Lecture 4: Laplace Transforms (4)

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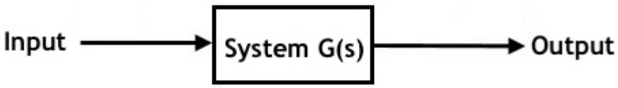
Further Mathematics, Signals and Systems

Today we shall cover:

- Determining the transfer function of a linear electronic system from the set of ODEs.
- Interpreting the transfer function to determine:
  1. the characteristic equation
  2. the order
  3. the stability of the system
  4. distortion

Motivation - Circuit analysis

- Electronic engineers can study a circuit diagram and derive a set of equations using Kirchoff's Laws.
- In simple cases, we could use the techniques from last week to determine the output, given a particular input.
- Often, we desire more general information about how the system will respond to different input signals.
- This can be achieved using **transfer functions**.



Definition: Transfer Function

Given a linear time-invariant system with:

- **input** $v_i(t)$
- **output** $v_o(t)$

The **Transfer Function** $G(s)$ of the system is the ratio of the transform of the output to the transform of the input.

Transfer Function $G(s)$

$$\bar{v}_o(s) = G(s) \times \bar{v}_i(s), \quad \text{or} \quad G(s) = \frac{\bar{v}_o(s)}{\bar{v}_i(s)}$$

The transfer function is independent of the input to the system.

Theory: Determining the Transfer Function

Given a set of multiple ODEs corresponding to the system,

To find the transfer function:

1. Take Laplace transforms of all equations.
2. Plan a sequence of steps (substituting and rearranging the equations) to eliminate all other s-dependent variables.
3. Enact the plan!
4. Re-arrange to the form $\bar{v}_o(s) = G(s) \times \bar{v}_i(s)$.
5. Read off $G(s)$.

Example 1

An electronic circuit has the following set of associated equations derived from Kirchoff's laws. The resistance R and capacitance C are positive constants, while i_1 and i_2 are time-dependent currents

$$(a) \qquad v_i(t) = 3Ri_1(t) + 2Ri_2(t)$$

$$(b) \qquad Ri_1(t) = Ri_2 + \frac{1}{C}\int_0^t i_2(t)\mathrm{d}t$$

$$(c) \qquad v_o(t) = \frac{1}{C}\int_0^t i_2(t)\mathrm{d}t + R\big(i_1(t) + i_2(t)\big)$$

Example 1 - determining $G(s)$

Taking Laplace transforms of all equations,

$$
\begin{aligned}
&(A) \qquad \bar{v}_i = 3R\bar{i}_1 + 2R\bar{i}_2 \\
&(B) \qquad R\bar{i}_1 = R\bar{i}_2 + \frac{1}{sC}\bar{i}_2 \\
&(C) \qquad \bar{v}_o = \frac{1}{sC}\bar{i}_2 + R(\bar{i}_1 + \bar{i}_2)
\end{aligned}
$$

Plan for obtaining $\bar{v}_o$ in terms of $\bar{v}_i$ only:

1. Use (B) to eliminate $\bar{i}_1$ from (A).
Call this equation (D), it has variables $\bar{v}_i$ and $\bar{i}_2$.
2. Use (B) to eliminate $\bar{i}_1$ from (C).
Call this equation (E), it has variables $\bar{v}_o$ and $\bar{i}_2$.
3. Then use (D) to substitute $\bar{i}_2$ for a function of $\bar{v}_i$ in (E).

Example 1 - determining $G(s)$

First simplify (B) and make $\bar{i}_1$ the subject:

$$\bar{i}_1 = \frac{1}{R}\left(R + \frac{1}{sC}\right)\bar{i}_2 = \frac{1}{sCR}(1 + sCR)\bar{i}_2$$

1. Substituting equation (B) into (A) yields equation (D):

$$\begin{aligned}\bar{v}_i &= 3R\frac{1}{sCR}(+sCR)\bar{i}_2 + R\bar{i}_2 \\ &= \frac{1}{sC}(3 + 4sCR)\bar{i}_2\end{aligned}$$

2. Substituting equation (B) into (C) yields equation (E):

$$\begin{aligned}\bar{v}_o &= \left(\frac{1}{sC} + R\right)\bar{i}_2 + \frac{R}{sCR}(sCR + 1)\bar{i}_2 \\ &= \frac{2}{sC}(1 + sCR)\bar{i}_2\end{aligned}$$

Example 1 - determining $G(s)$

Rearrange (D) to make $\bar{i}_2$ the subject:

$$\bar{i}_2 = \frac{sC}{3 + 4sCR}\bar{v}_i$$

3. Substitute (D) into (E) to eliminate $\bar{i}_2$:

$$\begin{aligned}\bar{v}_o &= \frac{2}{sC}(1 + sCR)\bar{i}_2 \\ &= \frac{2}{sC}(1 + sCR)\frac{sC}{3 + 4sCR}\bar{v}_i \\ &= \frac{2(1 + sCR)}{3 + 4sCR}\bar{v}_i\end{aligned}$$

Example 1 - determining $G(s)$

Thus we have eliminated both $\bar{i}_1$ and $\bar{i}_2$ and obtained a relationship between $\bar{v}_o$ and $\bar{v}_i$ only in terms of s.

Rearrange to obtain the transfer function:

$$G(s) = \frac{\bar{v}_o(s)}{\bar{v}_i(s)}$$

$$= \frac{2(1 + sCR)}{3 + 4sCR}$$

Now, what does the transfer function actually tell us about this system?

Theory: Interpreting the Transfer Function

When the transfer function is in the form:

$$G(s) = \frac{P(s)}{Q(s)}$$

(i.e. there should be no subfractions!)

Characteristic Equation of the system

$$Q(s) = 0$$

The largest power of s (equivalently, the number of solutions) in the characteristic equation is the **order** of the filter.

It limits the possible complexity of the system's behaviour.

Theory: Interpreting the Transfer Function

- A **stable** linear system will remain at rest unless it is excited by an external source, and will return to rest if such external influences are removed.
- The solutions to the characteristic equation of a system's transfer function determine the stability.

A system is stable if and only if:

All solutions of the characteristic equation have **negative** real part.

That is, $Re(s) < 0$ for all solutions s of $Q(s) = 0$.

These values of s determine the exponential functions in the time-domain solution for the output.

Example 1 - interpreting $G(s)$

As we found the transfer function to be:

$$G(s) = \frac{2(1 + sCR)}{3 + 4sCR}$$

From the denominator, the characteristic equation is:

$$3 + 4sCR = 0$$

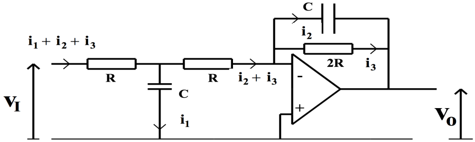
This has only one solution for s:

$$s_1 = \frac{-3}{4CR} < 0$$

For any positive values of C and R, this solution is real and negative, so $Re(s) < 0$ for all solutions.
Hence this **first-order** filter is **stable**.

Example 2: Determine the transfer function of this system

C, R are positive constants, while i_1, i_2, i_3 are time-dependent currents.



$$(a) \qquad v_i = R(i_1 + i_2 + i_3) + R(i_2 + i_3)$$

$$(b) \qquad \frac{1}{C}\int_0^t i_1 \mathrm{d}t = R(i_2 + i_3)$$

$$(c) \qquad \frac{1}{C}\int_0^t i_2 \mathrm{d}t = 2Ri_3$$

$$(d) \qquad v_o = -2Ri_3$$

Example 2 - determining $G(s)$

Taking Laplace transforms of each:

$$(A) \qquad \bar{v}_i = R(\bar{i}_1 + 2\bar{i}_2 + 2\bar{i}_3)$$

$$(B) \qquad \frac{1}{sC}\bar{i}_1 = R(\bar{i}_2 + \bar{i}_3)$$

$$(C) \qquad \frac{1}{sC}\bar{i}_2 = 2R\bar{i}_3$$

$$(D) \qquad \bar{v}_o = -2R\bar{i}_3$$

Goal: find a relationship between $\bar{v}_o$ and $\bar{v}_i$, and eliminate the other s-dependent variables $\bar{i}_1, \bar{i}_2, \bar{i}_3$:

1. Use (A) and (B) to obtain an equation with $\bar{v}_i$, $\bar{i}_2$ and $\bar{i}_3$.
2. Use (C) to eliminate $\bar{i}_2$, $\implies$ equation in terms of $\bar{v}_i$ and $\bar{i}_3$.
3. Use (D) to substitute $\bar{i}_3$ for a function of $\bar{v}_o$.

Example 2 - determining $G(s)$

1. Substituting (B) into (A) to remove $\bar{i}_1$:

$$\bar{v}_i = (\bar{i}_2 + \bar{i}_3)\left(2R + sCR^2\right)$$

2. Eliminating $\bar{i}_2$ using (C):

$$\bar{v}_i = \bar{i}_3 R(2 + sCR)(1 + 2sCR)$$

3. Substitute in (D) and rearrange to remove $\bar{i}_3$ and get a relationship between $\bar{v}_i$ and $\bar{v}_o$:

$$\bar{v}_o = \frac{-2}{(2 + sCR)(1 + 2sCR)}\bar{v}_i$$

Example 2 - determining $G(s)$

Therefore the transfer function for this system is:

$$G(s) = \frac{-2}{(2 + sCR)(1 + 2sCR)}$$

Set the denominator equal to zero for the characteristic equation:

$$Q(s) = 0 \quad \Longrightarrow \quad (2 + sCR)(1 + 2sCR) = 0$$

This is a second-order equation in s. The two solutions:

$$s_1 = \frac{-2}{CR} \quad \text{and} \quad s_2 = \frac{-1}{2CR}$$

$\therefore$ Both $s_1, s_2 < 0$ so the system is **stable**.

Theory - Convolution and Outputs

- If given a specific input function $v_i(t)$, determining the output in t-domain is not so simple as inverting the transfer function and multiplying by the input.

- Instead we first obtain the **Response Function** $g(t)$.
  This is the inverse Laplace transform of transfer function $G(s)$

- Then use the **Convolution Integral** to determine the output:

$$v_o(t) = \int_0^t g(t-z)v_i(z)\mathrm{d}z = \int_0^t g(z)v_i(t-z)\mathrm{d}z$$

Note: $0 < z < t$.

Example 2 - Convolution

Invert the transfer function to obtain the response function:

$$g(t) = \mathcal{L}^{-1}\left\{\frac{-2}{(2+sCR)(1+2sCR)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{3}\left(\frac{1}{2+sCR}\right) - \frac{4}{3}\left(\frac{1}{1+2sCR}\right)\right\}$$

using partial fractions

$$= \frac{2}{3CR}U(t)\left(e^{-2t/CR} - e^{-t/2CR}\right)$$

Example 2 - Convolution

Then given an input

$$v_i(t) = EU(t)$$

The convolution integral is:

$$\begin{aligned}
v_o(t) &= \int_0^t g(t-z)v_i(z)\mathrm{d}z \\
&= \int_0^t \frac{2}{3CR}U(t-z)(e^{-2(t-z)/CR} - e^{-(t-z)/2CR})EU(z)\mathrm{d}z \\
&= \frac{E}{3}\left\{4e^{-t/2CR} - e^{-2t/CR} - 3\right\}U(t)
\end{aligned}$$

Summary

After today, you should be able to . . .

- Write down the definition of a transfer function.
- Describe and carry out the procedure to determine the transfer function of a circuit, given a set (of 3,4 or 5) equations.
- Deduce the characteristic equation from the transfer function.
- Determine the order and stability of a system.
- Describe how to obtain the output of the system using a convolution integral.

This Week

This week's lecture corresponds to Section 2.8 of the Course Notes.

Before this week's tutorial:

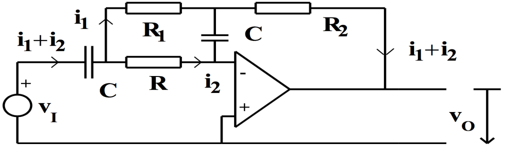
- Work through **Example 3** on these lecture slides!
- Attempt remaining questions on Tutorial Sheet 4.

In next week's lecture we will complete this topic (Section 2.9), and begin the next chapter on Fourier Series.

Next week's tutorial will also have questions like those we have seen today. You definitely want to be able to do these!

Bonus: Example 3

Determine the stability of this filter, based on an ideal op-amp.



It has the corresponding set of equations, with R, R_1, R_2, C positive constants and i_1, i_2 time-dependent currents:

$$(a) \qquad v_i(t) = \frac{1}{C}\int_0^t (i_1 + i_2)\mathrm{d}t + Ri_2$$

$$(b) \qquad R_1 i_1 = Ri_2 + \frac{1}{C}\int_0^t i_2 \mathrm{d}t$$

$$(c) \qquad v_o(t) = \frac{1}{C}\int_0^t i_2 \mathrm{d}t + R_2(i_1 + i_2)$$

Example 3 - determining $G(s)$

Taking Laplace transforms,

$$
\begin{aligned}
(A) \qquad & \bar{v}_i = \frac{1}{sC}(\bar{i}_1 + \bar{i}_2) + R\bar{i}_2 \\
(B) \qquad & R_1\bar{i}_1 = R\bar{i}_2 + \frac{1}{sC}\bar{i}_2 \\
(C) \qquad & \bar{v}_o = \frac{1}{sC}\bar{i}_2 + R_2(\bar{i}_1 + \bar{i}_2)
\end{aligned}
$$

Plan for obtaining $\bar{v}_o$ in terms of $\bar{v}_i$ only:

1. Use (B) to eliminate $\bar{i}_1$ from (A).
Call this equation (D), it has variables $\bar{v}_i$ and $\bar{i}_2$.
2. Use (B) to eliminate $\bar{i}_1$ from (C).
Call this equation (E), it has variables $\bar{v}_o$ and $\bar{i}_2$.
3. Then use (E) to substitute $\bar{i}_2$ for a function of $\bar{v}_o$ in (D).

Example 3 - determining $G(s)$

1. Substituting equation (B) into (A) yields equation (D):

$$\bar{v}_i = \bar{i}_2\left(\frac{1}{sC}+R\right)+\frac{1}{sC}\left(\frac{R}{R_2}\bar{i}_2+\frac{1}{sCR_1}\bar{i}_2\right) = \bar{i}_2\left(\frac{1}{Cs}+R+\frac{R}{sCR_1}+\frac{1}{C^2s^2R_1}\right)$$

2. Substituting equation (B) into (C) yields equation (E):

$$\bar{v}_o = \bar{i}_2\left(\frac{1}{sC}+R_2\right)+R_2\left(\frac{R}{R_1}\bar{i}_2+\frac{1}{sCR_1}\bar{i}_2\right) = \bar{i}_2\left(\frac{1}{sC}+R_2+\frac{RR_2}{R_1}+\frac{R_2}{sCR_1}\right)$$

3. Combining (D) and (E) to eliminate $\bar{i}_2$:

$$\bar{v}_o = \bar{v}_i\left\{\frac{(1/sC)+R_2+(RR_2/R_1)+(R_2/sCR_1)}{(1/sC)+R+(R/sCR_1)+(1/s^2C^2R_1)}\right\}$$

Example 3 - determining $G(s)$

Remove the subfractions by multiplying both the numerator and denominator by $s^2C^2R_1$:

$$\begin{aligned}\bar{v}_o &= \bar{v}_i\left\{\frac{sCR_1 + s^2C^2R_1R_2 + s^2C^2RR_2 + sCR_2}{sCR_1 + s^2C^2RR_1 + sCR + 1}\right\} \\ &= \bar{v}_i\left\{\frac{sC\big(s(CR_1R_2 + CRR_2) + (R_1 + R_2)\big)}{(sCR+1)(sCR_1+1)}\right\}\end{aligned}$$

Therefore the transfer function for this system is:

$$G(s) = \frac{sC\big(s(CR_1R_2 + CRR_2) + (R_1 + R_2)\big)}{(sCR+1)(sCR_1+1)}$$

Example 3 - interpreting $G(s)$

As we found the transfer function to be:

$$G(s) = \frac{sC\left(s(CR_1R_2 + CRR_2) + (R_1 + R_2)\right)}{(sCR + 1)(sCR_1 + 1)}$$

From the denominator, the characteristic equation is:

$$(sCR + 1)(sCR_1 + 1) = 0$$

This has solutions:

$$s_1 = \frac{-1}{CR} < 0 \quad \text{and} \quad s_2 = \frac{-1}{CR_1} < 0$$

Both are real and negative, so $Re(s) < 0$ for all solutions.
Hence this second-order filter is **stable**.

Extra Question

An electronic system is described by the following equations.

i_1, i_2, i_3 are the time-dependent currents.

R, C are positive constants.

$$v_i(t) = 3R(i_1 + i_2 + i_3) + 2Ri_2$$

$$\frac{1}{2C}\int_0^t i_1 \mathrm{d}t = 2Ri_2$$

$$3Ri_3 = 2Ri_2 + \frac{1}{C}\int_0^t i_2 \mathrm{d}t$$

$$v_o(t) = \frac{1}{C}\int_0^t i_2 \mathrm{d}t$$

Show that the transfer function is:

$$G(s) = \frac{1}{12s^2C^2R^2 + 7sCR + 1}$$