Lecture 6: Fourier Series (2/3)

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Further Mathematics, Signals and Systems

Lecture 6

Today we shall cover:

- Complex Fourier coefficients, known as phasors.
- The **complex form** of Fourier Series.
- A method for calculating the Fourier Series of a periodic function, using Laplace transforms and this complex form.
- We shall do **two examples** of this procedure.

Revision: Fourier Series

Fourier Series: representing a periodic function $f(t)$ by a combination of sine and cosine waves of different frequencies:

Fourier Series

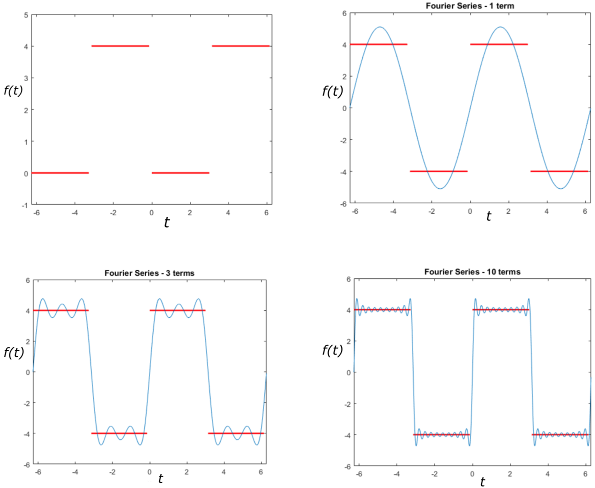
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

a_0, a_n, b_n are constants ("Fourier coefficients") that we want to calculate.

- $\frac{1}{2}a_0$ is the **DC level** of $f(t)$.
- $a_1 \cos(\omega t) + b_1 \sin(\omega t)$ is the **Fundamental mode**.
- $a_n \cos(n\omega t) + b_n \sin(n\omega t)$ is the n$^{\text{th}}$ **Harmonic**.

Fourier Series

As more terms in the Fourier Series are calculated, they add to give a better and better approximation to the true signal:



Fourier Series

So if we are given a function $f(t)$, how can we calculate the Fourier coefficients needed to write down its Fourier Series?

The standard method is to use . . .

Calculating Fourier Coefficients by Integration

Integral method for calculating Fourier Coefficients

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\mathrm{d}t$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\mathrm{d}t$$

You will **not** be expected to use this method on this module, but you should be aware of it. Instead, we will be using a method involving Laplace transforms to avoid integration and use what we have already learned.

Calculating Fourier Coefficients: The DC level

Recall that the DC level is $\frac{a_0}{2}$.

We **will** need to use the integral to calculate this:

Integral method for calculating Fourier Coefficients

$$a_0 = \frac{2}{T}\int_0^T f(t)\mathrm{d}t$$

Alternatively, the DC level $(a_0/2)$ is equivalent to the **average value of $f(t)$ over one complete cycle**. In simple cases this can be calculated from the graph instead of using the integral.

Complex Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

Using the relationship between trig. and exponential functions:

$$\cos(\phi) + j\sin(\phi) = e^{\phi j}$$

We can combine the sine and cosine terms, and obtain the **complex form** of the Fourier Series:

Complex form of the Fourier Series of $f(t)$

$$f(t) = \frac{a_0}{2} + Re\left\{ \sum_{n=1}^{\infty} A_n \, e^{jn\omega t} \right\}$$

Complex Fourier Series

Now there is only one sequence of complex coefficients, that we need to determine.

These are the **phasors**, A_n.

Relationship to the real coefficients:

$$A_n = a_n - jb_n$$

$$a_n = Re\{A_n\}, \qquad \text{and} \qquad b_n = -Im\{A_n\}$$

Phasors

The phasors can be calculated from a single integral, but they can also be obtained by **Laplace transforms**.

Use step functions to define a new function, g, that behaves like f during the first period $0 < t < T$, and is zero everywhere else:

$$g(t) = \begin{cases} f(t) & \text{for } 0 < t < T, \\ 0 & \text{otherwise.} \end{cases}$$

Then take the Laplace transform and substitute to get the phasor:

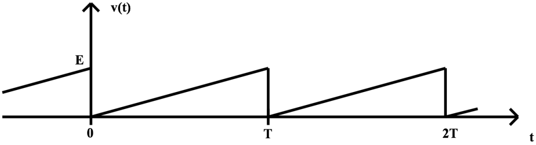
Phasor:

$$A_n = \frac{2}{T}\bar{g}(jn\omega)$$

Method: Determining complex Fourier Series of $f(t)$ using phasors and Laplace transforms

1. Define
$$g(t) = \begin{cases} f(t) & \text{for } 0 < t < T, \\ 0 & \text{otherwise.} \end{cases}$$
2. Obtain the Laplace transform $\bar{g}(s)$.
3. Change the variable to obtain $\bar{g}(jn\omega)$.
4. The phasor of the n^{th} harmonic is then $A_n = \frac{2}{T}\bar{g}(jn\omega)$.
5. Find the DC level either by integration or the average value.
6. State the complex form of the Fourier Series:
$$f(t) = \frac{a_0}{2} + Re\left\{ \sum_{n=1}^{\infty} A_n \, e^{jn\omega t} \right\}$$

Example 1 (Sawtooth Wave)



In this example, $f(t) = \frac{Et}{T}$ during the first cycle.

We want $g(t) = \frac{Et}{T}$ for time $0 < t < T$ and zero otherwise. This can be achieved using a combination of step changes that will turn "on" at time $t = 0$ and "off" at $t = T$:

$$g(t) = \frac{Et}{T}\Big(U(t) - U(t-T)\Big) = \frac{E}{T}\Big(tU(t) - tU(t-T)\Big)$$

Example 1 (Sawtooth Wave)

Writing the second term in delay form:

$$g(t) = \frac{E}{T}\Big(tU(t) - \big((t-T)+T\big)U(t-T)\Big)$$

Then taking Laplace transfoms:

$$\begin{aligned}
\bar{g}(s) &= \mathcal{L}\left\{\frac{E}{T}\Big(tU(t) - \big((t-T)+T\big)U(t-T)\Big)\right\} \\
&= \frac{E}{T}\Big(\mathcal{L}\Big\{tU(t)\Big\} - \mathcal{L}\Big\{\big((t-T)+T\big)U(t-T)\Big\}\Big) \\
&= \frac{E}{T}\Big(\mathcal{L}\{t\} - \mathcal{L}\{t+T\}\,\mathrm{e}^{-sT}\Big) \quad \text{using delay theorem} \\
&= \frac{E}{T}\left(\frac{1}{s^2} - \left[\frac{1}{s^2} + \frac{T}{s}\right]\mathrm{e}^{-sT}\right) = \frac{E}{s^2 T}\big(1 - (1+sT)\,\mathrm{e}^{-sT}\big)
\end{aligned}$$

Example 1 (Sawtooth Wave)

Changing the variable from s to $jn\omega$:

$$\begin{aligned}
\bar{g}(jn\omega) &= \frac{E}{(jn\omega)^2 T}\left(1 - (1 + jn\omega T)\,\mathrm{e}^{-jn\omega T}\right) \\
&= \frac{E}{(jn(2\pi/T))^2 T}\left(1 - \left(1 + jn\frac{2\pi}{T}T\right)\mathrm{e}^{-jn\frac{2\pi}{T}T}\right) \\
&\qquad \text{since} \quad \omega = \frac{2\pi}{T} \\
&= \frac{-ET^2}{4\pi^2 n^2 T}\left(1 - (1 + 2\pi nj)\,\mathrm{e}^{-2\pi nj}\right) \\
&\qquad \text{since} \quad j^2 = -1
\end{aligned}$$

Example 1 (Sawtooth Wave)

Simplifying further,

$$\bar{g}(jn\omega) = \frac{-ET}{4\pi^2 n^2}\left(1 - (1 + 2\pi nj)\,\mathrm{e}^{-2\pi nj}\right)$$

$$= \frac{-ET}{4\pi^2 n^2}\left(1 - (1 + 2\pi nj)\cdot 1\right)$$

$$\text{since} \quad \mathrm{e}^{-2jn\pi} = \cos(2\pi n) - j\sin(2\pi n) = 1 \quad \forall n \in \mathbb{N}$$

$$= \frac{-ET}{4\pi^2 n^2}(1 - 1 - 2\pi nj)$$

$$= \frac{2\pi njET}{4\pi^2 n^2} = j\frac{ET}{2\pi n}$$

Example 1 (Sawtooth Wave)

Hence the phasor of the n^{th} harmonic is:

$$\begin{aligned} A_n &= \frac{2}{T}\bar{g}(jn\omega) \\ &= \frac{2}{T}j\frac{ET}{2\pi n} \\ &= j\frac{E}{\pi n} \end{aligned}$$

Example 1 (Sawtooth Wave)

From looking at the graph, the average value of $f(t)$ over one cycle is clearly $\frac{E}{2}$.

Alternatively use integration:

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^T f(t)\mathrm{d}t = \frac{2}{T}\int_0^T \frac{E}{T}t\mathrm{d}t \\ &= \frac{2E}{T^2}\left[\frac{1}{2}t^2\right]_0^T = E \end{aligned}$$

So either way, the DC level is:

$$\frac{a_0}{2} = \frac{E}{2}$$

Example 1 (Sawtooth Wave)

$\therefore$ Complex form of the Fourier series:

$$f(t) = \frac{a_0}{2} + Re\left\{ \sum_{n=1}^{\infty} A_n\, e^{jn\omega t} \right\}$$

Substitute in a_0 and A_n:

$$= \frac{E}{2} + Re\left\{ \sum_{n=1}^{\infty} j\frac{E}{\pi n}\, e^{jn\omega t} \right\}$$

where $\omega = \frac{2\pi}{T}$

(you must state what ω is, since only T occured in the question)

Example 1 (Sawtooth Wave)

To obtain the regular Fourier coefficients:

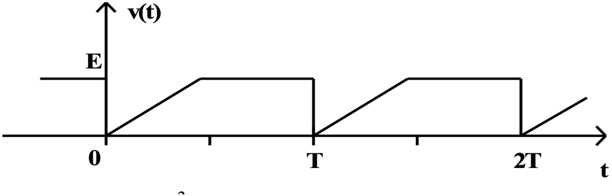
$$a_n = Re\{A_n\} = Re\left\{j\frac{E}{\pi n}\right\} = 0$$

$$b_n = -Im\{A_n\} = -Im\left\{j\frac{E}{\pi n}\right\} = \frac{-E}{\pi n}$$

So the Fourier series with real coefficients is:

$$f(t) = \frac{E}{2} - \frac{E}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2\pi nt}{T}\right)$$

Example 2 (Clipped Sawtooth Wave)



In the first cycle $0 < t < T$, this wave $f(t)$ is given by:

$$f(t) = \begin{cases} \frac{2Et}{T} & \text{for } 0 < t < \frac{T}{2}, \\ E & \text{for } \frac{T}{2} < t < T. \end{cases}$$

Determine the Fourier Series using phasors and Laplace transforms.

Example 2 (Clipped Sawtooth Wave)

First, use step functions to define g:

$$g(t) = \frac{2Et}{T}\left\{U(t) - U(t-T/2)\right\} + E\left\{U(t-T/2) - U(t-T)\right\}$$

Expand the brackets and gather terms with the same time-delay:

$$\begin{aligned} g(t) &= \frac{2Et}{T}U(t) - \frac{2Et}{T}U(t-T/2) + EU(t-T/2) - EU(t-T) \\ &= \frac{2Et}{T}U(t) + E\left(1 - \frac{2t}{T}\right)U(t-T/2) - EU(t-T) \end{aligned}$$

Example 2 (Clipped Sawtooth Wave)

The first term has no time delay, and the final term is in delay form. Therefore we just need to put the middle term in delay form:

$$\begin{aligned}
E\left(1-\frac{2t}{T}\right)U(t-T/2) &= E\left(1-\frac{2}{T}\left[\left(t-\frac{T}{2}\right)+\frac{T}{2}\right]\right)U(t-T/2) \\
&= E\left(1-\frac{2}{T}\left(t-\frac{T}{2}\right)-1\right)U(t-T/2) \\
&= -\frac{2E}{T}\left(t-\frac{T}{2}\right)U(t-T/2)
\end{aligned}$$

Hence:

$$g(t) = E\left\{\frac{2}{T}tU(t)-\frac{2}{T}\left(t-\frac{T}{2}\right)U(t-T/2)-U(t-T)\right\}$$

Example 2 (Clipped Sawtooth Wave)

Taking Laplace transforms (using the delay theorem twice):

$$
\begin{aligned}
\bar{g}(s) &= \mathcal{L}\left\{E\left(\frac{2}{T}tU(t) - \frac{2}{T}\left(t - \frac{T}{2}\right)U(t - T/2) - U(t - T)\right)\right\} \\
&= E\left(\frac{2}{T}\mathcal{L}\left\{tU(t)\right\} - \frac{2}{T}\mathcal{L}\left\{\left(t - \frac{T}{2}\right)U(t - T/2)\right\}\right. \\
&\qquad \left. -\mathcal{L}\left\{U(t - T)\right\}\right) \quad \text{by linearity} \\
&= E\left(\frac{2}{T}\mathcal{L}\{t\} - \frac{2}{T}\mathcal{L}\{t\}\,\mathrm{e}^{-sT/2} - \mathcal{L}\{1\}\,\mathrm{e}^{-sT}\right) \\
&= E\left(\frac{2}{Ts^2} - \frac{2}{Ts^2}\,\mathrm{e}^{-sT/2} - \frac{1}{s}\,\mathrm{e}^{-sT}\right)
\end{aligned}
$$

Then substituting s for $jn\omega$:

$$\begin{aligned}
\bar{g}(jn\omega) &= E\left\{ \frac{2}{T(jn\omega)^2}(1 - \mathrm{e}^{-jn\omega T/2}) - \frac{1}{jn\omega}\,\mathrm{e}^{-jn\omega T} \right\} \\
&= E\left\{ \frac{-2T^2}{4Tn^2\pi^2}(1 - \mathrm{e}^{-jn\pi}) + \frac{jT}{2\pi n}\,\mathrm{e}^{-2\pi jn} \right\} \\
&\qquad \text{since} \quad \omega = \frac{2\pi}{T}, \quad j^2 = -1, \quad \frac{1}{j} = -j \\
&= ET\left\{ \frac{1}{2\pi^2 n^2}(\mathrm{e}^{-jn\pi} - 1) + \frac{j}{2\pi n} \right\} \\
&\qquad \text{since} \quad \mathrm{e}^{-2jn\pi} = 1 \quad \forall n \in \mathbb{N}
\end{aligned}$$

Example 2 (Clipped Sawtooth Wave)

Since $e^{-jn\pi} = (-1)^n$,

$$\begin{aligned}\bar{g}(jn\omega) &= ET\left\{\frac{1}{2\pi^2 n^2}\left((-1)^n - 1\right) + \frac{j}{2\pi n}\right\} \\ &= \frac{ET}{2\pi^2 n^2}\left\{\left((-1)^n - 1\right) + j\pi n\right\}\end{aligned}$$

Therefore we have the phasor of the n^{th} harmonic:

$$A_n = \frac{2}{T}\bar{g}(jn\omega) = \frac{E}{\pi^2 n^2}\left\{(-1)^n - 1 + j\pi n\right\}$$

Example 2 (Clipped Sawtooth Wave)

From the graph, the average of $f(t)$ over one cycle is:

$$\left(\frac{1}{2} \times \frac{E}{2}\right) + \left(\frac{1}{2} \times E\right) = \frac{3E}{4}$$

Alternatively we can use the integral:

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^T f(t)\mathrm{d}t = \frac{2}{T}\left(\int_0^{T/2} \frac{2E}{T} t\mathrm{d}t + \int_{T/2}^T E\mathrm{d}t\right) \\ &= \frac{2E}{T}\left(\frac{2}{T}\left[\frac{1}{2}t^2\right]_0^{T/2} + \left[t\right]_{T/2}^T\right) \\ &= \frac{2E}{T}\left(\frac{2}{T}\frac{1}{2}\frac{T^2}{4} - 0 + T - \frac{T}{2}\right) = \frac{3E}{2} \end{aligned}$$

$$\therefore \text{DC level} = \frac{a_0}{2} = \frac{3E}{4}$$

Example 2 (Clipped Sawtooth Wave)

$\therefore$ Complex form of the Fourier Series of $f(t)$ is:

$$f(t) = \frac{a_0}{2} + Re\left\{ \sum_{n=1}^{\infty} A_n \, e^{jn\omega t} \right\}$$

$$= \frac{3E}{4} + Re\left\{ \frac{E}{\pi^2} \sum_{n=1}^{\infty} \frac{\left((-1)^n - 1 + j\pi n\right)}{n^2} \, e^{jn\omega t} \right\}$$

where $\omega = \frac{2\pi}{T}$

Example 2 (Clipped Sawtooth Wave)

To obtain the regular Fourier coefficients:

$$a_n = Re\{A_n\} = \frac{-E}{\pi^2 n^2}\left(1 - (-1)^n\right) = \begin{cases} \frac{-2E}{\pi^2 n^2} & \text{for odd } n, \\ 0 & \text{for even } n, \end{cases}$$

$$b_n = -Im\{A_n\} = \frac{-E}{\pi n} \qquad \forall n \in \mathbb{N}.$$

Then the real Fourier Series representation for $f(t)$ is:

$$f(t) = \frac{3E}{4} - \frac{2E}{\pi^2} \sum_{\text{odd } n \in \mathbb{N}} \frac{1}{n^2} \cos\left(\frac{2\pi nt}{T}\right) - \frac{E}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2\pi nt}{T}\right)$$

where $\omega = \frac{2\pi}{T}$

Special Exponential values

In this example we obtained special values of the complex exponential function. These arise by considering the role of the argument when writing complex numbers in Euler or polar form.

For any integer (whole number) n:

$$e^{2n\pi j} = \cos(2n\pi) + j\sin(2n\pi) = 1 + j \times 0 = 1$$

$$\begin{aligned} e^{n\pi j} &= \cos(n\pi) + j\sin(n\pi) \\ &= \cos(n\pi)+j\times 0 = \cos(n\pi) = (-1)^n = \begin{cases} -1 & \text{for odd } n, \\ +1 & \text{for even } n. \end{cases} \end{aligned}$$

Special Exponential values

In summary, the special results to look out for are:

For any integer (whole number) n:

$$e^{2n\pi j} = e^{-2n\pi j} = 1$$

$$e^{n\pi j} = e^{-n\pi j} = (-1)^n$$

$$e^{\frac{\pi}{2}nj} = j^n$$

Summary

After today, you should be able to . . .

- Define the complex form of Fourier Series.
- Explain what phasors are.
- Determine the complex Fourier Series for a periodic function using the method of phasors and Laplace transforms.

This Week

This lecture corresponds to Section 3.5-3.7 of the Course Notes.

Before this week's tutorial:

- Attempt Tutorial sheet 6 - practising what we have been doing today.

In the following lecture we will develop these techniques with some applications to circuit analysis.

Bonus information: odd and even functions!

Last week we learned to identify odd and even functions.

Why?

They allow us to take shortcuts when calculating the real coefficients a_n and b_n.

The Fourier Series consist of:

$$f(x) = \text{constant term} + \text{cosine terms} + \text{sine terms}$$

Notice that if we split up these three parts:

- The constant term is an **even** function.
- The cosine terms add up to give an **even** function.
- The sine terms add up to give an **odd** function.

Bonus information: odd and even functions!

If $f(t)$ is even, the sine terms must be missing:

$$b_n = 0$$

If $f(t)$ is odd, the constant term and cosine terms must be missing:

$$a_0 = a_n = 0$$