Lecture 9: Matrix Algebra (2/4)

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Further Mathematics, Signals and Systems

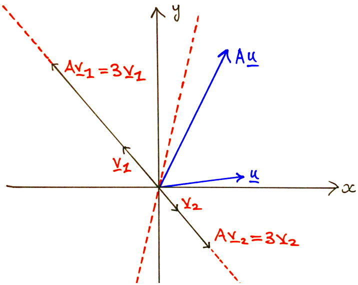
Lecture 9: Calculating eigenvalues and eigenvectors

Today we shall cover:

- Determining the eigenvalues of a 2×2 or 3×3 matrix.
- Determining the eigenvectors of a 2×2 or 3×3 matrix.

What are eigenvalues and eigenvectors?

When a 2×2 matrix A pre-multiplies a position vector, it is usually stretched and rotated. However, there exist "natural axes" of vectors (the eigenvectors) whose direction stays the same, and they are simply scaled by a constant (the eigenvalue).



$\underline{\mathbf{u}}$ is *not* an eigenvector of A, but both $\underline{\mathbf{v}}_1$ and $\underline{\mathbf{v}}_2$ *are*, with an eigenvalue of 3. In fact, *all* vectors on the red axes are eigenvectors.

How to calculate eigenvalues and eigenvectors (1)

To find the eigenvalues λ and eigenvectors $\underline{\mathbf{x}}$ of square matrix A, we first re-arrange the definition $A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}}$ to:

$$(A - \lambda I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

where I is the identity matrix.

First, find the eigenvalues by solving the following equation for λ:

Characteristic polynomial of A

$$\det\left(A - \lambda I\right) = 0$$

For a 2×2 matrix this will be a **quadratic equation**.

How to calculate eigenvalues and eigenvectors (2)

Then each eigenvalue $\lambda = \lambda_1, \lambda_2, \ldots,$ we then obtain a corresponding non-zero eigenvector $\underline{\mathbf{x}} = \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \ldots$

We can do this by substituting in the eigenvalue and solving:

To find the eigenvector:

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \qquad \text{or} \qquad (A - \lambda I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

for the column vector $\underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$

How to calculate eigenvectors

Recall from last week, that there are **infinitely-many** eigenvectors corresponding to each eigenvalue.

This means that when solving the set of equations to find $\underline{\mathbf{x}}$, there is *no single solution*. Instead, we can choose a value for one of the variables, and then use the equations to obtain the remainder.

This also results in redundancy among the equations.

In the 2×2 case, the two equations obtained will be the **same**.

Example 1: Eigenvalues and Eigenvectors of a 2×2 Matrix

Determine the eigenvalues and eigenvectors of the following 2×2 matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

Example 1: Eigenvalues and Eigenvectors of a 2×2 Matrix

First determine the eigenvalues:

$$
\begin{aligned}
A - \lambda I &= \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\
&= \begin{pmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{pmatrix}
\end{aligned}
$$

Therefore, we wish to solve the characteristic equation:

$$
|A - \lambda I| = 0 \quad \Longrightarrow \quad \begin{vmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{vmatrix} = 0
$$

Example 1: Eigenvalues and Eigenvectors of a 2×2 Matrix

Evaluating the determinant by taking the difference of the product of the diagonals:

$$(1-\lambda)(-4-\lambda)-(2)(3)=0,$$

and so the characteristic polynomial is:

$$\lambda^2+3\lambda-10=0$$

Solving this quadratic equation yields two distinct, real, integer roots:

$$\lambda_1=-5, \qquad \lambda_2=2$$

These are the two eigenvalues of matrix A.

Example 1: Eigenvalues and Eigenvectors of a 2×2 Matrix

Next we solve the eigenvectors one at a time.

For the first eigenvalue, $\lambda_1 = -5$, call the corresponding eigenvector $\underline{\mathbf{e}}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$

To find the values of the components x and y, we need to solve:

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \qquad \Longrightarrow \qquad A\underline{\mathbf{e}}_1 = -5\underline{\mathbf{e}}_1$$

$$\therefore \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -5 \begin{pmatrix} x \\ y \end{pmatrix}$$

The rows of this matrix equation yield a pair of equations.

Example 1: Eigenvalues and Eigenvectors of a 2×2 Matrix

$$\begin{aligned} x + 2y &= -5x \\ 3x - 4y &= -5y \end{aligned}$$

These two are actually the same equation, rearranged.

Solving either yields:

$$y = -3x$$

If we choose $x = 1$ (since *any* scalar multiple of the eigenvector will work), then $y = -3$.

So one eigenvector corresponding to $\lambda_1 = -5$ is:

$$\underline{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Example 1: Eigenvalues and Eigenvectors of a 2×2 Matrix

Using the same method, can you now find the second set of eigenvectors, associated with the other eigenvalue $\lambda_2 = 2$?

Example 1: Eigenvalues and Eigenvectors of a 2×2 Matrix

Call the corresponding eigenvector $\underline{\mathbf{e}}_2 = \begin{pmatrix} x \\ y \end{pmatrix}$

To find the values of the components x and y, we need to solve:

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \qquad \Longrightarrow \qquad A\underline{\mathbf{e}}_2 = 2\underline{\mathbf{e}}_2$$

$$\therefore \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

The rows of this matrix equation yield a pair of equations:

$$\begin{aligned} x + 2y &= 2x \\ 3x - 4y &= 2y \end{aligned}$$

Example 1: Eigenvalues and Eigenvectors of a 2×2 Matrix

Again these are actually the same equation. Can you see why?

We can rearrange either to:

$$-x + 2y = 0$$

and so

$$x = 2y$$

Choose $y = 1$, then it follows that $x = 2$ and so:

$$\underline{\mathbf{e}}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Example 2: Eigenvalues and Eigenvectors of a 3×3 Matrix

Consider the following 3×3 matrix A.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

We will calculate the three eigenvalues and associated eigenvectors for this matrix.

The method is the same, but the actual calculations will be a little more involved.

Example 2: Eigenvalues and Eigenvectors of a 3×3 Matrix

$$|A - \lambda I| = 0 \quad \text{gives} \quad \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} -1 & 2-\lambda \\ 0 & -1 \end{vmatrix} = 0$$

$$\therefore (1-\lambda)\big((2-\lambda)(1-\lambda) - (-1)(-1)\big) + \big((-1)(1-\lambda) - (-1)(0)\big)$$
$$+0\big((-1)(-1) - (2-\lambda)(0)\big) = 0$$

This reduces to:

$$(1-\lambda)(\lambda)(\lambda - 3) = 0$$

Hence there are three eigenvalues: $\lambda = 0, 1, 3$.

Example 2: Eigenvalues and Eigenvectors of a 3×3 Matrix

Note: In this case, we kept out the common factor of $(\lambda - 1)$. You may be *given* the eigenvalues and asked to **verify** them, meaning that you must obtain the characteristic polynomial, then show by substitution that the proposed value satisfies the equation.

e.g. If we had multiplied out the characteristic polynomial to obtain:

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0$$

Then to verify that $\lambda = 3$ is an eigenvalue:

$$\begin{aligned} (3)^3 - 4(3)^2 + 3(3) &= 27 - 4 \times 9 + 9 \\ &= 27 - 36 + 9 \\ &= 0 \end{aligned}$$

Example 2: Eigenvalues and Eigenvectors of a 3×3 Matrix

i) For the first eigenvalue $\lambda_1 = 0$, let $\underline{\mathbf{e}}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the corresponding eigenvector.

$$A\underline{\mathbf{e}}_1 = \lambda_1 \underline{\mathbf{e}}_1 \implies \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Multiplying out the rows gives three equations:

$$\begin{aligned} x - y \quad\quad &= 0 \\ -x + 2y - z &= 0 \\ -y + z &= 0 \end{aligned}$$

Example 2: Eigenvalues and Eigenvectors of a 3×3 Matrix

From the first equation, we obtain:

$$x - y = 0 \qquad \Longrightarrow \qquad x = y$$

From the third equation:

$$-y + z = 0 \qquad \Longrightarrow \qquad z = y$$

Thus, if we choose $y = 1$, then it follows that $x = 1$ and $z = 1$. As in the previous example, the final equation is redundant - but we should check by substitution that it agrees with our solution:

$$\begin{aligned} -x + 2y - z &= -(1) + 2(1) - (1) \\ &= -1 + 2 - 1 \\ &= 0 \quad \text{as expected!} \end{aligned}$$

Example 2: Eigenvalues and Eigenvectors of a 3×3 Matrix

Hence one eigenvector is:

$$\underline{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Any other vector with the same direction (same ratio between the components) would *also* be an eigenvector for $\lambda_1 = 0$.

For example:

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -31 \\ -31 \\ -31 \end{pmatrix}, \quad \begin{pmatrix} 2.007 \\ 2.007 \\ 2.007 \end{pmatrix}$$

Or we could say: $\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \forall \alpha \in \mathbb{R}$

to describe the set of all of them!

Example 2: Eigenvalues and Eigenvectors of a 3×3 Matrix

ii) $\lambda_2 = 1$:

$$A\underline{\mathbf{e}}_2 = \lambda_2\underline{\mathbf{e}}_2 \implies \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This gives three equations:

$$\begin{aligned} x - y \quad\quad &= x \\ -x + 2y - z &= y \\ -y + z &= z \end{aligned}$$

Example 2: Eigenvalues and Eigenvectors of a 3×3 Matrix

From the first equation $x - y = x$, we have:

$$y = 0$$

Substituting this into $-x + 2y - z = y$ gives:

$$-x - z = 0 \quad \Longrightarrow \quad z = -x$$

Choose $x = 1$, then $y = 0$ and $z = -1$, so an eigenvector is:

$$\underline{\mathbf{e}}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Example 2: Eigenvalues and Eigenvectors of a 3×3 Matrix

iii) **As an exercise now**

Can you determine an eigenvector $\underline{\mathbf{e}}_3$ corresponding to the third eigenvalue:

$$\lambda_3 = 3$$

(Don't worry if you get a different answer from your neighbours - ask, is the *direction* the same?)

Example 2: Eigenvalues and Eigenvectors of a 3×3 Matrix

iii) $\lambda_3 = 3$:

$$A\underline{\mathbf{e}}_3 = \lambda_3 \underline{\mathbf{e}}_3 \implies \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This gives three equations:

$$\begin{aligned} x - y \quad\quad &= 3x \\ -x + 2y - z &= 3y \\ -y + z &= 3z \end{aligned}$$

Example 2: Eigenvalues and Eigenvectors of a 3×3 Matrix

Simplifying,

$$\begin{aligned} -2x - y \qquad &= 0 \\ -x - y \; - z &= 0 \\ - y - 2z &= 0 \end{aligned}$$

From the first equation:

$$y = -2x$$

and from the third equation:

$$z = -\frac{1}{2}y$$

So choose $x = 1$, then $y = -2$ and then $z = 1$. Hence,

$$\underline{\mathbf{e}}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Example 2: Eigenvalues and Eigenvectors of a 3×3 Matrix

The complete solution to the problem is therefore:

$$\lambda_1 = 0, \qquad \underline{\mathbf{e}}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1, \qquad \underline{\mathbf{e}}_2 = \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_3 = 3, \qquad \underline{\mathbf{e}}_3 = \gamma \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

for *any* real values of the scalar constants α, β, γ.

Unit Vectors

A unit vector has magnitude equal to one.

Given any vector, $\underline{\mathbf{v}}$ we can find the unit vector in the same direction by:

$$\hat{\underline{\mathbf{v}}} = \frac{\underline{\mathbf{v}}}{|\underline{\mathbf{v}}|}$$

Consider the eigenvector:

$$\underline{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

It has magnitude:

$$|\underline{\mathbf{e}}_1| = \sqrt{(1)^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$$

So a unit vector in the same direction as $\underline{\mathbf{e}}_1$ is:

$$\hat{\underline{\mathbf{e}}}_1 = \frac{\underline{\mathbf{e}}_1}{|\underline{\mathbf{e}}_1|} = \frac{1}{\sqrt{(1)^2 + (-3)^2}} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Summary

After today, you should be able to . . .

- Calculate the **eigenvalues** and **eigenvectors** of 2×2 and 3×3 matrices.
- Explain what the eigenvalue-eigenvector pairs of a matrix are.
- Calculate **unit** eigenvectors.

This Week

This week's lecture corresponds to Section 4.2 of the Course Notes.

Before this week's tutorial:

- Attempt Tutorial sheet 9

In the following lecture we will think about representing systems of ODEs by matrix equations. These can then be solved in the final lecture using eigenvalues and eigenvectors.

Extra Question

From Tutorial Sheet 9, Question 4:

Determine the eigenvalues and eigenvectors of:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$