Lecture 8: Matrix Algebra (1/4)

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Further Mathematics, Signals and Systems

Lecture 8

Today we shall cover:

- Revision of matrix algebra (addition, multiplication, etc.).
- **Determinants** and inverses of matrices.
- An introduction to the concept of Eigenvalues and Eigenvectors.

Introduction

A matrix is a rectangular array for storing data. We often denote them in mathematics by a capital letter, or underlining.

The **order** of a matrix is a description of its dimensions - the number of rows, then the number of columns.

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix}$$ This is a 2×1 matrix. It is also a *vector*.

$$\begin{pmatrix} 1 & -2 & 8 \\ 3 & 1 & 4 \end{pmatrix}$$ This is a 2×3 matrix.

$$\begin{pmatrix} 2 & 0 & -1 & 6 \end{pmatrix}$$ This is a 1×4 matrix.

Matrix Addition and Subtraction

Only matrices with the same order can be added or subtracted.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \end{pmatrix} \quad \textbf{INVALID}$$

$$\begin{aligned} \begin{pmatrix} 4 & -1 & -2 \\ -2 & 3 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 3 \\ 1 & -4 & -6 \end{pmatrix} &= \begin{pmatrix} 4-2 & -1-0 & -2-3 \\ -2-1 & 3-(-4) & 5-(-6) \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 & -5 \\ -3 & 7 & 11 \end{pmatrix} \end{aligned}$$

Exercise

$$\text{Given} \quad A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$$

Find (if they exist):

$$A + B, \qquad A + C, \qquad C - A$$

Scalar Multiplication

Multiply a matrix by a scalar (a real or complex *number*, rather than a vector or matrix) simply by multiplying ("scaling") each element of the matrix by that scalar.

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

For example:

$$-3 \begin{pmatrix} 2 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} -3 \times 2 \\ -3 \times 8 \\ -3 \times -5 \end{pmatrix} = \begin{pmatrix} -6 \\ -24 \\ 15 \end{pmatrix}$$

The magnitude of the vector is changed, but the direction remains the same.

Matrix Multiplication

Matrix multiplication is a **non-commutative** operation. This means that $A \times B$ is *not* equivalent to $B \times A$ and does not necessarily yield the same result.

The order of matrix multiplication *can not be changed*.

In fact, one order might not even exist whilst the other does!

Matrix Multiplication

The number of columns in the first matrix must match the number of rows in the second.

If this is satisfied, the result is is given by the remaining dimensions - the same number of rows as the first matrix and columns as the second matrix.

$$\underset{2 \times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \quad \underset{2 \times 1}{\begin{pmatrix} 5 \\ 6 \end{pmatrix}} \quad = \begin{pmatrix} 1 \times 5 + 2 \times 6 \\ 3 \times 5 + 4 \times 6 \end{pmatrix} = \underset{2 \times 1}{\begin{pmatrix} 17 \\ 39 \end{pmatrix}}$$

Matrix Multiplication: Example 1

Let,

$$B = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$$

Calculate BC and CB if they exist.

As B is a 2×1 and C is a 2×2 matrix, BC does not exist as the columns of B do not match the number of rows of C.

However, CB does exist, and the result will be another 2×1 matrix:

$$CB = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \times 3 + 2 \times -2 \\ 4 \times 3 + 5 \times -2 \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$$

Matrix Multiplication

If we plot the vector before and after the “action” of matrix C on a 2-dimensional graph:

$$\begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$$

We can see that while scalar multiplication preserved the direction of the vectors, in this case matrix multiplication has both stretched *and* rotated the input vector.

We will revisit this idea when we introduce the concept of *eigenvectors*!

Matrix Multiplication: Example 2

Let,

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

Calculate AD if it exists:

The result will be another 2×2 matrix:

$$\begin{aligned} AD &= \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 \times 3 + 0 \times 0 & 2 \times 1 + 0 \times (-2) \\ -1 \times 3 + 1 \times 0 & -1 \times 1 + 1 \times (-2) \end{pmatrix} \\ &= \begin{pmatrix} 6 + 0 & 2 + 0 \\ -3 + 0 & -1 - 2 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ -3 & -3 \end{pmatrix} \end{aligned}$$

Exercise

Given

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \quad E = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad F = \begin{pmatrix} 1 & -3 \\ 0 & 4 \\ -2 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

Find (if they exist):

$$-4A, \qquad 3E, \qquad AE, \qquad EA, \qquad AF, \qquad FA, \qquad DA$$

Special matrices

The **zero matrix** is a square matrix where every entry is zero.

$$\underline{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \underline{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It acts like the number 0 in matrix addition and matrix multiplication, so:

$$A\underline{O} = \underline{O} = \underline{O}A \quad \text{for any matrix } A \text{ of suitable order.}$$

and

$$A + \underline{O} = A = \underline{O} + A \quad \text{for any matrix } A \text{ of suitable order.}$$

Special matrices

For each positive integer n, the $n \times n$ **identity matrix** consists of one's on the diagonal entries and zeroes elsewhere.

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

This is the only matrix which satisfies, for a matrix A of appropriate dimensions,

$$AI = IA = A$$

So it acts like a matrix version of the number "1" when it comes to multiplication.

Special matrices

Take a moment to appreciate the 2×2 and 3×3 identity matrices:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Very good.

Determinants

Square matrices (with dimensions $n \times n$) have a property called the **determinant**, denoted by $det(A)$ or $|A|$.

The determinant represents the scaling factor when the matrix is used to transform an image.

For a 2×2 matrix A, we simply multiply the diagonal entries:

Determinant of a 2×2 matrix:

$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example of determinant of a 2×2 matrix

Given the square matrix

$$B = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$$

The determinant is given by:

$$\begin{aligned} det(B) &= 3 \times 2 - (-1) \times 4 \\ &= 6 + 4 \\ &= 10 \end{aligned}$$

Exercise: Determinant of a 2×2 matrix

For the following square matrices, find the determinant of the matrix:

$$B = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Solution: Determinant of a 2×2 matrix

$$B = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

$$\det(B) = (1)(2) - (0)(-3) = 2$$

$$C = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\det(C) = (1)(-1) - (1)(-1) = 0$$

Inverse matrix

For a **square** matrix A, there may exist an inverse matrix A^{-1}

Inverse Matrix

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I$$

So an inverse matrix is analagous to the reciprocal of a number - it's what you multiply by to get back to 1 (or the identity):

$$5 \times \frac{1}{5} = 1$$

$$A \times A^{-1} = I$$

Calculating the inverse of a 2×2 matrix

For a general 2×2 square matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

Inverse of a 2×2 matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{or} \quad A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If the determinant of a square matrix is equal to zero, then that matrix has no inverse!

Example: Inverse of a 2×2 matrix

To find (if it exists) the inverse of 2×2 square matrix A:

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

First obtain the determinant:

$$det(A) = (1)(2) - (-1)(0) = 2$$

Then as the determinant is non-zero, the inverse exists and is:

$$A^{-1} = \frac{1}{det(A)} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}$$

Determinant of a 3×3 matrix

Work across the **top row** and multiply each entry by the determinant of the corresponding 2×2 co-matrix of the rows and columns that the current entry is *not* in.

Then **change the sign of the middle entry**.

Determinant of a 3×3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

Example: Determinant of a 3×3 matrix

Find the determinant of

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} det(A) &= \begin{vmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} \\ &= 3\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} - (0)\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 3\big(0 \times 1 - (-2) \times 1\big) - 0 + 2\big(2 \times 1 - 0 \times 0\big) \\ &= 3(0+2) + 2(2-0) \ = \ 10 \end{aligned}$$

Exercise

Find the determinant of:

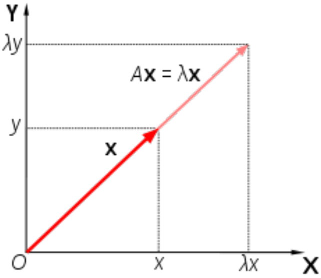
$$A = \begin{pmatrix} 1 & 4 & -1 \\ 1 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$$

Motivation - Eigenvalues and Eigenvectors

- When a square matrix A acts on a vector $\underline{\mathbf{x}}$, we obtain a new vector $A\underline{\mathbf{x}}$ that may be stretched and rotated in some way.
- In many cases, it is useful to consider solutions to:

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \quad \text{where} \quad \lambda \quad \text{is a scalar.}$$

- These are vectors (eigenvectors) $\underline{\mathbf{x}}$ whose direction is **preserved** when we multiply by matrix A. They are magnified by a scaling factor (eigenvalue) λ.



Theory

- For an $n \times n$ square matrix A, there are n (not necessarily distinct) eigenvalues.
- Every eigenvalue has a family of infinitely-many eigenvectors associated with it. They all have the **same direction**, but can be of **any magnitude**.
- This means that if $\underline{\mathbf{e}}_i$ is an eigenvector of A with eigenvalue λ_i, then so is *any scalar multiple* of $\underline{\mathbf{e}}_i$.
- We will use this to help us find the "easiest" example of an eigenvector in our examples.
- Eigenvalue-eigenvector pairs have many applications.
  e.g. to determine resonant frequencies of oscillating systems.

How to calculate eigenvalues and eigenvectors (1)

To find the eigenvalues λ and eigenvectors $\underline{\mathbf{x}}$ of matrix A, we first re-arrange the definition $A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}}$ to:

$$(A - \lambda I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

where I is the identity matrix.

Then to find the eigenvalues, solve the following equation for λ:

Characteristic polynomial of A

$$\det\left(A - \lambda I\right) = 0$$

For a 2×2 matrix this will be a **quadratic equation**.

How to calculate eigenvalues and eigenvectors (2)

Then for each eigenvalue $\lambda = \lambda_1, \lambda_2, \ldots,$ we then obtain a corresponding non-zero eigenvector $\underline{\mathbf{x}} = \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \ldots$

We can do this by substituting in the eigenvalue and solving:

To find the eigenvector:

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \qquad \text{or} \qquad (A - \lambda I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

for the column vector $\underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$

Summary

After today, you should be able to . . .

- (Revision) Add, subtract, and multiply matrices.
- Find the **inverse** of a 2×2 matrix.
- Find the **determinant** of 2×2 and 3×3 matrices.
- Explain what the eigenvalue-eigenvector pairs of a matrix are.

This Week

This week's lecture corresponds to Section 4.1 and 4.2.1 of the Course Notes.

Before this week's tutorial:

- Attempt Tutorial sheet 8

In the following lecture we will put these ideas into practice and calculate the eigenvalues and eigenvectors of some matrices.