1 - \mathrm{e}^{-t/CR}\right)U(t)$$

This is the output p.d. in the time-domain.

Example 2 (Solution (b))

ii) $e(t) = EU(t-T)$

This is in delay form with delay T, so using the delay theorem:

$$\bar{e}(s) = \frac{E}{s}e^{-sT}$$

Hence,

$$\bar{v}(s) = e^{-sT}\frac{E}{s(1+CRs)}$$

Inverting using the result of (i) and the inverse delay theorem:

$$v(t) = E\left(1 - e^{-(t-T)/CR}\right)U(t-T),$$

so a delayed input $e(t)$ has resulted in a delayed output $v(t)$.

Example 3 (Question)

Consider a system with input $f(t)$ and output $x(t)$, that is described by the following second-order nonhomogeneous ODE:

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 2x(t) = f(t)$$

with initial conditions

$$x(0) = 0 \quad \text{and} \quad \dot{x}(0) = 1$$

Using Laplace transforms, determine $x(t)$ when the input is:

$$f(t) = U(t-3)$$

Example 3 (Solution)

Taking Laplace transforms:

$$\bar{f}(s) = \mathcal{L}\left\{\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x\right\}$$

$$= \mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} + 3\mathcal{L}\left\{\frac{dx}{dt}\right\} + 2\mathcal{L}\{x\}$$

$$= \left(s^2\bar{x}(s) - sx(0) - \dot{x}(0)\right) + 3\left(s\bar{x}(s) - x(0)\right) + 2\bar{x}(s)$$

Using the initial conditions:

$$= s^2\bar{x}(s) - 0 - 1 + 3\big(s\bar{x}(s) - 0\big) + 2\bar{x}(s)$$

$$= s^2\bar{x}(s) - 1 + 3s\bar{x}(s) + 2\bar{x}(s)$$

$$= \bar{x}(s)\big(s^2 + 3s + 2\big) - 1$$

Example 3 (Solution)

Rearrange to obtain the transform of the output $\bar{x}(s)$ in terms of the transform of the general input $\bar{f}(s)$:

$$\bar{x}(s) = \frac{\bar{f}(s) + 1}{s^2 + 3s + 2}$$

Then given the specific input $f(t)$:

$$f(t) = U(t-3) \qquad \Longrightarrow \qquad \bar{f}(s) = \frac{e^{-3s}}{s}$$

and we substitute this into our equation for $\bar{x}(s)$:

$$\bar{x}(s) = \frac{1 + e^{-3s}/s}{s^2 + 3s + 2} = \frac{e^{-3s}}{s(s+1)(s+2)} + \frac{1}{(s+1)(s+2)}$$

Example 3 (Solution)

Finally, invert this to obtain the output $x(t)$. The terms are treated separately, as one has a time delay and the other does not.

$$x(t) = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s+1)(s+2)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$

Using partial fractions expansion,

$$\begin{aligned} \frac{1}{s(s+1)(s+2)} &= \frac{1}{2s} + \frac{-1}{s+1} + \frac{1}{2(s+2)} \\ \frac{1}{s(s+1)(s+2)} &= \frac{1}{s+1} + \frac{-1}{s+2} \end{aligned}$$

Hence,

$$\bar{x}(s) = e^{-3s}\left\{\frac{1}{2s} + \frac{-1}{s+1} + \frac{1}{2(s+2)}\right\} + \left\{\frac{1}{s+1} + \frac{-1}{s+2}\right\}$$

Example 3 (Solution)

Inverting the non-time-delayed part of the first term:

$$\mathcal{L}^{-1}\left\{\frac{1}{2s}+\frac{-1}{s+1}+\frac{1}{2(s+2)}\right\}=\frac{1}{2}-e^{-t}+\frac{e^{-2t}}{2}$$

Then the inverse delay theorem with a delay of 3 says swap t for $t-3$, then multiply by the delayed step function $U(t-3)$:

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s+1)(s+2)}\right\}=U(t-3)\left\{\frac{1}{2}-e^{-(t-3)}+\frac{e^{-2(t-3)}}{2}\right\}$$

The second term does not feature time-delay and is simpler:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}-\frac{1}{s+2}\right\}=U(t)\left\{e^{-t}-e^{-2t}\right\}$$

Example 3 (Solution)

Combining these gives our final answer:

$$x(t) = U(t-3)\left\{\frac{1}{2} - e^{-(t-3)} + \frac{e^{-2(t-3)}}{2}\right\} + U(t)\left\{e^{-t} - e^{-2t}\right\}$$

This example demonstrates some advantages of the Laplace transforms method of solving nonhomogeneous ODEs:

- It is not necessary to first solve the homogeneous ODE.
- Initial values are automatically accounted for in the solution rather than requiring a set of extra steps.

Summary

After today, you should be able to . . .

- Take the Laplace transforms of integrals and derivatives.
- Explain what it means to "solve an ODE".
- Follow the general procedure of solving an ODE using the method of Laplace transforms.

This Week

This week's lecture corresponds to Section 2.7 of the Course Notes.

Before this week's tutorial:

- Complete Tutorial Sheets 2 and 3.
- Attempt Question 1 of Tutorial Sheet 4.

In next week's lecture we will develop these ideas into a more systematic procedure for analysing the outputs of electronic circuits (Section 2.8).

Extra Question

Attempt Question 1 of Tutorial Sheet 4 in groups:

Use Laplace transforms to solve the following differential equation for $x(t)$:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 2x(t) = \sin(t)U(t)$$

with the initial condition $x(0) = 0$