# Lecture 10: Matrix Algebra (3/4)

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Further Mathematics, Signals and Systems

Today we shall cover:

- Obtaining the state variable description of systems of ODEs.
- This is a systematic way of writing a system of ODEs as a matrix equation.

# Motivation

- Analysing a circuit may result in a set of ODEs.
- It is possible to solve such systems (i.e. to find a formula for the output) using eigenvalues and eigenvectors.
- But for these techniques to be applied, the system needs to be written as a set of first-order ODEs in the form:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f(x_1, x_2, \ldots, x_n)$$

which can then be stated as a matrix equation:

$$\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}} + B\underline{\mathbf{u}}$$

- This can be achieved with a **state variable description**.

# Theory: The Goal

So, given an ordinary differential equation (ODE) initial value problem of order $n$.

For example:

$$\frac{\mathrm{d}^4 x(t)}{\mathrm{d}t^4} + a_3 \frac{\mathrm{d}^3 x(t)}{\mathrm{d}t^3} + a_2 \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_0 x(t) = 0$$

The goal is to rewrite this as a set of $n$ **first-order** ODEs:

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \dots \qquad \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \dots$$

$$\frac{\mathrm{d}x_3(t)}{\mathrm{d}t} = \dots \qquad \frac{\mathrm{d}x_4(t)}{\mathrm{d}t} = \dots$$

# Theory: The Method

1. We define a new set of variables, $x_i$, called **state variables**.
2. Every variable, *apart from external inputs*, generates an additional state variable for each derivative.
3. Rearrange to a set of equations for the derivative of each state variable, in terms of the state variables and external inputs.

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \ldots, \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = \ldots, \qquad \frac{\mathrm{d}x_3}{\mathrm{d}t} = \ldots$$

4. Representing these as a single matrix equation:

**The state variable description**

$$\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}} + B\underline{\mathbf{u}}$$

# Theory: External control inputs

**The state variable description**

$$\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}} + B\underline{\mathbf{u}}$$

- $\underline{\mathbf{x}}$ is the vector containing the state variables $x_1, x_2, \ldots, x_n$.
- $\underline{\dot{\mathbf{x}}}$ is the vector containing the first derivative of each state variable $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$.
- External control inputs are encoded in their own vector $B\underline{\mathbf{u}}$.
- If there is no control input for the problem, the state variable description is just:

$$\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}}$$

# Theory: Output vector

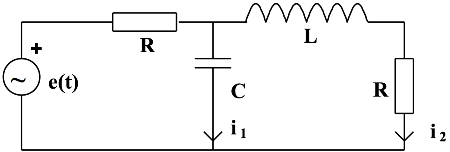
- We may also choose to define some output measurements.
- An output vector $\underline{\mathbf{y}}$ can encode both the **control inputs** and the **measured outputs**, constructed from a linear combination of the state variable and control input vectors:

**Output vector:**

$$\underline{\mathbf{y}} = C\underline{\mathbf{x}} + D\underline{\mathbf{u}}$$

# Example 1

Determine the state variable description of the circuit shown.




The currents in this circuit are described by a pair of ODEs:

$$\frac{\mathrm{d}i_1(t)}{\mathrm{d}t} = -\frac{i_1(t)}{CR} - \frac{\mathrm{d}i_2(t)}{\mathrm{d}t} + \frac{1}{R}\frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2 i_2(t)}{\mathrm{d}t^2} = -\frac{R}{L}\frac{\mathrm{d}i_2(t)}{\mathrm{d}t} + \frac{i_1(t)}{LC}$$

$L$, $C$ and $R$ are positive constants and $e(t)$ is an **external input**.

# Example 1 - Solution

- This is in effect a third-order system:
- Dependent variable $i_1$ is differentiated **once** in the equations, so it generates **one** state variable.
- Dependent variable $i_2$ is differentiated **twice**, and so generates **two** state variables.

$$x_1 \equiv i_1, \qquad x_2 \equiv i_2, \qquad x_3 \equiv \frac{\mathrm{d}i_2}{\mathrm{d}t}$$

Differentiating these:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\mathrm{d}i_1}{\mathrm{d}t}, \qquad \frac{dx_2}{dt} = \frac{\mathrm{d}i_2}{\mathrm{d}t}, \qquad \frac{\mathrm{d}x_3}{\mathrm{d}t} = \frac{\mathrm{d}^2 i_2}{\mathrm{d}t^2}$$

# Example 1 - Solution

**We want a set of equations for the derivative of each state variable, in terms only of the state variables (and the input), and not featuring any derivatives.**

Substituting into the original equations:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -\frac{x_1}{CR} - x_3 + \frac{1}{R}\frac{\mathrm{d}e}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_3$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = -\frac{R}{L}x_3 + \frac{x_1}{LC}$$

**Note:** Any set of linear ODEs can be manipulated into this canonical form of first order ODEs written explicity as:

$$\dot{x}_i = f(x_1, x_2, \ldots, x_n)$$

# Example 1 - Solution

To see how we can represent this set of equations in matrix form, we need to rewrite them with the coefficients of all state variables (including zeros) in the **same correct order** ($x_1$, then $x_2$, then $x_3$):

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -\frac{1}{CR}x_1 + 0x_2 - x_3 + 1 \times \frac{1}{R}\frac{\mathrm{d}e}{\mathrm{d}t} \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= 0x_1 + 0x_2 + 1x_3 + 0 \times \frac{1}{R}\frac{\mathrm{d}e}{\mathrm{d}t} \\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= \frac{x_1}{LC} + 0x_2 - \frac{R}{L}x_3 + 0 \times \frac{1}{R}\frac{\mathrm{d}e}{\mathrm{d}t}
\end{aligned}$$

# Example 1 - Solution

Then putting these equations as the rows of a matrix:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{CR}x_1 + 0x_2 - x_3 + 1 \times \frac{1}{R}\frac{\mathrm{d}e}{\mathrm{d}t} \\ 0x_1 + 0x_2 + 1x_3 + 0 \times \frac{1}{R}\frac{\mathrm{d}e}{\mathrm{d}t} \\ \frac{x_1}{LC} + 0x_2 - \frac{R}{L}x_3 + 0 \times \frac{1}{R}\frac{\mathrm{d}e}{\mathrm{d}t} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{CR}x_1 + 0x_2 - x_3 \\ 0x_1 + 0x_2 + 1x_3 \\ \frac{x_1}{LC} + 0x_2 - \frac{R}{L}x_3 \end{pmatrix} + \begin{pmatrix} 1 \times \frac{1}{R}\frac{\mathrm{d}e}{\mathrm{d}t} \\ 0 \times \frac{1}{R}\frac{\mathrm{d}e}{\mathrm{d}t} \\ 0 \times \frac{1}{R}\frac{\mathrm{d}e}{\mathrm{d}t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{CR} & 0 & -1 \\ 0 & 0 & 1 \\ \frac{1}{LC} & 0 & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \left(\frac{1}{R}\frac{\mathrm{d}e}{\mathrm{d}t}\right)$$

# Example 1 - Solution

Hence: $\quad \underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}} + B\underline{\mathbf{u}}$

where

$$\underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} -\frac{1}{CR} & 0 & -1 \\ 0 & 0 & 1 \\ \frac{1}{LC} & 0 & -\frac{R}{L} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{\mathbf{u}} = \left( \frac{1}{R} \frac{\mathrm{d}e}{\mathrm{d}t} \right)$$

- The column vector $\underline{\mathbf{x}}$ is the vector of **state variables**.
- $\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}} + B\underline{\mathbf{u}}$ is the **state variable description** or **state variable equations**.
- $A\underline{\mathbf{x}}$ represents the internal feedback (or internal control).
- $B\underline{\mathbf{u}}$ encodes the single external "control".

# Example 2

A control system is modelled by the following fourth-order ordinary differential equation:

$$\frac{\mathrm{d}^4 x(t)}{\mathrm{d}t^4} + a_3 \frac{\mathrm{d}^3 x(t)}{\mathrm{d}t^3} + a_2 \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_0 x(t) = 0$$

where $x(t)$ is a scalar and the **output** of the system.

There is **no external control input** in this example.

In the case of a system described by a single $n^{th}$ order ordinary differential equation without external control, the state equations will be of the form $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$, and $A$ will have special properties.

# Example 2 - Solution

The dependent variable $x$ is differentiated **four** times, so it generates **four** state variables:

$$x_1 \equiv x, \qquad x_2 \equiv \frac{\mathrm{d}x}{\mathrm{d}t}, \qquad x_3 \equiv \frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \qquad x_4 \equiv \frac{\mathrm{d}^3x}{\mathrm{d}t^3}$$

Differentiating:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}, \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \qquad \frac{\mathrm{d}x_3}{\mathrm{d}t} = \frac{\mathrm{d}^3x}{\mathrm{d}t^3}, \qquad \frac{\mathrm{d}x_4}{\mathrm{d}t} = \frac{\mathrm{d}x^4}{\mathrm{d}t^4}$$

Then substitute in the original ODE, and rearrange to obtain equations for $\frac{\mathrm{d}x_i}{\mathrm{d}t}$ in terms of $x_i$.

# Example 2 - Solution

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_3$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_4$$

$$\frac{\mathrm{d}x_4}{\mathrm{d}t} = -a_0x_1 - a_1x_2 - a_2x_3 - a_3x_4$$

This may be written as $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$, where:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{pmatrix}, \quad \underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

# Example 2 - Solution

In this case, there is no $D\underline{\mathbf{u}}$ term because there is no control input.

The state variable matrix $A$ is in **companion form**.

## Companion form

A **companion matrix** $A$ consists of 1's in the entries that are one above the diagonal, any real numbers in the bottom row entries, and zeros elsewhere.

This is a standard feature of $A$ when $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$ is obtained from a single $n^{th}$ order ODE.

# Example 2 - Solution

The output, as specified, is the original variable $x$ which is identical to state variable $x_1$. The output vector is:

$$\underline{\mathbf{y}} = \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} x_1 \end{pmatrix} = \begin{pmatrix} 1x_1 + 0x_2 + 0x_3 + 0x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\therefore \underline{\mathbf{y}} = C\underline{\mathbf{x}}$$

$$\text{where} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

As with the state variable description,

No control input $\implies$ no $D\underline{\mathbf{u}}$ term in the output.

# Eigenmode solutions to systems without external control

A system without external control has a state variable representation $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$, where $A$ is a square matrix.

**Eigenmodes**

For each eigenvalue $\lambda_i$ and eigenvector $\underline{\mathbf{b}}_i$ pair for $A$, the corresponding **eigenmode** is:

$$\underline{\mathbf{x}}(t) = c_i\ \underline{\mathbf{b}}_i\, \mathrm{e}^{\lambda_i t} \qquad \text{where } c_i \text{ is any scalar.}$$

These time-dependent functions describe natural vibrations of the system where all parts move at the same frequency.

# Example of calculating eigenmodes

The state variables for a certain electronic system are given by:

$$\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}}, \quad \text{where} \quad A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

The eigenvalue and eigenvector pairs for $A$ are:

$$\lambda_1 = 1, \quad \underline{\mathbf{b}}_1 = \alpha \begin{pmatrix} 1 \\ -2 \end{pmatrix}; \qquad \lambda_2 = 6, \quad \underline{\mathbf{b}}_2 = \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore there are two eigenmodes:

$$\underline{\mathbf{x}}_1 = \mathrm{e}^{t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}; \qquad \underline{\mathbf{x}}_2 = \mathrm{e}^{6t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

or any scalar multiples of each of these.

# Eigenmode solutions to systems without external control

What does it mean to "solve" a system?

# Eigenmode solutions to systems without external control

- "Solving" means obtaining a function for the state variables $\underline{\mathbf{x}}$, so that if we know some initial conditions then we can predict the value of all of the state variables at any future time.
- Adding the eigenmodes of the state variable matrix, scaled by some particular constants, gives the full solution.

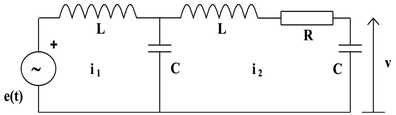
**Solution from eigenmodes:**

$$\underline{\mathbf{x}}(t) = \sum_{i=1}^{n} c_i \; \underline{\mathbf{b}}_i \, e^{\lambda_i t} \qquad \text{(but what should the value of } c_i \text{ be?)}$$

- To actually find the specific values of the constants $c_i$, we will need a diagonalisation technique that we shall see next week.

# Example 3

Determine the state variable description of this circuit:




$$e(t) = L\frac{\mathrm{d}i_1(t)}{\mathrm{d}t} + \frac{1}{C}\int_0^t \big(i_1(t) - i_2(t)\big)\mathrm{d}t$$

$$\frac{1}{C}\int_0^t \big(i_2(t) - i_1(t)\big)\mathrm{d}t + Ri_2(t) + L\frac{\mathrm{d}i_2(t)}{\mathrm{d}t} + \frac{1}{C}\int_0^t i_2(t)\mathrm{d}t = 0$$

$$v(t) = \frac{1}{C}\int_0^t i_2(t)\mathrm{d}t$$

$e(t)$ is an input; $R$, $C$, $L$ are constants; and the output is $z = L\frac{\mathrm{d}i_2}{\mathrm{d}t}$

# Example 3 - Solution

Before introducing the state variables, differentiate these equations to remove the integrals:

$$\frac{\mathrm{d}e}{\mathrm{d}t} = L\frac{\mathrm{d}^2 i_1}{\mathrm{d}t^2} + \frac{1}{C}(i_1 - i_2)$$
$$\frac{1}{C}(i_2 - i_1) + R\frac{\mathrm{d}i_2}{\mathrm{d}t} + L\frac{\mathrm{d}^2 i_2}{\mathrm{d}t^2} + \frac{1}{C}i_2 = 0$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{C}i_2$$

Rearranging,

$$\begin{aligned}
\frac{\mathrm{d}^2 i_1}{\mathrm{d}t^2} &= -\frac{1}{LC}i_1 + \frac{1}{LC}i_2 + \frac{1}{L}\frac{\mathrm{d}e}{\mathrm{d}t} \\
\frac{\mathrm{d}^2 i_2}{\mathrm{d}t^2} &= \frac{1}{LC}i_1 - \frac{2}{LC}i_2 - \frac{R}{L}\frac{\mathrm{d}i_2}{\mathrm{d}t} \\
\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{1}{C}i_2
\end{aligned}$$

# Example 3 - Solution

$i_1$ and $i_2$ are each differentiated **twice**, and so generate **two** state variables **each**.

$v$ is differentiated **once**, generating **one** additional state variable.

Hence define five state variables $x_1, \ldots, x_5$:

$$x_1 \equiv i_1, \qquad x_2 \equiv \frac{\mathrm{d}i_1}{\mathrm{d}t}, \qquad x_3 \equiv i_2, \qquad x_4 \equiv \frac{\mathrm{d}i_2}{\mathrm{d}t}, \qquad x_5 \equiv v.$$

Differentiating:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\mathrm{d}i_1}{\mathrm{d}t}, \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{\mathrm{d}^2 i_1}{\mathrm{d}t^2}, \qquad \frac{\mathrm{d}x_3}{\mathrm{d}t} = \frac{\mathrm{d}i_2}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x_4}{\mathrm{d}t} = \frac{\mathrm{d}^2 i_2}{\mathrm{d}t^2}, \qquad \frac{\mathrm{d}x_5}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}t}$$

# Example 3 - Solution

Substitute in the original equations to obtain equations for $\frac{\mathrm{d}x_i}{\mathrm{d}t}$ in terms of $x_i$:

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2 \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= -\frac{1}{LC}x_1 + \frac{1}{LC}x_3 + \frac{1}{L}\frac{\mathrm{d}e}{\mathrm{d}t} \\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= x_4 \\
\frac{\mathrm{d}x_4}{\mathrm{d}t} &= \frac{1}{LC}x_1 - \frac{2}{LC}x_3 - \frac{R}{L}x_4 \\
\frac{\mathrm{d}x_5}{\mathrm{d}t} &= \frac{1}{C}x_3
\end{aligned}
$$

and we have $z = Lx_4$ as a measured output.

# Example 3 - Solution

Explicitly writing in all the coefficients:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 0x_1 + 1x_2 + 0\ x_3 + 0\ x_4 + 0x_5 + 0\frac{\mathrm{d}e}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\frac{1}{LC}x_1 + 0x_2 + \frac{1}{LC}x_3 + 0\ x_4 + 0x_5 + \frac{1}{L}\frac{\mathrm{d}e}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = 0x_1 + 0x_2 + 0\ x_3 + 1\ x_4 + 0x_5 + 0\frac{\mathrm{d}e}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x_4}{\mathrm{d}t} = \frac{1}{LC}x_1 + 0x_2 - \frac{2}{LC}x_3 - \frac{R}{L}x_4 + 0x_5 + 0\frac{\mathrm{d}e}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x_5}{\mathrm{d}t} = 0x_1 + 0x_2 + \frac{1}{C}\ x_3 + 0\ x_4 + 0x_5 + 0\frac{\mathrm{d}e}{\mathrm{d}t}$$

# Example 3 - Solution

The state variable description is therefore:

$$\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}} + B\underline{\mathbf{u}}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{-1}{LC} & 0 & \frac{1}{LC} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{LC} & 0 & \frac{-2}{LC} & \frac{-R}{L} & 0 \\ 0 & 0 & \frac{1}{C} & 0 & 0 \end{pmatrix}, \quad \underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{\mathbf{u}} = \left(\frac{1}{L}\frac{\mathrm{d}e}{\mathrm{d}t}\right)$$

# Example 3 - Solution

There is one true output $z = Lx_4$ and the control input is $\frac{1}{L}\frac{\mathrm{d}e}{\mathrm{d}t}$.

The output vector $\underline{\mathbf{y}}$ can encode both of these:

$$
\begin{aligned}
\underline{\mathbf{y}} &= \begin{pmatrix} z \\ \frac{1}{L}\frac{\mathrm{d}e}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} Lx_4 \\ \frac{1}{L}\frac{\mathrm{d}e}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} Lx_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{L}\frac{\mathrm{d}e}{\mathrm{d}t} \end{pmatrix} \\
&= \begin{pmatrix} 0x_1 + 0x_2 + 0x_3 + Lx_4 + 0x_5 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 \end{pmatrix} + \begin{pmatrix} 0 \times \frac{1}{L}\frac{\mathrm{d}e}{\mathrm{d}t} \\ \frac{1}{L}\frac{\mathrm{d}e}{\mathrm{d}t} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left( \frac{1}{L}\frac{\mathrm{d}e}{\mathrm{d}t} \right)
\end{aligned}
$$

# Example 3 - Solution

or

$$\underline{\mathbf{y}} = C\underline{\mathbf{x}} + D\underline{\mathbf{u}}$$

where

$$C = \begin{pmatrix} 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \underline{\mathbf{u}} = \left( \frac{1}{L} \frac{\mathrm{d}e}{\mathrm{d}t} \right)$$

# Summary

After today, you should be able to . . .

- Determine the appropriate number of state variables for a system.
- Obtain the state variable description.
- Obtain a suitable output vector if appropriate.
- Write down the eigenmodes for a system without external control.

# This Week

This week's lecture corresponds to Section 4.3 of the Course Notes.

Before this week's tutorial:

- Attempt Tutorial sheet 10

In the following lecture we will use these ideas to help us solve systems of ODEs.

# Extra Question: Tutorial Sheet 10, Question 1(a)

A control system with internal feedback has external control inputs $e_1(t)$ and $e_2(t)$. The output voltages $v_1(t)$ and $v_2(t)$ obey:

$$\frac{\mathrm{d}^2 v_1}{\mathrm{d}t^2} = 4v_1 - 5v_2 + 5e_1$$
$$\frac{\mathrm{d}v_2}{\mathrm{d}t} = 6v_1 - 6v_2 + 3\frac{\mathrm{d}v_1}{\mathrm{d}t} + 2e_2$$

It is possible to measure both inputs and $v_1$ only.

Obtain the state variable equations in the form $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}} + B\underline{\mathbf{u}}$ where $A$ is a $3 \times 3$ matrix, $\underline{\mathbf{x}}$ is a 3-dimensional vector of suitably-defined state variables, $B$ is a $3 \times 2$ matrix, and $\underline{\mathbf{u}}$ is a 2-dimensional vector of the external controls.

Also obtain a suitable output vector (encoding the true output and the control input) in the form $\underline{\mathbf{y}} = C\underline{\mathbf{x}} + D\underline{\mathbf{u}}$.