Lecture 5: Laplace Transforms and introduction to Fourier Series (1/3)

Dr Gavin M Abernethy

Further Mathematics, Signals and Systems

Lecture 5

Today we shall cover:

- Analysing the transfer function of a circuit to determine the Bode plot and estimate the type of filter.
- Preliminary material for the next topic (Fourier Series):
  - Periodicity
  - Odd and Even functions
- The definition and core concept of Fourier Series.

Revision: Transfer Function

Last week we learned how to calculate the transfer function of a linear system:

$$G(s) = \frac{\bar{v}_o(s)}{\bar{v}_i(s)}$$

In addition to the order and stability, $G(s)$ can tell us how the system affects the input to determine an output.

So what does it say?

To determine this, we will use **Amplitude Bode Plots**.

Drawing Bode Plots

To sketch the amplitude bode plot from the transfer function $G(s)$:

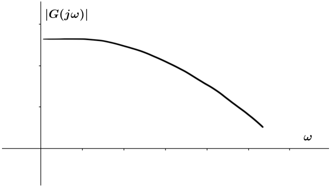
1. Replace s with $j\omega$.
2. Consider the **absolute value** of this function $|G(j\omega)|$
3. How does it behave when ω is extremely **small** $(\omega << 1)$?
4. How does it behave when ω is extremely **large** $(\omega >> 1)$?
5. Draw a rough sketch of $|G(j\omega)|$ against frequency ω.

Types of Filter

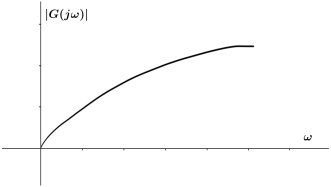
- High-pass (allows only high frequencies through)
- Low-pass (allows only frequencies below a certain value through)
- All-pass (allows all frequencies through)
- Band pass (allows only frequencies within a particular interval through)
- Band-eliminate/band-stop (allows all frequencies through except for a given interval).

We can determine the type of filter after the bode plot is drawn.

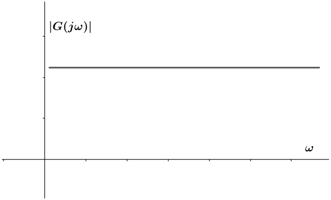
Amplitude Bode Plots: First order



(a) Low-pass

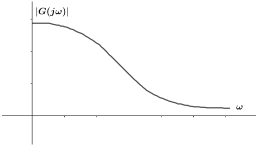


(b) High-pass

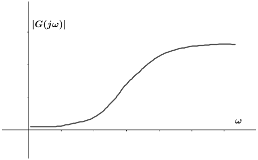


(c) All-pass

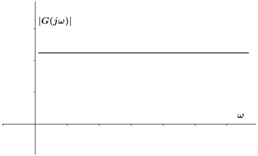
Amplitude Bode Plots: Second order



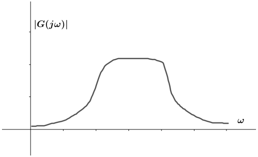
(d) Low-pass



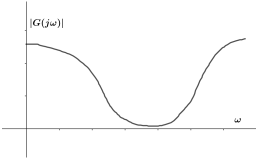
(e) High-pass



(f) All-pass



(g) Band-pass



(h) Band-eliminate

Example: Determining the type of Filter

A system has the transfer function:

$$G(s) = \frac{-4}{1 + sRC}$$

First let $s = j\omega$, then:

$$\begin{aligned} |G(j\omega)| &= \left|\frac{-4}{1 + j\omega RC}\right| = \frac{|-4|}{|1 + j\omega RC|} \\ &= \frac{4}{|1 + j\omega RC|} \end{aligned}$$

Example: Determining the type of Filter

Analysing this function at very low and very high frequencies (ω):

$$\text{When } \omega << 1: \quad |G(j\omega)| = \frac{4}{|1 + (\text{smaller})|} \approx \frac{4}{1} = 4$$

$$\begin{aligned}\text{When } \omega >> 1: \quad |G(j\omega)| &= \frac{4}{|(\text{smaller}) + j\omega RC|} \approx \frac{4}{|j\omega RC|} \\ &= \frac{4}{\omega RC} \to 0 \text{ as } \omega \to \infty\end{aligned}$$

$G(s)$ is positive for low frequencies and approaches zero for high frequencies:

Conclusion: **Low-pass filter**.

Exercise: Determining the type of Filter

A system has the transfer function:

$$G(s) = \frac{sCR}{1 + sCR}$$

By approximating the amplitude bode plot, determine the nature of this filter.

Part 2

Fourier Series

- Periodicity
- Odd and Even functions
- The core concept of Fourier Series
- The definition of Fourier Series

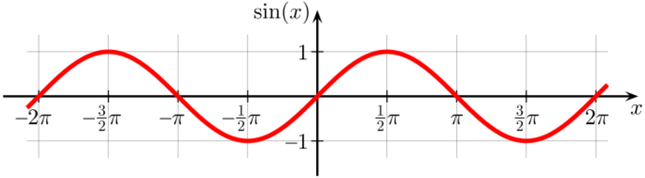
Jean-Baptiste Fourier (1768-1830)

"[Mathematics] brings together phenomena the most diverse, and discovers the hidden analogies which unite them"

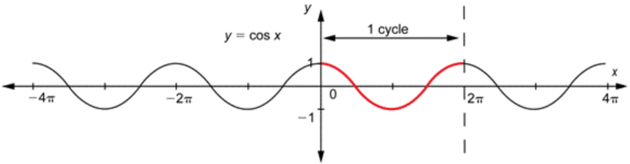
The Analytical Theory of Heat

Periodic functions

Sine, cosine and tangent are examples of **periodic** functions, meaning that they repeat a pattern forever.



The **period** of both sine and cosine is 2π as this is the minimum time required before the pattern begins to repeat.



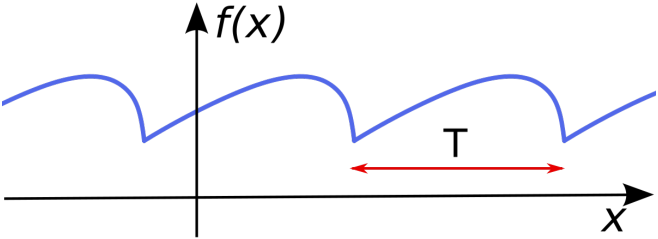
Periodic functions

Periodic function

A function $f(t)$ is periodic with period T if for all values of t, and for any integer m:

$$f(t + mT) = f(t)$$

The minimum time required for one full cycle is the **period** T.



Periodic functions

The number of full cycles per unit of time (usually seconds) is called the **frequency** and given by $f = T^{-1}$.

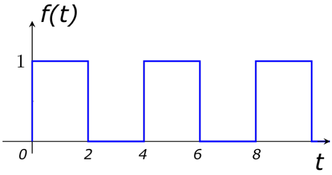
However it is often useful to use the **angular frequency** ω, measured in radians per second.

Angular Frequency

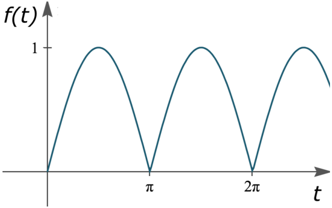
$$\omega = \frac{2\pi}{T} \qquad \text{or} \qquad T = \frac{2\pi}{\omega}$$

As the period of oscillation T increases, the frequency (both angular and regular) decreases and vice versa.

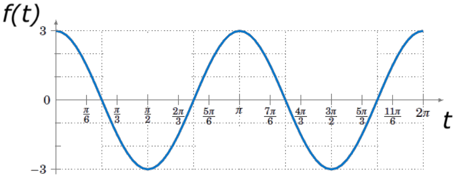
Determine the period and angular frequency of the following peridic functions:



(i) Pulse wave



(j) $|\sin(t)|$



(k) $3\cos(2t)$

Periodic functions

Note that if we have a sine or cosine wave of the form:

$$f(t) = a\sin(mt) \quad \text{or} \quad f(t) = a\cos(mt)$$

where a and $m > 0$ are constants,

Then the angular frequency is just:

$$\omega = m$$

Odd and Even functions

Functions can be classified as:

- **odd**
- **even**
- both (in a few trivial cases)
- neither

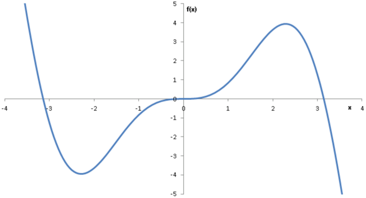
Being able to recognise such functions will enable us to take shortcuts when calculating Fourier Series.

Odd functions

Odd Functions

An **odd** function is one where $f(-x) = -f(x)$.

The graph has rotational symmetry of 180° about the origin.



Example: $f(x) = x^2 \sin(x)$ is odd. To demonstrate this:

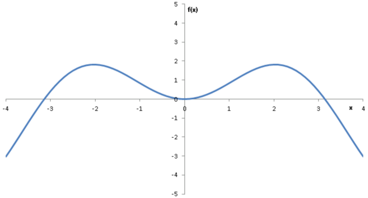
$$\begin{aligned} f(2) &= 2^2 \sin(2) = 3.637 \\ f(-2) &= (-2)^2 \sin(-2) = -3.637 = -f(2) \end{aligned}$$

Even functions

Even Functions

An **even** function is one where $f(-x) = f(x)$.

The graph has reflective symmetry about the vertical axis.



Example: $f(x) = x\sin(x)$ is even. To demonstrate this:

$$\begin{aligned} f(2) &= 2\sin(2) = 1.819 \\ f(-2) &= (-2)\sin(-2) = 1.819 = f(2) \end{aligned}$$

Odd and Even functions

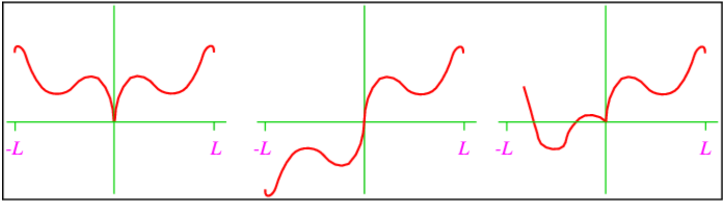
- Examples of odd functions include:

$$x, \quad x^3, \quad x^5, \quad \text{and} \quad \sin(mx)$$

- Examples of even functions include:

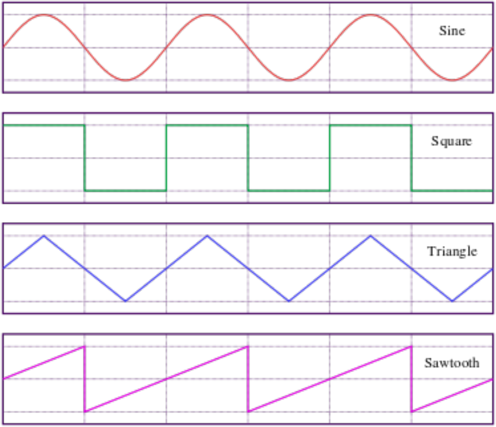
$$17, \quad x^2, \quad x^4, \quad \text{and} \quad \cos(mx)$$

- Which of the functions below are odd, even, or neither?



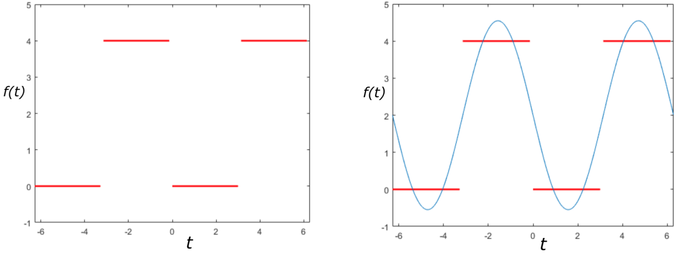
Motivation for Fourier Series

Often when analysing electrical or acoustic signals, we encounter discountinuous periodic signals:



Motivation for Fourier Series

It can be useful to approximate these signals by fitting a continuous sine or cosine curve:



This is the core idea of Fourier Series:
We can represent (almost) any periodic function by some **combination of sine and cosine waves of different frequencies**.

Fourier Series

Given a periodic function $f(t)$ with period T and angular frequency ω, it can be represented by a *unique* series of sines and cosines:

Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

This is the **Fourier Series** representation of $f(t)$.

a_0, a_n and b_n are constants, called the Fourier coefficients. When we consider a new function $f(t)$, the only things that change are the period T and the values of these Fourier coefficients a_0, a_n, b_n

$$f(t) = \text{constant term} + \text{cosine terms} + \text{sine terms}$$

Our goal is therefore to calculate the values of the coefficients a_0, a_n and b_n for whatever function we are interested in.

We will learn a method for doing this next week.

The terms in the series have certain names:

- $\frac{1}{2}a_0$ is the **DC level** of $f(t)$.
- $a_1 \cos(\omega t) + b_1 \sin(\omega t)$ is the first harmonic. Also called the **Fundamental mode**.
- $a_n \cos(n\omega t) + b_n \sin(n\omega t)$ is the n^{th} **Harmonic**. It has angular frequency $n\omega$.

Fourier Series

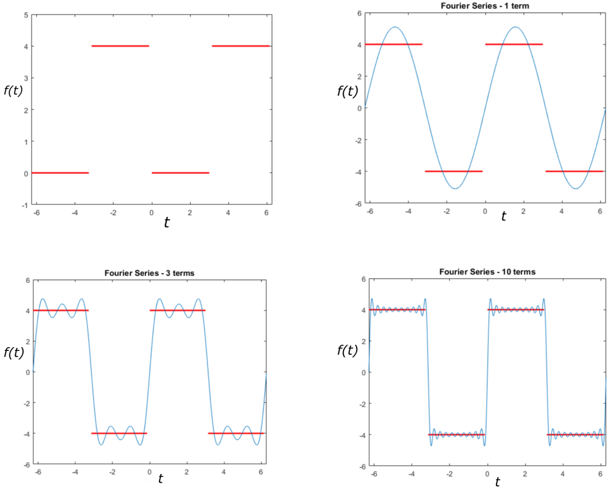
- As the Fourier Series is infinite, we will only calculate the first few terms in the series, called the partial sum.
- The N^{th} **Partial Sum** for a function $f(t)$ consists of the first N terms of the Fourier Series (count each harmonic as a single term):

$$f_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N-1} \left(a_n \cos(n\omega t) + b_n \sin(n\omega t) \right)$$

- As we increase the number of terms, the partial sum converges towards the full (infinite) Fourier Series and thus the true function.

Fourier Series

So as more terms in the Fourier Series are calculated, a more accurate approximation to the true signal is found:



Summary

After today, you should be able to . . .

- Estimate the **Bode plot** and **filter type** of a system from its transfer function.
- Identify the **period**, **angular frequency**, and amplitude of a periodic function.
- Recognise an **odd** or **even** function.
- Explain the core idea of Fourier Series.
- Recall the formula/definition of the Fourier Series of a periodic function.

This Week

This lecture corresponds to Section 2.9-3.3 of the Course Notes.

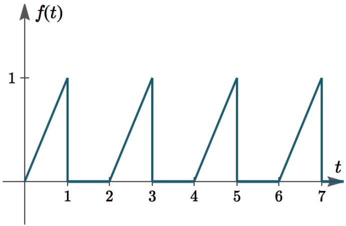
Before this week's tutorial:

- Study the additional examples (2.37 and 2.38) in the Course Notes of determining the Bode Plots and filter types.
- Attempt Tutorial sheet 5 - this concludes the section on Laplace transforms with exam-style questions.

In the following lecture we will learn a method for calculating these Fourier coefficients of a periodic function.

Extra Question

Consider the following function $f(t)$:



1. What is the period of this function?
2. What is its angular frequency?
3. Draw what this function would look like if it was odd.
4. Draw what this function would look like if it was even.
5. Draw an example of what this function could look like if it was neither odd nor even.