Lecture 11: Matrix Algebra (4/4)

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Further Mathematics, Signals and Systems

Lecture 11

Today we shall cover:

- Modal matrices.
- The diagonalisation process to determine the state transition matrix for an ODE system.
- How to obtain the solutions to the system from the state transition matrix and some initial conditions.

You will need to be able to:

- Obtain the state variable description of an ODE system.
- Calculate eigenvalues and eigenvectors of square matrices.
- Perform matrix multiplication, and invert 2×2 matrices.

Motivation

Consider this relatively simple first-order linear ODE:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = kx(t)$$

where k is a constant and the initial condition $x(0) = x_0$ is known.

Last year, you saw that the solution to this initial value problem is:

$$x(t) = x_0 \, \mathrm{e}^{kt}$$

(This could be obtained by Laplace transforms or by separation of variables.)

Theory - the State Transition Matrix

Now consider a system of *multiple* such first-order ODEs:

$$\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$$

Extending the previous result, the solution to this vector problem turns out to be:

$$\underline{\mathbf{x}}(t) = \mathrm{e}^{At}\,\underline{\mathbf{x}}(0)$$

where e^{At} is the **exponential matrix** or **state transition matrix**.

With some initial conditions $\underline{\mathbf{x}}(0)$, we can use this formula to predict the state variables $\underline{\mathbf{x}}(t)$ at any future time.

The state transition matrix is obtained using a **diagonalisation process** involving a modal matrix T of the matrix A.

Modal matrices

Modal matrices

Given an $n \times n$ matrix A, a **modal matrix** T is constructed column-by-column using the eigenvectors of A:

$$T = \left(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \ldots, \underline{\mathbf{e}}_n,\right)$$

where each column vector $\underline{\mathbf{e}}_i$ is the i^{th} eigenvector of A. The actual ordering of eigenvectors is not important so long as **the ordering always matches with the corresponding eigenvalues**.

There are infinitely many modal matrices, since the order of the eigenvectors is interchangable, and the eigenvectors themselves are not unique. Each column of a modal matrix of A could be scaled separately to give another matrix that is still a modal matrix of A.

Method: Obtaining the state transition matrix

Given a system of n linear first-order ODEs formulated as $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$, where A is a square $n \times n$ matrix:

1. Find eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $\underline{\mathbf{e}}_1, \ldots, \underline{\mathbf{e}}_n$ of A.
2. Construct the diagonal matrix of eigenvalues $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the $n \times n$ modal matrix T where the i^{th} column consists of the eigenvector of A corresponding to the eigenvalue in the i^{th} diagonal entry of D.

$$D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \quad \text{and} \quad T = \left(\underline{\mathbf{e}}_1, \ldots, \underline{\mathbf{e}}_n\right)$$

Method: Obtaining the state transition matrix

3. Construct the diagonal matrix of exponentials e^{Dt}

$$e^{Dt} = diag(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}) = \begin{pmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{pmatrix}$$

4. Perform two matrix multiplications to calculate e^{At}

State Transition Matrix:

$$e^{At} = T\, e^{Dt}\, T^{-1}$$

5. The solution is given by:

Solution of the state varaibles:

$$\underline{\mathbf{x}}(t) = e^{At}\, \underline{\mathbf{x}}(0)$$

Example 1

A simple continuous-time model of population dynamics for two species is given by:

$$\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}, \quad \text{where} \quad A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \quad \text{with initial conditions } \underline{\mathbf{x}}(0).$$

The eigenvalue and eigenvector pairs of A are:

$$\lambda_1 = 1, \quad \underline{\mathbf{b}}_1 = \alpha \begin{pmatrix} 1 \\ -2 \end{pmatrix}; \qquad \text{and} \qquad \lambda_2 = 6, \quad \underline{\mathbf{b}}_2 = \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Obtain the solution of the state variables.

Example 1 - Solution

Take $D = diag(1, 6)$, then a modal matrix of A is: $T = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$$e^{Dt} = diag(e^t, e^{6t}) = \begin{pmatrix} e^t & 0 \\ 0 & e^{6t} \end{pmatrix}$$

Calculate the inverse of T:

$$T^{-1} = \frac{1}{(1)(1) - (2)(-2)} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

Then performing matrix multiplication,

$$T\, e^{Dt} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{6t} \end{pmatrix} = \begin{pmatrix} e^t & 2\, e^{6t} \\ -2\, e^t & e^{6t} \end{pmatrix}$$

Example 1 - Solution

Second matrix multiplication:

$$
\begin{aligned}
e^{At} &= (T\, e^{Dt})\, T^{-1} \\
&= \frac{1}{5} \begin{pmatrix} e^{t} & 2\, e^{6t} \\ -2\, e^{t} & e^{6t} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} \left(e^{t} + 4\, e^{6t} \right) & \left(-2\, e^{t} + 2\, e^{6t} \right) \\ \left(-2\, e^{t} + 2\, e^{6t} \right) & \left(4\, e^{t} + e^{6t} \right) \end{pmatrix}
\end{aligned}
$$

Example 1 - Solution

Then $\underline{\mathbf{x}}(t) = e^{At}\,\underline{\mathbf{x}}(0)$, and so:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} \left(e^{t} + 4e^{6t}\right) & \left(-2e^{t} + 2e^{6t}\right) \\ \left(-2e^{t} + 2e^{6t}\right) & \left(4e^{t} + e^{6t}\right) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

Formula for each state variable (in terms of initial conditions):

$$\begin{aligned} x_1(t) &= \frac{1}{5}\left\{\left(e^{t} + 4e^{6t}\right)x_1(0) + \left(-2e^{t} + 2e^{6t}\right)x_2(0)\right\} \\ x_2(t) &= \frac{1}{5}\left\{\left(-2e^{t} + 2e^{6t}\right)x_1(0) + \left(4e^{t} + e^{6t}\right)x_2(0)\right\} \end{aligned}$$

Example 2

An electronic control system is described by the following set of state variable equations:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2, \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_3, \qquad \frac{\mathrm{d}x_3}{\mathrm{d}t} = \frac{9}{2}x_1 - \frac{7}{2}x_3$$

We use the process of diagonalisation to obtain the state transition matrix and hence obtain solutions for x_1, x_2, x_3.

Example 2 - Solution

First write these three first-order ODEs in the form $\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}}$, where:

$$\underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{9}{2} & 0 & -\frac{7}{2} \end{pmatrix}$$

and we can obtain the eigenvalues and eigenvectors of A:

$$\lambda_1 = 1, \ \underline{\mathbf{b}}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \qquad \lambda_2 = -3, \ \underline{\mathbf{b}}_2 = \beta \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix};$$

$$\lambda_3 = -\frac{3}{2}, \ \underline{\mathbf{b}}_3 = \gamma \begin{pmatrix} 4 \\ -6 \\ 9 \end{pmatrix};$$

Example 2 - Solution

Hence, define a modal matrix of A and calculate its inverse:

$$T = \begin{pmatrix} 1 & 1 & 4 \\ 1 & -3 & -6 \\ 1 & 9 & 9 \end{pmatrix} \implies T^{-1} = \frac{1}{60}\begin{pmatrix} 27 & 27 & 6 \\ -15 & 5 & 10 \\ 12 & -8 & -4 \end{pmatrix}$$

We also define the diagonal matrices:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3/2 \end{pmatrix} \quad \text{and} \quad e^{Dt} = \begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-3t/2} \end{pmatrix}$$

Example 2 - Solution

Twice performing matrix multiplication,

$$
\begin{aligned}
e^{At} &= T\, e^{Dt}\, T^{-1} \\
&= \frac{1}{60}\begin{pmatrix} 1 & 1 & 4 \\ 1 & -3 & -6 \\ 1 & 9 & 9 \end{pmatrix}\begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-3t/2} \end{pmatrix}\begin{pmatrix} 27 & 27 & 6 \\ -15 & 5 & 10 \\ 12 & -8 & -4 \end{pmatrix}
\end{aligned}
$$

$$
= \frac{1}{60}\begin{bmatrix} \left(27\,e^{t} - 15\,e^{-3t} + 48\,e^{-3t/2}\right) & \left(27\,e^{t} + 5\,e^{-3t} - 32\,e^{-3t/2}\right) & \left(6\,e^{t} + 10\,e^{-3t} - 16\,e^{-3t/2}\right) \\ \left(27\,e^{t} + 45\,e^{-3t} - 72\,e^{-3t/2}\right) & \left(27\,e^{t} - 15\,e^{-3t} + 48\,e^{-3t/2}\right) & \left(6\,e^{t} - 30\,e^{-3t} + 24\,e^{-3t/2}\right) \\ \left(27\,e^{t} - 135\,e^{-3t} + 108\,e^{-3t/2}\right) & \left(27\,e^{t} + 45\,e^{-3t} - 72\,e^{-3t/2}\right) & \left(6\,e^{t} + 90\,e^{-3t} - 36\,e^{-3t/2}\right) \end{bmatrix}
$$

Example 2 - Solution

Then the solution is $\underline{\mathbf{x}}(t) = \mathrm{e}^{At}\,\underline{\mathbf{x}}(0)$, where $\underline{\mathbf{x}}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}$

Hence:

$$x_1(t) = \frac{1}{60}\Big\{ \left(27\,\mathrm{e}^{t} - 15\,\mathrm{e}^{-3t} + 48\,\mathrm{e}^{-3t/2}\right) x_1(0) + \left(27\,\mathrm{e}^{t} + 5\,\mathrm{e}^{-3t} - 32\,\mathrm{e}^{-3t/2}\right) x_2(0) + \left(6\,\mathrm{e}^{t} + 10\,\mathrm{e}^{-3t} - 16\,\mathrm{e}^{-3t/2}\right) x_3(0) \Big\}$$

Example 2 - Solution

and similarly:

$$x_2(t) = \frac{1}{60}\Big\{ \big(27\,e^{t} + 45\,e^{-3t} - 72\,e^{-3t/2}\big)x_1(0) + \big(27\,e^{t} - 15\,e^{-3t} + 48\,e^{-3t/2}\big)x_2(0) + \big(6\,e^{t} - 30\,e^{-3t} + 24\,e^{-3t/2}\big)x_3(0)\Big\}$$

$$x_3(t) = \frac{1}{60}\Big\{ \big(27\,e^{t} - 135\,e^{-3t} + 108\,e^{-3t/2}\big)x_1(0) + \big(27\,e^{t} + 45\,e^{-3t} - 72\,e^{-3t/2}\big)x_2(0) + \big(6\,e^{t} + 90\,e^{-3t} - 36\,e^{-3t/2}\big)x_3(0)\Big\}$$

Summary

After today, you should be able to . . .

- Explain what a modal matrix is, and be able to construct one from the eigenvectors of a matrix.
- Use the diagonalisation method to obtain the state transition matrix for a system of the form $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$.
- Determine solutions to the state variables from the state transition matrix and some initial conditions.
- Explain how eigenvalues and eigenvectors have helped us to predict the behaviour of a circuit from its equations.

This Week

This week's lecture corresponds to Section 4.4 of the Course Notes.

Before this week's tutorial:

- Attempt Tutorial sheet 11

Extra Question - Tutorial Sheet 11, Q1:

The state variable description of a certain electronic system is $\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}}$, where:

$$\underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}$$

Use the diagonalisation method to determine the state transition matrix.

Hence obtain solutions for $x_1(t)$ and $x_2(t)$.