Lecture 7: Fourier Series (3/3)

Dr Gavin M Abernethy

Further Mathematics, Signals and Systems

Lecture 7

Today we shall cover:

- Applying Fourier Series to circuit analysis.
- In particular, if we are given an **input signal** and the system's **transfer function**:

  1. We can determine the Fourier series of the **input**.
  2. Then (easily!) calculate the Fourier series of the **output**.

We shall be using the complex form of Fourier Series.

Complex Fourier Series

Complex form of the Fourier Series:

$$f(t) = \frac{a_0}{2} + Re\left\{ \sum_{n=1}^{\infty} A_n \, e^{jn\omega t} \right\}$$

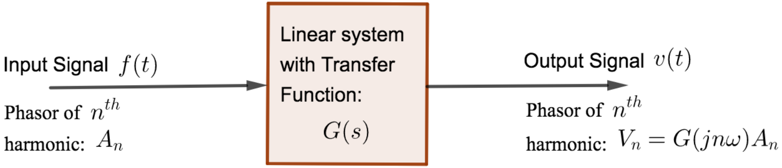
The complex Fourier coefficients ("phasors") are obtained by Laplace transforms:

Phasor of the n^{th} harmonic:

$$A_n = \frac{2}{T}\bar{g}(jn\omega)$$

Concept: Use of Fourier Series in Circuit analysis

- Let $f(t)$ be a periodic input to a linear system.
- What is the effect of the system on the input, encoded in its transfer function $G(s)$?
- The n^{th} harmonic is modified by the frequency response function $G(jn\omega)$.



Concept: Use of Fourier Series in Circuit analysis

Let A_n be the phasor for the n^{th} harmonic of the input.
Then the output phasor V_n is given by:

Phasor of the n^{th} harmonic of the output:

$$V_n = G(jn\omega)A_n$$

So the Fourier Series of the output $v(t)$ is:

Complex Fourier Series of output signal:

$$v(t) = G(0)\frac{a_0}{2} + Re\left\{ \sum_{n=1}^{\infty} G(jn\omega)A_n\, e^{jn\omega t} \right\}$$

This is one reason why Fourier Series (esp. complex form) is useful.

Distortion

- Input phasors are modified by the frequency response function to produce the corresponding output phasor.
- The modulus of A_n is multiplied by the modulus of $G(jn\omega)$ to give the amplitude of the n^{th} harmonic of the output.
- The phase angle (the argument) of $G(jn\omega)$ is added to the phase angle of A_n to give the phase angle of the n^{th} harmonic V_n of the output.

Distortion

Amplitude Distortion

If the modulus of $G(jn\omega)$ depends on n, the amplitude of each harmonic may be scaled by a different factor.

Phase Distortion

If the phase angle of $G(jn\omega)$ depends on n, each phase angle may be altered by a different amount.

Often both types of distortion occur. Either will result in the output waveform having a different shape to the input waveform.

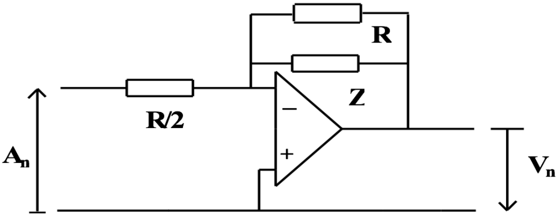
Method: Determining Fourier Series of an output signal

Given a linear circuit and an input signal $f(t)$, we wish to determine the output signal $v(t)$:

1. Determine the transfer function $G(s)$ of the system.
2. Using the method of Laplace transforms, determine the complex Fourier Series of input $f(t)$.
3. Evaluate the transfer function at $s = 0$ to find $G(0)$.
4. Multiply the DC level of $f(t)$ by $G(0)$ to determine the DC level of the output.
5. Multiply the phasors A_n of the input by the frequency response function $G(jn\omega)$ to determine the phasors of the output.
6. Write down the complex Fourier Series of output $v(t)$.

Example 1

Consider the circuit shown.



The transfer function for this system is:

$$G(s) = 2\left(\frac{1 + sCR}{1 + 2sCR}\right)$$

Example 1

The frequency response function is therefore:

$$G(jn\omega) = 2\left(\frac{1 + jn\omega CR}{1 + 2jn\omega CR}\right)$$

and at $s = 0$:

$$G(0) = 2\left(\frac{1+0}{1+0}\right) = 2$$

Therefore, given a general input signal with Fourier series:

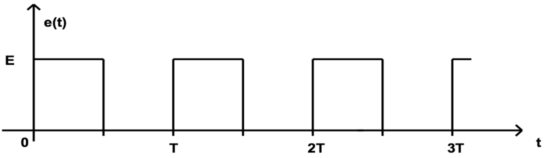
$$e(t) = \frac{a_0}{2} + Re\left\{\sum_{n=1}^{\infty} A_n \, e^{jn\omega t}\right\}$$

the output from this system will be:

$$v(t) = a_0 + Re\left\{2\sum_{n=1}^{\infty} \frac{1 + jn\omega CR}{1 + 2jn\omega CR} A_n \, e^{jn\omega t}\right\}.$$

Example 1

For example, consider if the input signal was this pulse wave $e(t)$:



In this case the input phasor is:

$$A_n = a_n - jb_n = \frac{-jE}{\pi n}\left(1 - \cos(\pi n)\right) = \begin{cases} \frac{-2Ej}{\pi n} & \text{for odd } n, \\ 0 & \text{for even } n, \end{cases}$$

and input DC level: $\dfrac{a_0}{2} = \dfrac{E}{2}$

Example 1

Hence for this specific input, the output phasor is:

$$V_n = G(jn\omega)A_n = 2\left(\frac{1 + jn\omega CR}{1 + 2jn\omega CR}\right) \times \begin{cases} \frac{-2Ej}{\pi n} & \text{for odd } n, \\ 0 & \text{for even } n, \end{cases}$$

The output DC level is:

$$G(0) \times \frac{a_0}{2} = 2 \times \frac{E}{2} = E$$

So the Fourier Series of the output signal is:

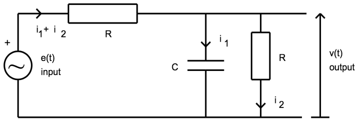
$$v(t) = E + Re\left\{\frac{-4Ej}{\pi} \sum_{odd\ n\in\mathbb{N}} \frac{(1 + jn\omega CR)}{n(1 + 2jn\omega CR)} e^{jn\omega t}\right\}, \qquad \omega = \frac{2\pi}{T}$$

This process consolidates much of what we have studied on this module:

- Finding and manipulating the **transfer function** of a system (you must be able to do this!)
- Using Laplace transforms to find the **complex Fourier Series** of an input signal.
- Combining these to determine the Fourier Series of the **output signal**.

Example 2 - Practice

Consider this circuit:



$$e(t) = R(i_1(t) + i_2(t)) + Ri_2(t)$$

$$\frac{1}{C}\int_0^t i_1(t)\mathrm{d}t = Ri_2(t)$$

$$v(t) = Ri_2(t)$$

R and C are positive constants, $e(t)$ is the input signal and $v(t)$ is the output signal. Given a general input,

$$e(t) = \frac{a_o}{2} + Re\left\{\sum_{n=1}^{\infty} A_n\, e^{jn\omega t}\right\}$$

Determine the Fourier Series of the corresponding output $v(t)$.

Example 2 - Practice

Taking Laplace transforms of each equation:

$$(1) \qquad \bar{e}(s) = R(\bar{i}_1 + 2\bar{i}_2)$$

$$(2) \qquad \frac{1}{sC}\bar{i}_1 = R\bar{i}_2$$

$$(3) \qquad \bar{v}(s) = R\bar{i}_2$$

Then to find the transfer function...

1. Rearrange (2), then substitute it into (1) to eliminate $\bar{i}_1$:

$$\text{From (2):} \quad \bar{i}_1 = sCR\bar{i}_2$$

$$\begin{aligned} \therefore \bar{e}(s) &= R(\bar{i}_1 + 2\bar{i}_2) = R(sCR\bar{i}_2 + 2\bar{i}_2) \\ &= R\bar{i}_2(sCR + 2) \end{aligned}$$

Example 2 - Practice

2. Subsitute in (3) to eliminate $\bar{i}_2$:

$$\bar{e}(s) = \bar{v}(s)(sCR + 2)$$

Finally, rearrange this to the transfer function:

$$G(s) = \frac{\bar{v}(s)}{\bar{e}(s)} = \frac{1}{sCR + 2}$$

$\therefore$ Frequency response function:

$$G(jn\omega) = \frac{1}{jn\omega CR + 2}.$$

Example 2 - Practice

Let A_n be the phasors of the n^{th} harmonics of the input with angular frequency $n\omega$.

Then the phasor of the n^{th} harmonic of the output is:

$$V_n = G(jn\omega)A_n = \left(\frac{1}{2 + jn\omega CR}\right)A_n$$

The DC level of the output is:

$$\frac{V_0}{2} = G(0) \times \frac{1}{2}a_0 = \left(\frac{1}{2 + 0 \cdot CR}\right) \times \frac{1}{2}a_0 = \frac{1}{4}a_0$$

Example 2 - Practice

Hence if the Fourier series of the input signal is

$$e(t) = \frac{a_0}{2} + Re\left\{ \sum_{n=1}^{\infty} A_n \, e^{jn\omega t} \right\}$$

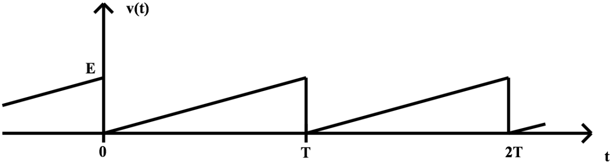
then the output will be:

$$\begin{aligned} v(t) &= G(0)\frac{1}{2}a_0 + Re\left\{ \sum_{n=1}^{\infty} G(jn\omega) A_n \, e^{jn\omega t} \right\} \\ &= \frac{a_0}{4} + Re\left\{ \sum_{n=1}^{\infty} \frac{1}{2 + jn\omega CR} A_n \, e^{jn\omega t} \right\}, \qquad \omega = \frac{2\pi}{T} \end{aligned}$$

$G(jn\omega)$ is complex and frequency dependent,
$\implies$ expect both amplitude and phase distortion.

Example 2 - Practice

Consider if the input was the sawtooth waveform:



Last week we saw that this has phasor:

$$A_n = \frac{Ej}{\pi n}$$

and DC level:

$$\frac{a_0}{2} = \frac{E}{2}$$

Example 2 - Practice

Hence for this specific input, the output phasor is:

$$V_n = G(jn\omega)A_n = \frac{1}{2 + jn\omega CR} \times \frac{Ej}{\pi n}$$

$$= \frac{Ej(2 - jn\omega CR)}{\pi n(2 + jn\omega CR)(2 - jn\omega CR)} = \frac{E(2j + n\omega CR)}{\pi n(4 + n^2\omega^2 C^2 R^2)}$$

The output DC level is:

$$G(0) \times \frac{a_0}{2} = \frac{1}{2} \times \frac{E}{2} = \frac{E}{4}$$

$\therefore$ Complex Fourier Series of the output signal:

$$v(t) = \frac{E}{4} + Re\left\{ \sum_{n=1}^{\infty} \frac{E(2j + n\omega CR)}{\pi n(4 + n^2\omega^2 C^2 R^2)} e^{jn\omega t} \right\}, \quad \omega = \frac{2\pi}{T}$$

Summary

After today, you should be able to . . .

- (From last week) Determine the complex Fourier Series of an input signal from the waveform.
- (From week 4) Find the transfer function from the equations of a circuit.
- Determine the Fourier Series of the output signal of a circuit, given the input.

This Week

This week's lecture corresponds to Section 3.8 of the Course Notes.

Before this week's tutorial:

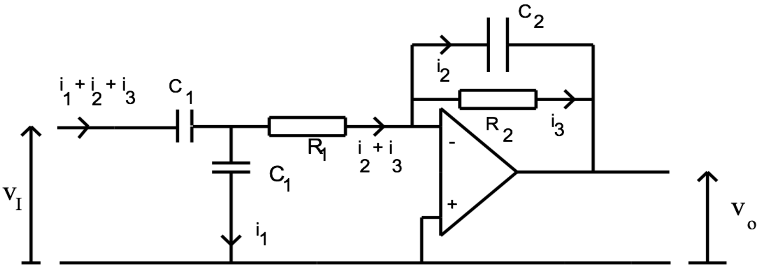
- Attempt Tutorial sheet 7

In the following lecture we will move on to the final topic:

Matrix algebra

Extra Question

Consider the following circuit:



Extra Question

These equations describe the currents i_1, i_2, i_3, input voltage v_i and output voltage v_o, where R_1, R_2, C_1, C_2 are positive constants.

$$
\begin{aligned}
v_i(t) &= \frac{1}{C_1}\int_0^t (i_1(t)+i_2(t)+i_3(t))\mathrm{d}t + R_1\left(i_2(t)+i_3(t)\right)\mathrm{d}t \\
\frac{1}{C_1}\int_0^t i_1(t)\mathrm{d}t &= R_1\left(i_2(t)+i_3(t)\right) \\
R_2 i_3(t) &= \frac{1}{C_2}\int_0^t i_2(t)\mathrm{d}t \\
v_o(t) &= -R_2 i_3(t)
\end{aligned}
$$

For a general input $v_i(t)$, find the Fourier Series of the output $v_o(t)$

Extra Question

The transfer function is:

$$G(s) = \frac{\bar{v}_o}{\bar{v}_i} = \frac{-sC_1R_2}{(1+2sC_1R_1)(1+sC_2R_2)}.$$

Therefore, if the Fourier series for the input wave is:

$$v_i(t) = \frac{a_0}{2} + Re\left\{ \sum_{n=1}^{\infty} A_n\, e^{jn\omega t} \right\}, \qquad \omega = \frac{2\pi}{T},$$

then the corresponding Fourier series for the output is:

$$\begin{aligned} v_o(t) &= \frac{a_0}{2}G(0) + Re\left\{ \sum_{n=1}^{\infty} G(jn\omega)A_n\, e^{jn\omega t} \right\} \\ &= Re\left\{ \sum_{n=1}^{\infty} \frac{-jn\omega C_1R_2}{(1+2jn\omega C_1R_1)(1+jn\omega C_2R_2)} A_n\, e^{jn\omega t} \right\} \end{aligned}$$