Further Mathematics, Signals and Systems

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Contents

1 Introduction and Instructions to Students

1.1 Content

This module consists of three main topics in mathematics, that all have some application to analysing electronic circuits:

- Laplace transforms. These were introduced in the first year engineering mathematics course.
- Fourier analysis.
- Matrix algebra, eigenvalues and eigenvectors.

Everything you need to know for assessment is included within the lecture slides, this set of lecture notes, and the tutorial sheets. However, we will not have time to discuss everything fully in the actual lecture (e.g. working through every example) as we will use the lectures to carefully understand core concepts. It is both assumed and essential that students work through the notes *and especially the tutorial sheets* outside of lectures, and it is your responsibility to make sure you understand everything. Try to work through the solutions yourself, and then seek help if you cannot understand the solution.

1.2 Extra reading

Any "engineering mathematics" textbook will cover most of the topics that we will study in at least as much detail. Most will also include sections on partial fractions and any other prerequisite mathematical knowledge. Suitable examples that may help you include:

- *Engineering Mathematics* by Stroud
- *Advanced Engineering Mathematics* by Kreyszig
- *Modern Engineering Mathematics* by James
- *Advanced Modern Engineering Mathematics* by James
- *Mathematical Methods for Physics and Engineering* by Riley, Hobson and Bence
- *Higher Engineering Mathematics* by Bird
- *National Engineering Mathematics* by Yates

In particular, Glyn James' *Advanced Modern Engineering Mathematics* very comprehensively explains the ideas studied in this course. I would recommend that you look at this one if you would like more information on a topic.

1.3 So what should I do now?

This is quite a lot to take in. To break it down, the best thing you can do each week is:

1. Attend the lecture, and take part in the exercises when prompted.
2. At home, read through the lecture slides again.
3. Then attempt the questions marked "Essential" in that week's Tutorial Sheet (I will have highlighted which sheet you should work on at the end of the lecture).
4. If you are finding the questions difficult, look at the corresponding section in these course notes to find some additional worked examples to follow along.
5. Then attend your assigned tutorial. Compare your solutions with other students, ask the lecturer for help, and perhaps present some of your solutions to the class if you would like. I will have fully-worked solutions available at the tutorial.
6. If you still aren't sure, **ask**! I want you to understand this material and succeed!

For revision at the end of the module, there is a revision checklist at the back of these course notes.

2 Laplace Transforms

This technique is named after Pierre-Simon Laplace (1749-1827), a French mathematician. He was mainly concerned with problems in astronomy and statistics, such as celestial mechanics (planetary motion) which was a scientific problem of very great interest in the 18th century. Laplace's interests also included the concept of causal determinism, publishing one of the first papers on the subject, which resulted in "Laplace's Demon" being attributed to his ideas. He also taught Napoleon at the military academy, who later temporarily named him Minister of the Interior. The techniques we will use in this course were mostly developed by Oliver Heaviside, a 19th-20th century English electrical engineer who was seeking a systematic method of solving ODEs with constant coefficients.

You should remember taking Laplace transforms of simple functions in the first year Engineering Mathematics course. As before, a formulae sheet is provided with the table of standard transforms.

The Laplace transform has many uses, but one of the most common is as a tool for solving sets of ordinary differential equations (ODEs) which describe linear systems, and we will learn how to do this later in the course.

2.1 Revision: Introduction to Laplace Transforms

2.1.1 Definition of Laplace Transform

Let f be a function of time, t, that is zero for $t < 0$.

> The Laplace transform of $f(t)$ is denoted $\mathcal{L}\{f(t)\}$ or $\bar{f}(s)$.

The Laplace transform is defined (i.e. we can talk about it) only for $t > 0$. It is defined by the integral:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

where e^{-st} is called the kernel of the transformation. When we take the Laplace transform of $f(t)$ we shift our interest from the time-domain to the s-domain (s is sometimes called the complex frequency), where certain problems become easier to solve. In particular, problems involving differential equations can be transformed into something much easier to manipulate. We can then use the inverse Laplace transform to move back to the time-domain with the results that we wanted.

2.1.2 Examples from First Principles

Example 2.1. *Derive the Laplace transform of*

$$f(t) = \begin{cases} 0 & \textit{if } t < 0; \\ 1 & \textit{if } t > 0. \end{cases}$$

Then,

$$\begin{aligned} \bar{f}(s) &= \mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)dt \\ &= \int_0^\infty \mathrm{e}^{-st}\, 1dt = \left[\frac{-1}{s}\,\mathrm{e}^{-st}\right]_0^\infty \\ &= \left\{\frac{-1}{s}0\right\} - \left\{\frac{-1}{s}1\right\} \quad \textit{provided} \quad s > 0 \end{aligned}$$

Hence,

$$\bar{f}(s) = \frac{1}{s} \quad \textit{for} \quad s > 0$$

Note that the condition $s > 0$ is required, as the exponential only e^{-st} only takes a finite value (zero) as $t \to \infty$ if it is a decaying exponential. That is, if $-s < 0$.

In practice, when determining Laplace transforms, the condition on s for the transform to be valid can usually be found by looking for any occurences of $(s - \alpha)$ in the denominator of the transform, and then requiring $s > \alpha$.

Example 2.2. *Derive the Laplace transform of*

$$f(t) = \begin{cases} 0 & \text{for } t < 0; \\ \mathrm{e}^{at} & \text{for } t > 0. \end{cases}$$

$$\begin{aligned} \bar{f}(s) &= \mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t) dt \\ &= \int_0^\infty \mathrm{e}^{-st}\, \mathrm{e}^{at}\, dt = \int_0^\infty \mathrm{e}^{-(s-a)t}\, dt \\ &= \left[\frac{-1}{s-a} \mathrm{e}^{-(s-a)t} \right]_0^\infty = \left\{ \frac{-1}{s-a} 0 \right\} - \left\{ \frac{-1}{s-a} 1 \right\} \quad \text{provided} \quad s > a \end{aligned}$$

Hence,

$$\bar{f}(s) = \frac{1}{s-a} \quad \text{for} \quad s > a$$

In practice, you will be provided with a table of standard Laplace transforms, and will use those as tools for solving the more complicated problems that we will face. As an exercise, use the tables to obtain the Laplace transforms of the previosu two examples.

2.1.3 The Linearity Property

For constants a and b, and time-dependent functions $f(t)$ and $g(t)$:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

This is called the "linearity" of Laplace transforms, and essentially says that if I want to take the Laplace transform of two functions that are added together, I can just take the Laplace transform of each and then add them together. And if I want to take the tranform of a function that is multipled **by a constant**, then I can take the transform of my original function and just multiply it by that constant.

It is very important to realise that this is only for constants. It is *not* true that the

Laplace transform of $f(t) \times g(t)$ is the same as the transform of $f(t)$ multiplied by the transform of $g(t)$!

We can use linearity to obtain the Laplace transform of a complicated-looking function if it is just a linear combination of terms that we already know the transforms of.

Example 2.3. *Using tables, determine the Laplace transform of*

$$x(t) = \begin{cases} 0 & \textit{for } t < 0; \\ 3t - 2t\,\mathrm{e}^{-4t} & \textit{for } t > 0. \end{cases}$$

$$\begin{aligned} \bar{x}(s) &= \mathcal{L}\{3t - 2t\,\mathrm{e}^{-4t}\} = 3\mathcal{L}\{t\} - 2\mathcal{L}\{t\,\mathrm{e}^{-4t}\} \\ &= 3\left(\frac{1}{s^2}\right) - 2\left(\frac{1}{(s+4)^2}\right) = \frac{3(s+4)^2 - 2s^2}{s^2(s+4)^2} \\ &= \frac{s^2 + 24s + 48}{s^2(s+4)^2} \end{aligned}$$

for $s > 0$*.*

Example 2.4. *Using tables, determine the Laplace transform of*

$$v(t) = \begin{cases} 0 & \textit{for } t < 0; \\ 4\cos(2t) + 5\sin(2t) & \textit{for } t > 0. \end{cases}$$

$$\begin{aligned} \bar{v}(s) &= \mathcal{L}\{4\cos(2t) + 5\sin(2t)\} = 4\mathcal{L}\{\cos(2t)\} + 5\mathcal{L}\{\sin(2t)\} \\ &= 4\frac{s}{s^2 + 2^2} + 5\frac{2}{s^2 + 2^2} \\ &= \frac{4s + 10}{s^2 + 4} \end{aligned}$$

and this is valid for $s > 0$*.*

Example 2.5. *Using tables, determine the Laplace transform of*

$$v(t) = \begin{cases} 0 & \text{for } t < 0; \\ \frac{E}{T}\{t - T + T\,\mathrm{e}^{-t/T}\} & \text{for } t > 0 \end{cases}$$

where E and T are positive constants.

$$\begin{aligned}
\bar{v}(s) &= \mathcal{L}\{v(t)\} = \mathcal{L}\left\{\frac{E}{T}\{t - T + T\,\mathrm{e}^{-t/T}\}\right\} \\
&= \frac{E}{T}\big(\mathcal{L}\{t\} - T\mathcal{L}\{1\} + T\mathcal{L}\{\mathrm{e}^{-t/T}\}\big) = \frac{E}{T}\left\{\frac{1}{s^2} - \frac{T}{s} + T\frac{1}{s + 1/T}\right\} \\
&= \frac{E}{T}\left\{\frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1}\right\} = \frac{E}{T}\left\{\frac{(sT+1) - sT(sT+1) + s^2T^2}{s^2(sT+1)}\right\} \\
&= \frac{E}{T}\left\{\frac{sT + 1 - s^2T^2 - sT + s^2T^2}{s^2(sT+1)}\right\} = \frac{E}{T}\left\{\frac{1}{s^2(sT+1)}\right\}
\end{aligned}$$

So we have,

$$\bar{v}(s) = \frac{E}{Ts^2(sT+1)}$$

2.1.4 Inversion

When we have the Laplace transform of a function, say $\bar{f}(s)$, and wish to obtain the corresponding function in the time-domain (i.e $f(t)$), we need to use the inverse transform. In the simplest cases, we look up the relevant entry in the Laplace transform tables and work in the reverse direction.

Remember that the Laplace transform is only defined for functions which are zero when $t < 0$, so we have to give our final answer as a piecewise function - meaning that we separately write how the function behaves for $t > 0$ and $t < 0$. Later on we will see a nicer way to indicate this by using step functions.

Example 2.6. *Invert the Laplace transform*

$$\bar{u}(s) = \frac{1}{s+3}$$

The relevant entry is $\mathcal{L}^{-1}\left\{\frac{1}{s+\alpha}\right\} = \mathrm{e}^{-\alpha t}$ with $\alpha = 3$. Therefore,

$$u(t) = \begin{cases} 0 & \text{for } t < 0; \\ \mathrm{e}^{-3t} & \text{for } t > 0. \end{cases}$$

Example 2.7. *Invert the Laplace transform*

$$\bar{x}(s) = \frac{1}{s(s-5)}$$

The relevant entry is $\bar{f}(s) = \frac{\alpha}{s(s+\alpha)}$ *with* $\alpha = -5$. *Therefore,*

$$x(t) = \begin{cases} 0 & \text{for } t < 0; \\ -\frac{1}{5}(1 - \mathrm{e}^{5t}) & \text{for } t > 0. \end{cases} = \begin{cases} 0 & \text{for } t < 0; \\ \frac{1}{5}(\mathrm{e}^{5t} - 1) & \text{for } t > 0. \end{cases}$$

Example 2.8. *Invert the Laplace transform*

$$\bar{v}(s) = \frac{E}{sCR + 1} \qquad \text{where } E, C, R \text{ are positive constants.}$$

The relevant entry is $\mathcal{L}^{-1}\left\{\frac{1}{1+sT}\right\} = \frac{1}{T}\,\mathrm{e}^{-t/T}$ *with* $T = CR$ *and scaled by* E. *Hence,*

$$v(t) = E\left(\frac{1}{CR}\,\mathrm{e}^{-t/(CR)}\right) = \frac{E}{CR}\,\mathrm{e}^{-t/(CR)} \quad \text{for} \quad t > 0.$$

Alternatively, we could have performed some manipulation:

$$\bar{v}(s) = \frac{E}{CR}\left(\frac{1}{s + (1/CR)}\right)$$

and then used the entry for $\bar{f}(s) = \frac{1}{s+\alpha}$ *with* $\alpha = \frac{1}{CR}$.

More complicated examples may first involve using partial fractions or completing the square. In particular, if we are given an example which has a polynomial on the denominator, we will try to factorise it and then use partial fractions. If it is a quadratic function which cannot be factorised, then try completing the square. We will consider some examples of this next.

Example 2.9 (Completing the Square)**.** *Invert the Laplace transform*

$$\bar{y}(s) = \frac{2s + 7}{s^2 + 6s + 34}$$

The denominator $s^2 + 6s + 34$ *cannot be factorised neatly, so we complete the square and then separate the numerator:*

$$\bar{y}(s) = \frac{2s + 7}{(s+3)^2 + 5^2} = \frac{2s}{(s+3)^2 + 5^2} + \frac{7}{(s+3)^2 + 5^2}$$

Then we use the inverse transforms for $\frac{s}{(s+\alpha)^2+\omega^2}$ *and* $\frac{\omega}{(s+\alpha)^2+\omega^2}$ *with* $\alpha = 3$ *and* $\omega = 5$*, and scale them appropriately to obtain:*

$$\begin{aligned} y(t) &= 2\,\mathrm{e}^{-3t}\left\{\cos(5t) - \frac{3}{5}\sin(5t)\right\} + 7\frac{1}{5}\,\mathrm{e}^{-3t}\sin(5t) \\ &= \mathrm{e}^{-3t}\left\{2\cos(5t) + \frac{1}{5}\sin(5t)\right\} \quad \textit{for} \;\; t > 0 \end{aligned}$$

Alternatively, we could have put the transform in the form:

$$\bar{y}(s) = \frac{2(s+3)}{(s+3)^2+5^2} + \frac{1}{(s+3)^2+5^2}$$

and used the inverse transforms for $\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$ *and* $\frac{\omega}{(s+\alpha)^2+\omega^2}$ *to obtain the same result.*

Example 2.10. *Invert the Laplace transform*

$$\bar{v}(s) = \frac{s}{s^2+4s+13}$$

The denominator $s^2+4s+13$ *doesn't factorise, so completing the square:*

$$\bar{v}(s) = \frac{s}{(s+2)^2-2^2+13} = \frac{s}{(s+2)^2+3^2}$$

Then using the inverse transform $\mathcal{L}^{-1}\left\{\frac{s}{(s+\alpha)^2+\omega^2}\right\} = \mathrm{e}^{-\alpha t}\left(\cos(\omega t) - \frac{\alpha}{\omega}\sin(\omega t)\right)$ *with* $\alpha = 2$ *and* $\omega = 3$ *we obtain:*

$$v(t) = \mathrm{e}^{-2t}\left(\cos(3t) - \frac{2}{3}\sin(3t)\right) \quad \textit{for} \;\; t > 0$$

Example 2.11. *Invert the Laplace transform*

$$\bar{x}(s) = \frac{7}{s^2+2s+17}$$

The denominator $s^2+2s+17$ *doesn't have an integer factorisation, so completing the square:*

$$\bar{x}(s) = \frac{7}{(s+1)^2-1^2+17} = \frac{7}{4}\left(\frac{4}{(s+1)^2+4^2}\right)$$

Then using the inverse transform $\mathcal{L}^{-1}\left\{\frac{\omega}{(s+\alpha)^2+\omega^2}\right\} = \mathrm{e}^{-\alpha t}\sin(\omega t)$ *with* $\alpha = 1$ *and* $\omega = 4$ *we obtain:*

$$x(t) = \frac{7}{4}\,\mathrm{e}^{-t}\sin(4t) \quad \textit{for} \;\; t > 0$$

Example 2.12. *Invert the Laplace transform*

$$\bar{q}(s) = \frac{5}{(s+2)(s^2+3^2)}$$

Now, this appears to be similar in form to the entry

$$\frac{\alpha^2+\omega^2}{(s+\alpha)(s^2+\omega^2)}$$

with $\alpha = 2$ and $\omega = 3$, but some manipulation is required to put it exactly in that form.

$\alpha^2 + \omega^2 = 2^2 + 3^2 = 13$, but we have only 5 on the numerator, so:

$$\bar{q}(s) = \frac{5}{13}\left(\frac{13}{(s+2)(s^2+3^2)}\right) = \frac{5}{13}\left(\frac{2^2+3^2}{(s+2)(s^2+3^2)}\right)$$

Hence,

$$\begin{aligned} q(t) &= \frac{5}{13}\left\{ \mathrm{e}^{-2t} + \frac{2}{3}\sin(3t) - \cos(3t) \right\} \\ &= \frac{5}{39}\left\{ 3\,\mathrm{e}^{-2t} + 2\sin(3t) - 3\cos(3t) \right\} \quad \textit{for} \quad t > 0. \end{aligned}$$

For more difficult cases, we may be able to obtain the partial fractions expansion of the Laplace transform and then invert the individual partial fractions.

Example 2.13 (Partial Fractions)**.** *Invert the Laplace transform*

$$\bar{f}(s) = \frac{1}{s(sCR+1)}$$

where C, R are positive constants.

First we use use partial fractions:

$$\frac{1}{s(sCR+1)} = \frac{A}{s} + \frac{B}{sCR+1}$$

Cross-multiplying,

$$1 = A(sCR+1) + Bs$$

Choosing $s = 0$ reduces the equation to $A = 1$, and then choosing $s = -1/CR$, we obtain:

$$1 = \frac{-1}{CR}B \qquad \Longrightarrow \qquad B = -CR$$

Hence, the transformed function is equivalent to:

$$\bar{f}(s) = \frac{1}{s} - \frac{CR}{sCR+1}$$

Then we can obtain the inverse Laplace transform of each term:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{CR}{sCR+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{sCR+1}\right\} \\ &= 1 - CR\frac{1}{CR}\,\mathrm{e}^{-t/CR} \\ &= 1 - \mathrm{e}^{-t/CR} \end{aligned}$$

Example 2.14. *Invert the Laplace transform*

$$\bar{v}(s) = \frac{E(4sCR+3)}{s(sCR+2)(sCR+5)}$$

As usual, assume that C, R are positive constants and t is the time variable.

Using partial fractions,

$$\frac{(4sCR+3)}{s(sCR+2)(sCR+5)} = \frac{A}{s} + \frac{B}{sCR+2} + \frac{C}{sCR+5}$$

and we obtain $A = \frac{3}{10}$, $B = \frac{5CR}{6}$, and $C = \frac{-17CR}{15}$. Hence,

$$\bar{v}(s) = E\left\{\frac{3}{10s} + \frac{5CR}{6(sCR+2)} - \frac{17CR}{15(sCR+5)}\right\}$$

and so inverting each term gives:

$$\begin{aligned} v(t) &= E\left\{\frac{3}{10} + \frac{5}{6}\,\mathrm{e}^{-2t/(CR)} - \frac{17}{15}\,\mathrm{e}^{-5t/(CR)}\right\} \\ &= \frac{E}{30}\left\{9 + 25\,\mathrm{e}^{-2t/(CR)} - 34\,\mathrm{e}^{-5t/(CR)}\right\} \quad \textit{for} \quad t > 0. \end{aligned}$$

Example 2.15. *Invert the Laplace transform:*

$$\bar{g}(s) = \frac{E(sCR+1)}{s(sCR+2)^2}$$

Using partial fraction expansion:

$$\frac{E(sCR+1)}{s(sCR+2)^2} = \frac{A}{s} + \frac{B}{sCR+2} + \frac{D}{(sCR+2)^2}$$

$$E(sCR+1) = A(sCR+2)^2 + Bs(sCR+2) + Ds$$

Choosing $s = 0$ yields $E = 4A$, and so $A = E/4$. Setting $s = -2/CR$ results in $D = ECR/2$. We could choose another value of s and solve for B, or by equating the coefficients of s^2 find that $0 = C^2R^2A + CRB$ and so using our result for A obtain $B = (-ECR)/4$. Hence,

$$\bar{g}(s) = \left(\frac{E}{4}\right)\frac{1}{s} + \left(\frac{-E}{4}\right)\frac{1}{s+(2/CR)} + \left(\frac{E}{2CR}\right)\frac{1}{(s+(2/CR))^2}$$

and so inverting,

$$\begin{aligned} g(t) &= \frac{E}{4} - \frac{E}{4}\,\mathrm{e}^{-2t/CR} + \frac{E}{2CR}t\,\mathrm{e}^{-2t/CR} \\ &= \frac{E}{4CR}\left\{CR + (2t - CR)\,\mathrm{e}^{-2t/CR}\right\} \end{aligned}$$

for $t > 0$, and zero otherwise.

2.2 Computational Approach

In MATLAB, taking the Laplace transform of a function is straightforward. You need to have the Symbolic Math Toolbox installed. Then declare the symbolic variable (let's say t) and define the symbolic function f:

```
syms t;

f(t) = 5*t*exp(t);

laplace(f);
```

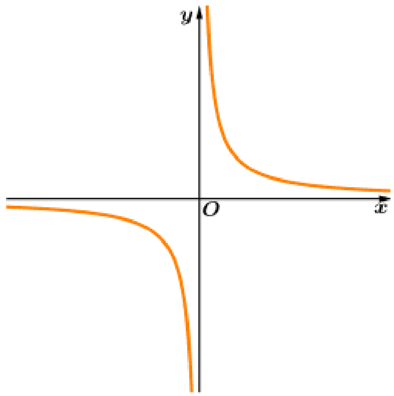
Similarly, to determine the inverse Laplace transform of a function $F(s)$:

```
syms s;

F(s) = 1/s^2;

ilaplace(F);
```

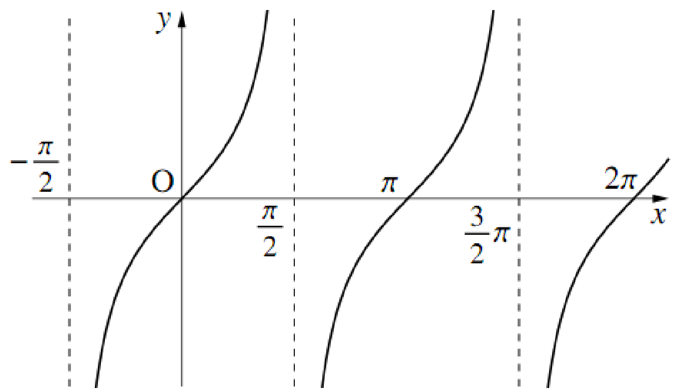
Use this to check your answers.

2.3 Discontinuous Functions

Generally speaking, a function $f(x)$ is discontinuous at a point x_0 if there is a break in the graph of the function at that point. For example, $y = \frac{1}{x}$ is discontinuous at $x = 0$, and $y = \tan(x)$ is discontinuous at infinitely many points ($\frac{\pi}{2} + n\pi$ for all integers n).



(a) $y = \frac{1}{x}$



(b) $y = \tan(x)$

Figure 1: Examples of functions with a discontinuity

An important class of functions with a discontinuity are called "step functions".

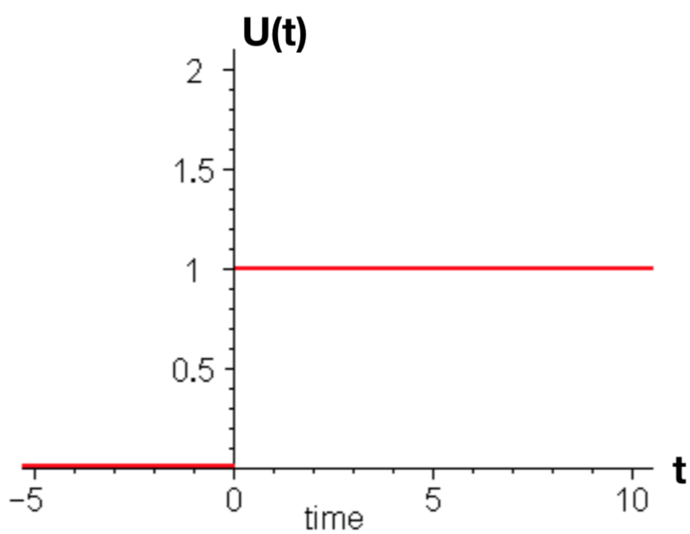


Figure 2: Heaviside's Unit function

In particular, the Unit Step Function, or Heaviside's unit function, $U(t)$ (although note that some texts may use $H(t)$), has a discontinuity at $t = 0$:

$$U(t) = \begin{cases} 0 & \text{if } t < 0; \\ 1 & \text{if } t > 0. \end{cases}$$

For our purposes, it will not be necessary to define $U(0)$, although some textbooks use $U(0) = 1$. Combining $U(t)$ with other functions acts as a "switching function" that switches on at $t = 0$. We will use it instead of explicitly refering to the domain of t.

Example 2.16. *For example, instead of writing:*

$$v(t) = \begin{cases} 0 & \textit{if } t < 0; \\ e^{-3t}(2cos(5t) + \frac{1}{5}sin(5t)) & \textit{if } t > 0. \end{cases}$$

we can now just use:

$$v(t) = e^{-3t}(2\cos(5t) + \frac{1}{5}\sin(5t))U(t)$$

to indicate that the function "switches on" at time $t = 0$.

Recall that we previously said that Laplace transforms are defined for functions that are zero for $t < 0$, and thus when obtaining an inverse Laplace transform we usually state "valid for $t > 0$". From here, instead of writing this condition, instead when we obtain the inverse transform of any function $\bar{f}(s)$ *without time-delay* (see later in this section!) we multiply the final answer $f(t)$ by $U(t)$ to indicate that it starts at $t = 0$.

Example 2.17. *Find the Laplace transform of $f(t) = 3tU(t)$*
This is the same as finding the Laplace transform of

$$f(t) = \begin{cases} 0 & \textit{if } t < 0; \\ 3t & \textit{if } t > 0. \end{cases}$$

And so just as before,

$$\bar{f}(s) = \mathcal{L}\{3tU(t)\} = \mathcal{L}\{3t\} = 3\frac{1}{s^2}$$

valid for $s > 0$.

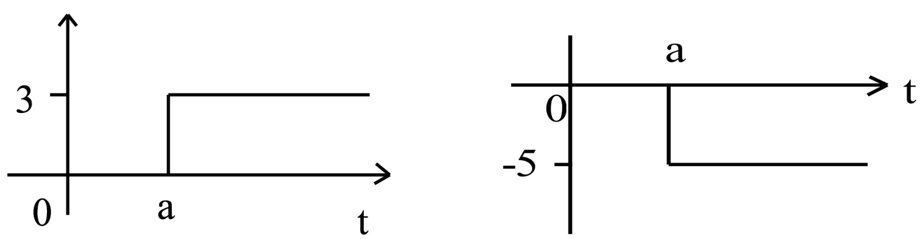
We can generalise the step function to switch on at time $t = a$:

$$U(t-a) = \begin{cases} 0 & \text{if } t < a; \\ 1 & \text{if } t > a. \end{cases}$$

and the usual function transformations apply. For example,

$$3U(t-a) = \begin{cases} 0 & \text{if } t < a; \\ 3 & \text{if } t > a. \end{cases}$$

This would represent a step up of size 3 at time a, while $-5U(t-a)$ indicates a step down of 5 at time $t = a$.



(a) $y = 3U(t-a)$ (b) $y = -5U(t-a)$

2.3.1 Applications of $U(t)$

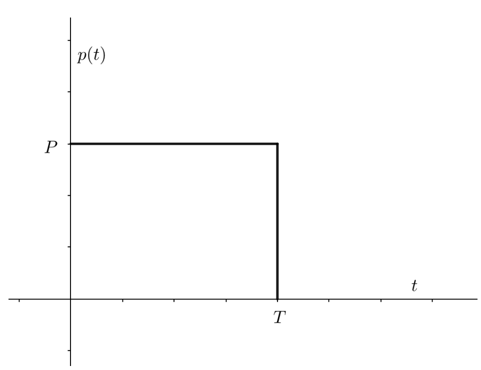
1. A constant e.m.f. of value E is applied to a circuit starting at time $t = 0$:

   $$e(t) = EU(t)$$

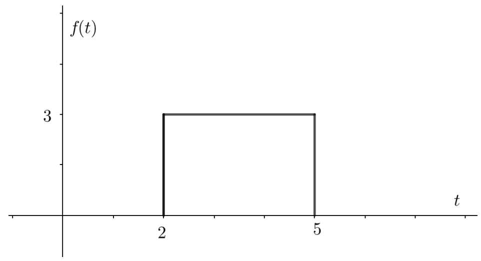
2. We can combine multiple step functions to switch a function on for an interval, and then off again. For example, a constant force P is applied for the first T time units:

   $$p(t) = PU(t) - PU(t-T) = P(U(t) - U(t-T))$$

   Step up at $t = 0$ and then step down after time T.



3. Similarly, $f(t) = 3U(t-2) - 3U(t-5) = 3\Big(U(t-2) - U(t-5)\Big)$ describes a signal of constant strength 3 that begins at time $t = 2$ and ends at time $t = 5$.



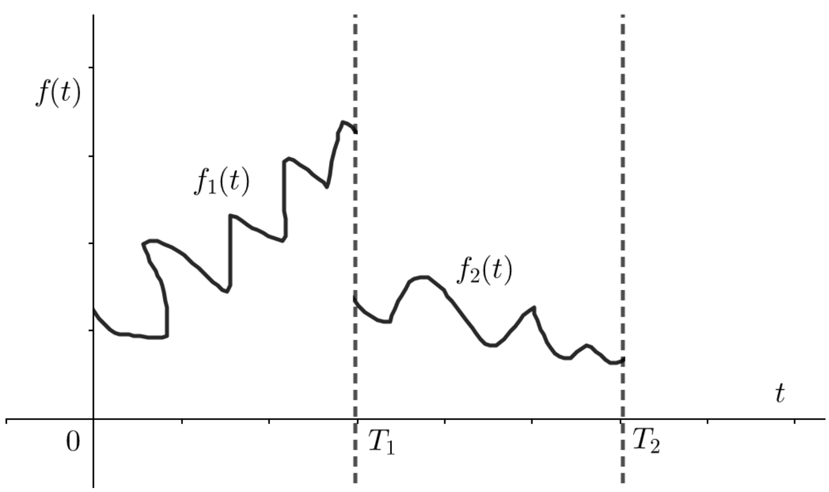
4. A capacitor is charged from the discharged state through a resistor for a time T by a constant DC voltage supply:

$$i(t) = \frac{E}{R}e^{-t/CR}(U(t) - U(t-T))$$

5. It doesn't just have to be constant functions. We can combine step functions to switch on and off functions with any general behaviour. This will be used later in the course to describe piecewise functions (functions that obey certain behaviour for a period, and then change to a different kind of behaviour).

   Consider a function f that is behaves like f_1 for the interval $[0, T_1]$, but then changes to act like f_2 during the next interval $[T_1, T_2]$ before switching off. This is described by:

$$f(t) = f_1(t)\Big(U(t) - U(t-T_1)\Big) + f_2(t)\Big(U(t-T_1) - U(t-T_2)\Big)$$



2.4 Laplace Transforms of Functions with a Delay

Note from the Laplace transform tables that there are transforms given for step functions. In particular,

$$\mathcal{L}\{U(t)\} = \frac{1}{s} \quad \text{and} \quad \mathcal{L}\{U(t-T)\} = \frac{\mathrm{e}^{-sT}}{s}$$

So for example, $\mathcal{L}\{5U(t-3)\} = 5\frac{1}{s}\,\mathrm{e}^{-3s}$.

Next, we will learn how to take Laplace transforms of more complicated functions involving a time-shifted step function.

In order to take the Laplace transform of a function with a delay, we need to re-write it into a specific "delay form". This means manipulating it into a function of the time with delay, usually $g(t-T)$, so that a substitution can be made between real time and a time-delayed variable. This is best explained by an example:

$$\begin{aligned} f(t) &= tU(t-2) && \text{NOT delay form.} \\ &= ((t-2)+2)U(t-2) && \text{In delay form.} \end{aligned}$$

Once we have put the delayed equation into Delay Form, there is a special transform for that which we can use, known as the Heaviside or **delay theorem**:

$$\mathcal{L}\{g(t-T)U(t-T)\} = e^{-sT}\mathcal{L}\{g(t)\}$$

If you have an equation with a time delay, you **must** ensure that it is in Delay Form before this transformation can be applied.

This theorem is the first really substantial new piece of content in this course. **You must know and be able to use the delay theorem!**

As an immediate consequence of this theorem, we can derive the theorem $\mathcal{L}\{U(t-T)\} = \frac{e^{-sT}}{s}$ above.

2.4.1 Method for finding Laplace Transforms of functions with delay

Let us work through a procedure for an example, $f(t) = 3tU(t-2)$:

1. First, we look at the step function $U(t-2)$ to identify the delay is of value 2.

2. Then we consider the other part of the term (i.e. *what is multiplied by this step function*), which in this case is $3t$. Let's call this part the function $g(t-2)$. Thus,

   $$g(t-2) = 3t$$

   If all occurences of t in this part are written explicitly as $t-2$ then we already have delay form and can skip the Step 3.

   Note: you can call it $h(t-2)$, $v(t-2)$, etc. if you wish. All that matters is that we do not choose a name that is already being used in the question, which in this case would just be f and U.

3. In this case, $g(t-2)$ is *not* already written in delay form. However, we can remedy this by replacing all the instances of t with $\big((t-2)+2\big)$. Hence in this case we get:

   $$g(t-2) = 3\big((t-2)+2\big) = 3(t-2)+6$$

   This is now in Delay Form!

4. Once $g(t-2)$ is in delay form, we obtain $g(t)$ by replacing all instances of $(t-2)$ with t. That is, we are changing the input for the function g, and so it performs the same set of transformations to t instead of to $t-2$. Thus we find:

   $$g(t) = 3t+6$$

5. Finally, the Delay Theorem says that we take the Laplace transform of $g(t)$ (that we have finally obtained in the previous step), and multiply by $\mathrm{e}^{-s\times\text{delay}}$. Thus we have:

$$
\begin{aligned}
\bar{f}(s) &= \mathcal{L}\big\{3tU(t-2)\big\} && \text{This is the problem.}\\
&= \mathcal{L}\big\{g(t-2)U(t-2)\big\} && \text{Naming the other part in Step 2.}\\
&= \mathcal{L}\big\{g(t)\big\}\,\mathrm{e}^{-2s} && \text{According to the delay theorem.}\\
&= \mathcal{L}\big\{3t+6\big\}\,\mathrm{e}^{-2s} && \text{Determined in Step 4.}\\
&= \left(\frac{3}{s^2}+\frac{6}{s}\right)\mathrm{e}^{-2s} && \text{Using the transform tables.}\\
&= \frac{3}{s^2}\big(1+2s\big)\,\mathrm{e}^{-2s} && \text{Simplifying.}
\end{aligned}
$$

2.4.2 Examples

Example 2.18. *Find the Laplace Transform of:*

$$f(t) = 3t^2 U(t-1)$$

First we identify from the step function $U(t-1)$ that this function has a delay of value 1.

Thus we name the other parts of the term $g(t-1)$:

$$g(t-1) = 3t^2$$

This is not in delay form, as there is a t that is not written as $t-1$. So we will replace this t with $\big((t-1)+1\big)$:

$$\begin{aligned} g(t-1) &= 3\big((t-1)+1\big)^2 \\ &= 3\Big((t-1)^2 + 2(t-1) + 1\Big) \\ &= 3(t-1)^2 + 6(t-1) + 3 \end{aligned}$$

This is now in delay form.

Now, the Delay Theorem says that I need to take the transform of $g(t)$, not *$g(t-1)$. So I replace $t-1$ with t to obtain $g(t)$:*

$$g(t) = 3t^2 + 6t + 3$$

(The purpose of delay form is to make this step a straight swap!)

So now we take Laplace Transforms of $g(t)$:

$$\begin{aligned} \mathcal{L}\big\{g(t)\big\} &= \mathcal{L}\big\{3t^2 + 6t + 3\big\} = 3\mathcal{L}\big\{t^2\big\} + 6\mathcal{L}\big\{t\big\} + 3\mathcal{L}\big\{1\big\} \\ &= 3\left(\frac{2!}{s^3}\right) + 6\left(\frac{1}{s^2}\right) + 3\left(\frac{1}{s}\right) = \frac{3}{s^3}\big(2 + 2s + s^2\big) \end{aligned}$$

Then the Delay Theorem tells us that the final answer is $\bar{f}(s) = \mathcal{L}\big\{g(t)\big\}\,\mathrm{e}^{-1s} = \mathcal{L}\big\{g(t)\big\}\,\mathrm{e}^{-s}$ since the delay has value of 1:

$$\bar{f}(s) = \frac{3}{s^3}\big(2 + 2s + s^2\big)\,\mathrm{e}^{-s}$$

Example 2.19. *Given*

$$f(t) = (t-4)U(t-4)$$

This is already in delay form, so declare $g(t-4) = t-4$*, then* $g(t) = t$*. Since*

$$\bar{g}(s) = \mathcal{L}\{t\} = \frac{1}{s^2},$$

then by the delay theorem we have:

$$\bar{f}(s) = \mathcal{L}\{(t-4)U(t-4)\} = \mathcal{L}\Big\{t\Big\} \times \mathrm{e}^{-4s} = \frac{1}{s^2}\,\mathrm{e}^{-4s}$$

Example 2.20. *Find the Laplace transform of:*

$$f(t) = \mathrm{e}^{-a(t-5T)}\,U(t-5T).$$

From the step function $U(t-5T)$*, we can see that the delay is of size* $5T$*.*

Let's first declare $g(t-5T) = \mathrm{e}^{-a(t-5T)}$*. This is already in delay form, so we can immediately replace* $t-5T$ *with* t *and obtain* $g(t)$*:*

$$g(t) = \mathrm{e}^{-at}$$

Using the result $\mathcal{L}\{\mathrm{e}^{-at}\} = \frac{1}{s+a}$ *from the tables, then according to the delay theorem:*

$$\bar{f}(s) = \mathcal{L}\{\mathrm{e}^{-a(t-5T)}\,U(t-5T)\} = \mathcal{L}\Big\{\ \mathrm{e}^{-at}\ \Big\} \times \mathrm{e}^{-5Ts} = \frac{\mathrm{e}^{-5Ts}}{s+a}$$

These two examples were already given to us in delay form. However, the following examples are not yet in delay form and so will need to be manipulated first:

Example 2.21. *Given*

$$f(t) = (3t+1)U(t-2)$$

Then we restate the problem in the form $f(t) = g(t-2)U(t-2)$*, and so:*

$$g(t-2) = 3t+1$$

Since this involves a t and not explicitly $t-2$, this is not in delay form. Therefore, we rearrange:

$$g(t-2) = 3\big((t-2)+2\big) + 1 = 3(t-2) + 7$$

Using the delay theorem, we require $\bar{g}(s)$, which is the Laplace transform of $g(t)$:

$$\begin{aligned} g(t) &= 3t + 7 \\ \therefore \bar{g}(s) &= \mathcal{L}\big\{3t+7\big\} = 3\mathcal{L}\{t\} + 7\mathcal{L}\{1\} = \frac{3}{s^2} + \frac{7}{s} \end{aligned}$$

And the delay form tells us that the solution is $\bar{f}(s) = \bar{g}(s)\,\mathrm{e}^{-2s}$, hence:

$$\bar{f}(s) = \left(\frac{3}{s^2} + \frac{7}{s}\right)\mathrm{e}^{-2s} = \frac{\mathrm{e}^{-2s}}{s^2}(3+7s)$$

Example 2.22. *Obtain the Laplace transform of*

$$v(t) = E\,\mathrm{e}^{-t/CR}\,U(t-2T) \quad \textit{where} \quad E, C, R, T \ \textit{are constants.}$$

Writing in delay form:

$$v(t) = E\,\mathrm{e}^{-\big((t-2T)+2T\big)/CR}\,U(t-2T)$$

So $v(t) = f(t-2T)U(t-2T)$, where

$$f(t) = E\,\mathrm{e}^{-(t+2T)/CR} = E\,\mathrm{e}^{-2T/CR}\,\mathrm{e}^{-t/CR}$$

Obtaining the Laplace transform of this:

$$\bar{f}(s) = E\,\mathrm{e}^{-2T/CR}\,\frac{1}{s+1/CR} = ECR\,\mathrm{e}^{-2T/CR}\,\frac{1}{sCR+1}$$

and so we have

$$\bar{v}(s) = \frac{ECR}{sCR+1}\,\mathrm{e}^{-2T/CR}\,\mathrm{e}^{-2Ts} = \frac{ECR}{sCR+1}\,\mathrm{e}^{-2T(s+1/CR)}$$

Note: Remember that if the step function is just $U(t)$ then there is no time-delay and we do not need to use the delay theorem!

2.4.3 Example with Multiple Delays

Example 2.23. *Find the Laplace transform of the following function:*

$$f(t) = 5t\big(U(t-2) - 3U(t-5)\big)$$

Since this function contains multiple delays, we will need to expand the brackets and treat each delay separately - putting the bits multiplied by each step function into the corresponding delay form:

$$\begin{aligned} f(t) &= 5t\big(U(t-2) - 3U(t-5)\big) \\ &= 5tU(t-2) - 15tU(t-5) \\ &= 5\big((t-2)+2\big)U(t-2) - 15\big((t-5)+5\big)U(t-5) \end{aligned}$$

Now we are ready to take the Laplace Transform by applying the delay theorem twice:

$$\begin{aligned} \bar{f}(s) &= \mathcal{L}\Big\{\big((t-2)+2\big)U(t-2) - 15\big((t-5)+5\big)U(t-5)\Big\} \\ &= 5\mathcal{L}\Big\{\big((t-2)+2\big)U(t-2)\Big\} - 15\mathcal{L}\Big\{\big((t-5)+5\big)U(t-5)\Big\} \\ &= 5\mathcal{L}\big\{t+2\big\}\,\mathrm{e}^{-2s} - 15\mathcal{L}\big\{t+5\big\}\,\mathrm{e}^{-5s} \qquad \textit{by two applications of the delay theorem} \\ &= 5\left(\frac{1}{s^2}+\frac{2}{s}\right)\mathrm{e}^{-2s} - 15\left(\frac{1}{s^2}+\frac{5}{s}\right)\mathrm{e}^{-5s} \\ &= \frac{5}{s^2}\Big(\big(1+2s\big)\,\mathrm{e}^{-2s} - 3\big(1+5s\big)\,\mathrm{e}^{-5s}\Big) \end{aligned}$$

2.5 Inverting Laplace Transforms involving a Delay

If a term in a Laplace transformation includes a factor of the form e^{-sT} then you know that the inverse will involve a delay of T time units.
In this case we will effectively reverse the delay theorem and then apply it:

$$\mathcal{L}^{-1}\{e^{-sT}\bar{f}(s)\} = f(t-T)U(t-T).$$

2.5.1 Method

Let us follow a procedure to invert the following time-delayed function:

$$\bar{v}(s) = \frac{2\,\mathrm{e}^{-7s}}{s+3}$$

1. Identify a factor of e^{-sT}, where T is any constant (it could be a number like 3, 5, 97, etc. or it could be symbolic like T, $3T$, $40T$, $5kT$ etc.).This means that there will be a delay of time T.

   In this example the presence of e^{-7s} indicates that there will be a delay of 7.

2. Next we ask what is multiplied by the exponential e^{-sT}?
   In particular, we need to identify $\bar{f}(s)$, where:

   $$\bar{v}(s) = \bar{f}(s)e^{-sT}$$

   If there are multiple exponentials, and thus a variety of different time-delays, split the function up and consider each time-delay separately.

   In the case of the example $\bar{v}(s)$, we have:

   $$\bar{f}(s) = \frac{2}{s+3}$$

3. This is the function that we need to find the Inverse Laplace Transform of initially:

   $$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\}$$

   This step often involves using either partial fractions or completing the square.

   In this example,

   $$f(t) = \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = 2\,\mathrm{e}^{-3t} \qquad \text{from the transform tables.}$$

4. Change the variable from t to $t-T$, and so replace every occurrence of t in $f(t)$ with $t-T$. This gives us the function $f(t-T)$.

   Hence for this example, which has delay 7, we need to replace every occurence of t in $f(t)$ with $t-7$ and thus obtain $f(t-7)$:

   $$f(t-7) = 2\,\mathrm{e}^{-3(t-7)}$$

5. Finally, multiply by the step function $U(t-T)$, to give $v(t) = f(t-T)U(t-T)$ as in the formula above.

   So for the example we multiply $f(t-7)$ by $U(t-7)$ to obtain the final answer:

   $$v(t) = \mathcal{L}^{-1}\left\{\frac{2\,\mathrm{e}^{-7s}}{s+3}\right\} = 2\,\mathrm{e}^{-3(t-7)}\,U\big(t-7\big)$$

A summary of the method is as follows:

1. Identify a factor of e^{-sT}. This means that there will be a delay of T. Identify $\bar{f}(s)$, where $\bar{v}(s) = \bar{f}(s)e^{-sT}$.
2. Invert this part to obtain $f(t) = \mathcal{L}^{-1}\big\{\bar{f}(s)\big\}$.
3. Change the variable from t to $t-T$, and so replace every occurrence of t in $f(t)$ with $t-T$. This gives us $f(t-T)$.
4. Finally, multiply by the step function $U(t-T)$, to give $v(t) = f(t-T)U(t-T)$ as in the formula above.

2.5.2 Examples

Example 2.24. *Find the inverse Laplace Transform of:*

$$\bar{f}(s) = \frac{1}{s+2}\,\mathrm{e}^{-2sT}$$

First we identify from the exponential function involving s (i.e. the e^{-2sT}) that the final answer will involve a delay of $2T$ as that is what is multiplied by $-s$ in the power.

Thus we name the other part of the term (i.e. what is multiplied by this exponential) $\bar{g}(s)$:

$$\bar{g}(s) = \frac{1}{s+2}$$

and we obtain the inverse of this to obtain:

$$g(t) = \mathcal{L}^{-1}\{\bar{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \mathrm{e}^{-2t} \quad \textit{from the tables.}$$

However, the inverse delay theorem says that our solution will involve $g(t-2T)$*, rather than* $g(t)$*, so we find this by replacing* t *with* $t-2T$*:*

$$g(t-2T) = \mathrm{e}^{-2(t-2T)}$$

Finally, we need to multiply $g(t-2T)$ *by a step function with the same delay, so multiply by* $U(t-2T)$ *to obtain the final answer:*

$$f(t) = \mathrm{e}^{-2(t-2T)}\,U(t-2T)$$

Example 2.25. *Given*

$$\bar{v}(s) = \frac{s}{s^2+\omega^2}\,\mathrm{e}^{-3s},$$

then the factor e^{-3s} *means that* $v(t)$ *involves a delay of 3 units.*

Then we separately consider the other factor is $\bar{f}(s) = \frac{s}{s^2+\omega^2}$*, which we see from the tables inverts to* $f(t) = \cos(\omega t)$*. Hence,*

$$v(t) = \cos(\omega(t-3))U(t-3)$$

Example 2.26. *Given*

$$\bar{i}(s) = \frac{2\,\mathrm{e}^{-s}}{s+3},$$

then the factor e^{-s} *means that* $i(t)$ *involves a delay of 1.*
The other factor is $\bar{f}(s) = \frac{2}{s+3} = 2\left(\frac{1}{s+3}\right)$*, which inverts to* $f(t) = 2\,\mathrm{e}^{-3t}$*.*

Therefore we have,

$$i(t) = 2\,\mathrm{e}^{-3(t-1)}\,U(t-1)$$

Example 2.27. *Given*

$$\bar{q}(s) = \frac{Q\alpha\,\mathrm{e}^{-sT}}{s(s+\alpha)},$$

then the factor e^{-sT} *means that* $q(t)$ *involves a delay of* T.
The other factor is $\bar{f}(s) = \frac{Q\alpha}{s(s+\alpha)} = Q\left(\frac{\alpha}{s(s+\alpha)}\right)$, *which (either from the tables, or using partial fractions first) inverts to* $f(t) = Q\big(1 - \mathrm{e}^{-\alpha t}\big)$. *Therefore,*

$$q(t) = Q\big(1 - \mathrm{e}^{-\alpha(t-T)}\big)U(t-T)$$

Example 2.28. *Given*

$$\bar{v}(s) = \frac{E(s^2 + (sT+1)\omega^2)}{Ts^2(s^2+\omega^2)}\,\mathrm{e}^{-sT}$$

Then there will be a delay of time T *involved because of the term* e^{-sT}, *and the other factor is:*

$$\bar{f}(s) = \frac{E}{T}\left(\frac{s^2 + (sT+1)\omega^2}{s^2(s^2+\omega^2)}\right)$$

Using the methods of partial fractions (and ignoring the factor $\frac{E}{T}$ *for the time being),*

$$\frac{s^2 + (sT+1)\omega^2}{s^2(s^2+\omega^2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+\omega^2}$$

and so multiplying both sides by the denominator gives:

$$s^2 + (sT+1)\omega^2 = As(s^2+\omega^2) + B(s^2+\omega^2) + (Cs+D)s^2$$

Now we need to find A, B, C, D. *We could do this by choosing various values of* s *and obtaining simultaneous equations to solve, but in this particular case it is easier to use the method of equating the coefficients of the different powers of* s *(so consider* s^3, s^2, *etc.):*

$$\begin{aligned}
s^3:&\qquad 0 = A + C\\
s^2:&\qquad 1 = B + D\\
s^1:&\qquad T\omega^2 = A\omega^2 \implies A = T\\
s^0(i.e. constants):&\qquad \omega^2 = \omega^2 B \implies B = 1
\end{aligned}$$

Hence we have $A = T$, $B = 1$, $C = -T$ *and* $D = 0$. *Therefore,*

$$\bar{f}(s) = \frac{E}{T}\left\{\frac{T}{s} + \frac{1}{s^2} - \frac{sT}{s^2+\omega^2}\right\}$$

Inverting each of these terms, and bringing back the factor $\frac{E}{T}$*:*

$$f(t) = \frac{E}{T}\big\{T + t - T\cos(\omega t)\big\}$$

So using the inversion formula $\mathcal{L}^{-1}\{e^{-sT}\bar{f}(s)\} = f(t-T)U(t-T)$, *we substitute in* $t \to t - T$ *and multiply by the step function* $U(t-T)$ *to obtain the final answer:*

$$\begin{aligned} v(t) &= \frac{E}{T}\big\{T + (t-T) - T\cos\big(\omega(t-T)\big)\big\}U(t-T) \\ &= \frac{E}{T}\big\{t - T\cos\big(\omega(t-T)\big)\big\}U(t-T) \\ &= E\left\{\frac{t}{T} - \cos\big(\omega(t-T)\big)\right\}U(t-T) \end{aligned}$$

2.5.3 Examples with Multiple Delays

Example 2.29. *Determine the inverse Laplace transform of:*

$$\bar{f}(s) = \frac{3\,\mathrm{e}^{-2s} + 2s\,\mathrm{e}^{-3s}}{s^2}$$

Expanding this fraction,

$$\begin{aligned}\bar{f}(s) &= \frac{3}{s^2}\,\mathrm{e}^{-2s} + \frac{2s}{s^2}\,\mathrm{e}^{-3s} \\ &= 3\frac{1}{s^2}\,\mathrm{e}^{-2s} + 2\frac{1}{s}\,\mathrm{e}^{-3s}\end{aligned}$$

we can see that the two time-delay exponentials are multiplied by fundamentally different functions: $\frac{1}{s^2}$ *and* $\frac{1}{s}$*. Because of this, we will have to treat the two delays separately, and apply the inverse delay theorem twice.*

Let's name what is multiplied by the first exponential $\bar{g}_1(s)$*, and what is multiplied by the second exponential* $\bar{g}_2(s)$*:*

$$\bar{g}_1(s) = \frac{3}{s^2}, \qquad \bar{g}_2(s) = \frac{2}{s}$$

Then taking the inverse Laplace transform of each of these:

$$\begin{aligned}g_1(t) &= \mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 3t \\ g_2(t) &= \mathcal{L}^{-1}\left\{\frac{2}{s}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 2 \times 1 = 2\end{aligned}$$

And finally we apply the inverse delay theorem to each of the two delayed terms (replacing t *with* $t-2$ *or* $t-3$ *and multiplying by the corresponding step function):*

$$f(t) = g_1(t-2)U(t-2) + g_2(t-3)U(t-3)$$

So the solution is:

$$f(t) = 3(t-2)U(t-2) + 2U(t-3)$$

Example 2.30. *Determine the inverse Laplace transform of:*

$$\bar{f}(s) = \frac{3\,\mathrm{e}^{-2s} + \mathrm{e}^{-3s}}{s+1}$$

If we write this as

$$\bar{f}(s) = \frac{1}{s+1}\left(3\,\mathrm{e}^{-2s} + \mathrm{e}^{-3s}\right)$$

we can see that the two different time-delay exponentials are multiplied by effectively the same function in terms of s (the constant 3 does not change the form of what is multiplied by the first exponential).

This means that we can take a shortcut. Let's call the shared part of the term without the exponentials $\bar{g}(s)$:

$$\bar{g}(s) = \frac{1}{s+1}$$

So now the problem looks like:

$$\bar{f}(s) = 3\bar{g}(s)\,\mathrm{e}^{-2s} + \bar{g}(s)\,\mathrm{e}^{-3s}$$

Then the inverse Laplace transform of $\bar{g}(s)$ is:

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = \mathrm{e}^{-t}$$

Finally we treat the two different time delays separately, and apply the inverse delay theorem to each (replacing t with $t-2$ or $t-3$ and multiplying by the corresponding step function):

$$f(t) = 3\,\mathrm{e}^{-(t-2)}\,U(t-2) + \mathrm{e}^{-(t-3)}\,U(t-3)$$

2.6 Computational Approach

To manipulate functions involving a delay in MATLAB, we need to invoke the Heaviside command. For example, to obtain the transform of $f(t) = (3t+1)U(t) + 17t^2U(t-5T)$:

```
syms t T;

assume(T > 0);

H1=heaviside(t);

H2=heaviside(t-5*T);

laplace((3*t+1)*H1+17*t.^2*H2)
```

2.7 Solving Differential and Integral Equations using Laplace Transforms

We are building up to the objective of using Laplace transforms to solve systems of equations that could model an electronic circuit. The formulae for calculating potential differences relating to certain electronic components are described in the box.

Modelling linear electronic components:

- Capacitor with capacitance C and current i passing through it:

$$\text{Potential difference} = \frac{1}{C}\int_0^t i dt + \text{ initial p.d.}$$

- Resistor with resistance R and current i passing through it:

$$\text{Potential difference} = iR.$$

- Inductor with inductance L and current i passing through it:

$$\text{Potential difference} = \mathcal{L}\frac{di}{dt}$$

Several of these involve derivatives or integrals. Therefore, if we are analysing the equations that arise from circuits containing these components, we will need to know how to take the Laplace transforms of the derivatives or integrals of time-dependent variables.

First, let's consider how to take the Laplace transform of the first derivative of a function. From first principles:

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = \int_0^\infty e^{-st}\frac{dx}{dt}dt = [e^{-st}x(t)]_0^\infty + s\int_0^\infty e^{-st}x(t)dt = 0 - x(0) + s\bar{x}(s)$$

Here we use a integration by parts and assume that $e^{-st}x(t) \to 0$ as $s \to \infty$, which will certainly be true for any finite-valued function $x(t)$.

Therefore we have the formula,

$$\mathcal{L}\{\dot{x}\} = \mathcal{L}\left\{\frac{dx}{dt}\right\} = s\bar{x}(s) - x(0)$$

Repeated application of integration by parts, combined with a proof by induction, yields the more general formula for the Laplace transform of the nth derivative of a function:

$$\begin{aligned} \mathcal{L}\left\{\frac{d^n x}{dt^n}\right\} &= s^n \bar{x}(s) - s^{n-1}x(0) - s^{n-2}\left.\frac{dx}{dt}\right|_{t=0} - s^{n-3}\left.\frac{d^2x}{dt^2}\right|_{t=0} - \cdots - \left.\frac{d^{n-1}x}{dt^{n-1}}\right|_{t=0} \\ &= s^n \bar{x}(s) - \sum_{k=0}^{n-1} s^{(n-1)-k}\left.\frac{d^k x}{dt^k}\right|_{t=0} \end{aligned}$$

In particular, $n = 2$ gives the Laplace transform of a second-derivative of a function:

$$\mathcal{L}\left\{\frac{d^2 x}{dt^2}\right\} = s^2\bar{x}(s) - sx(0) - \dot{x}(0)$$

Now let's try to find the transform of the integral of a function. Let

$$z(t) = \int_0^t x(t)dt$$

Then clearly $z(0) = 0$ as it is an integral over zero range. By definition,

$$\frac{dz}{dt} = x(t)$$

so we can use the result for the Laplace transform of a derivative to obtain:

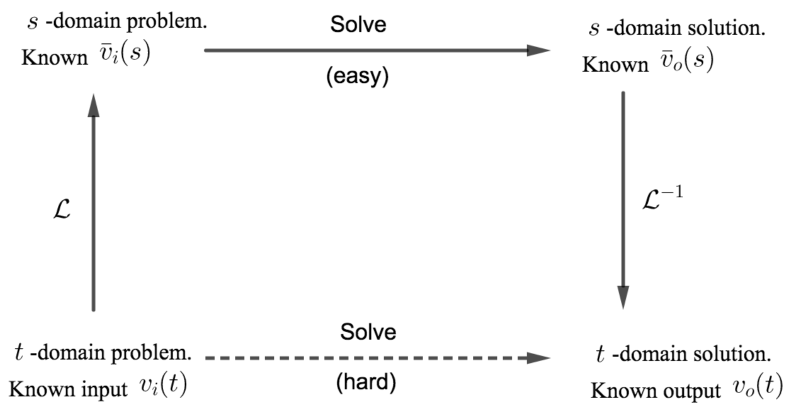
$$\bar{x}(s) = s\bar{z}(s) - z(0) = s\bar{z}(s) - 0$$

Therefore we have the formula:

$$\mathcal{L}\left\{\int_0^t x(t)dt\right\} = \frac{\bar{x}(s)}{s}$$

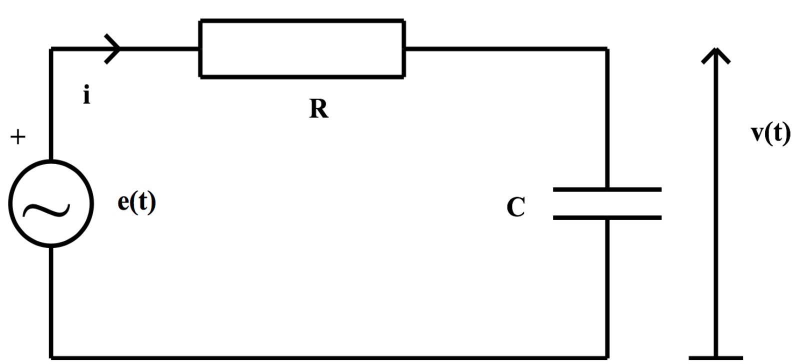
As usual, these results are given in the provided table of Laplace transforms.

When we first introduced Laplace transforms, a motivating factor was that we can move our problem into the complex frequency domain where they become much easier to solve. That is what we will be doing now. Consider the diagram:



We will begin to apply what we have learned about using Laplace transforms to analyse electronic circuits using this sort of procedure. In general, we will be given a problem in the time-domain that consists of a set of equations involving an input, an output, and possibly some-other time-dependent variables. The general idea displayed in the diagram is that first we will take Laplace transforms of each equation to obtain a statement of the problem in the s-domain. Then we will "solve" the problem in this domain by rearranging and substituting the equations in order to obtain an equation for the transform of the ouput solely in terms of the transform of the input (with all other s-dependent variables eliminated). Finally, we may be given a specific input function. We obtain its Laplace transform, subsititute this into the equation for the system that we have obtained in order to get the transform of the output corresponding to this specific input, and finally perform the inverse Laplace transform to obtain the output in the time-domain - this is the solution.

Example 2.31. *A series circuit consisting of an AC voltage source with voltage $e(t)$, a resistor of resistance R, and a capacitor with capacitance C has current i.*



Consider the potential differences in the circuit:

$$e(t) = Ri + \frac{1}{C}\int_0^t i dt,$$

where $v(t) = \frac{1}{C}\int_0^t idt$ *is the potential difference across the capacitor. Say we wish to find the output* $v(t)$*, given an input voltage* $e(t)$*. This can be done by taking Laplace transforms. First, we need to find* $\bar{i}(s)$*:*

$$\begin{aligned} e(t) &= Ri + \frac{1}{C}\int_0^t idt \\ \therefore \bar{e}(s) &= R\bar{i}(s) + \frac{1}{Cs}\bar{i}(s) \\ &= \bar{i}(s)\big(R + \frac{1}{Cs}\big) \\ \therefore \bar{i}(s) &= \frac{\bar{e}(s)}{R + \frac{1}{Cs}} \end{aligned}$$

Then we can find $\bar{v}(s)$ *in terms of both* $\bar{e}(s)$ *and* $\bar{i}(s)$*, and make a substitution to obtain* $\bar{v}(s)$ *solely in terms of* $\bar{e}(s)$ *(i.e., output voltage as a function of input voltage):*

$$\begin{aligned} e(t) &= Ri + v(t) \\ \therefore \bar{e}(s) &= R\bar{i}(s) + \bar{v}(s) \\ &= R\frac{\bar{e}(s)}{R + \frac{1}{Cs}} + \bar{v}(s). \\ \therefore \bar{v}(s) &= \bar{e}(s)\big(1 - \frac{R}{R + 1/Cs}\big) \\ &= \bar{e}(s)\frac{1}{1 + CRs}. \end{aligned}$$

Now, let us determine the output voltage $v(t)$ *given the following inputs:*
i) $e(t) = EU(t)$
ii) $e(t) = EU(t - T)$
iii) $e(t) = \frac{Et}{T}(U(t) - U(t - T))$.

Solutions:
i) $\bar{e}(s) = \frac{E}{s}$*, so we have:*

$$\bar{v}(s) = \frac{E}{s(1 + CRs)} = \frac{E/CR}{s(1 + 1/CR)}.$$

and inverting this using $\mathcal{L}\left\{\frac{\alpha}{s(1+\alpha)}\right\} = 1 - \mathrm{e}^{-\alpha t}$ *from the transform tables:*

$$v(t) = E(1 - e^{-t/CR})U(t).$$

ii) $\bar{e}(s) = \frac{E}{s}e^{-sT}$*, so we have:*

$$\bar{v}(s) = e^{-sT}\frac{E}{s(1 + CRs)}$$

Inverting using the result of (i) and the delay transform theorem,

$$v(t) = U(t - T)E(1 - e^{-(t-T)/CR}),$$

so we see that a delayed input $e(t)$ *has resulted in a delayed output* $v(t)$*.*
iii) $e(t) = \frac{E}{T}(tU(t) - ((t - T) + T)U(t - T))$*, and so*

$$\bar{e}(s) = \frac{E}{T}(\frac{1}{s^2} - e^{-sT}(\frac{1}{s^2} + \frac{T}{s})) = \frac{E}{Ts^2}(1 - (1 + sT)e^{-sT})$$

Therefore

$$\bar{v}(s) = \frac{E}{(1 + CRs)Ts^2}(1 - (1 + sT)e^{-sT}).$$

To carry out the inversion, we need to split the terms up, separating by the presence or absence of delay, and the order of s:

$$\bar{v}(s) = \frac{E}{T}\left\{\frac{1}{s^2(sCR + 1)} - \frac{e^{-sT}}{s^2(sCR + 1)} - \frac{Te^{-sT}}{s(sCR + 1)}\right\}$$

Consider the first term inside the bracket (remember E *and* T *are constants). Using partial fractions, we obtain:*

$$\frac{1}{s^2(sCR + 1)} = \frac{-CR}{s} + \frac{1}{s^2} + \frac{C^2R^2}{CRs + 1}$$

so applying the inversion to each fraction yields the following:

$$\mathcal{L}^{-1}\{\frac{1}{s^2(sCR + 1)}\} = U(t)(-CR + t + CRe^{-t/CR})$$

Since the second term is just a time-delayed version of the first, we can use the inverse delay transform to immediately obtain:

$$\mathcal{L}^{-1}\{e^{-sT}\frac{1}{s^2(sCR + 1)}\} = U(t - T)\big(-CR + (t - T) + CRe^{-(t-T)/CR}\big)$$

For the third term, applying partial fractions to the non-time delayed part gives us:

$$\frac{1}{s(sCR+1)} = \frac{1}{s} - \frac{CR}{CRs+1}$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s(sCR+1)}\right\} = tU(t) - \frac{CR}{CR}e^{-t/CR}U(t) = U(t)(1-e^{-t/CR})$$

and so the time-delayed version will give:

$$\mathcal{L}^{-1}\left\{\frac{Te^{-sT}}{s(sCR+1)}\right\} = TU(t-T)\left(1-e^{-(t-T)/CR}\right)$$

Putting it all together and gathering the step functions, we have the final result:

$$v(t) = \frac{E}{T}\left\{U(t)\big(t-CR+CRe^{-t/CR}\big) - U(t-T)\big(t-CR+(CR-T)e^{-(t-T)/CR}\big)\right\}.$$

Example 2.32. *Consider a system with input $f(t)$ and output $x(t)$, that satisfies the following:*

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = f(t)$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$. Using Laplace transforms, determine $x(t)$ when $f(t) = U(t-3)$.

Taking Laplace transforms of both sides of the equation:

$$\begin{aligned}
\bar{f}(s) &= \mathcal{L}\left\{\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x\right\} \\
&= \mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} + 3\mathcal{L}\left\{\frac{dx}{dt}\right\} + 2\mathcal{L}\{x\} \qquad \textit{by linearity,} \\
&= (s^2\bar{x}(s) - sx(0) - \dot{x}(0)) + 3(s\bar{x}(s) - x(0)) + 2\bar{x}(s) \\
&= s^2\bar{x}(s) - 0 - 1 + 3(s\bar{x}(s) - 0) + 2\bar{x}(s) \\
&= s^2\bar{x}(s) - 1 + 3s\bar{x}(s) + 2\bar{x}(s) \\
&= \bar{x}(s)(s^2 + 3s + 2) - 1
\end{aligned}$$

Rearranging, we obtain the transform of the output $\bar{x}(s)$ in terms of the transform of the input $\bar{f}(s)$:

$$\bar{x}(s) = \frac{\bar{f}(s)+1}{s^2+3s+2}$$

Then given the specific input $f(t)$:

$$f(t) = U(t-3) \quad \Longrightarrow \quad \bar{f}(s) = \frac{e^{-3s}}{s}$$

and we substitute this into our equation for $\bar{x}(s)$:

$$\bar{x}(s) = \frac{1+e^{-3s}/s}{s^2+3s+2} = \frac{e^{-3s}}{s(s+1)(s+2)} + \frac{1}{(s+1)(s+2)}$$

Finally, we want to invert this to obtain the output $x(t)$ in the time-domain. Now, there are two terms here that we will have to treat separately when inverting, because one of them has a time delay (because of the e^{-3s}) and the other does not. In particular,

$$x(t) = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s+1)(s+2)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$

For the first term, with time-delay, we need to find the inverse of the non-time-delayed part of that term. *That is, we need to find the inverse Laplace transform of $\frac{1}{s(s+1)(s+2)}$ for the first term. Using partial fractions expansion, we find that:*

$$\begin{aligned}\frac{1}{s(s+1)(s+2)} &= \frac{1}{2s} + \frac{-1}{s+1} + \frac{1}{2(s+2)} \\ \frac{1}{s(s+1)(s+2)} &= \frac{1}{s+1} + \frac{-1}{s+2}\end{aligned}$$

Hence,

$$\bar{x}(s) = e^{-3s}\left(\frac{1}{2s} + \frac{-1}{s+1} + \frac{1}{2(s+2)}\right) + \left(\frac{1}{s+1} + \frac{-1}{s+2}\right)$$

Inverting the non-time-delayed part of the first term:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{2s} + \frac{-1}{s+1} + \frac{1}{2(s+2)}\right\} \\ &= \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}\end{aligned}$$

and so using the inverse delay theorem with a delay of 3 means that we swap t for $t-3$, and then multiply by the delayed step function $U(t-3)$. This gives:

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s+1)(s+2)}\right\} = U(t-3)\left(\frac{1}{2} - e^{-(t-3)} + \frac{e^{-2(t-3)}}{2}\right)$$

Since the second term did not feature time-delay, it is more straightforward. We simply obtain the inverse:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} \\ &= U(t)\big(e^{-t} - e^{-2t}\big)\end{aligned}$$

Then putting both results together again gives us our final answer:

$$x(t) = U(t-3)\left(\frac{1}{2} - e^{-(t-3)} + \frac{e^{-2(t-3)}}{2}\right) + U(t)\big(e^{-t} - e^{-2t}\big)$$

This second example shows the advantages of using the Laplace transforms method to solve nonhomogeneous ODEs. In particular, it is not necessary to first solve the homogeneous ODE, and initial values are automatically accounted for in the solution rather than requiring a set of extra steps to solve the simultaneous equations.

Next we will study a more systematic method of solving systems of multiple such equations.

2.8 Transfer Functions for Linear Systems

First, let's go over some general theory and define some key concepts. Later they will be demonstrated in a worked example.

- Transfer function: Consider a linear time-invariant system with input $v_i(t)$ and output $v_o(t)$. The **Transfer Function** $G(s)$ of the system is the ratio of the transform of the output to the transform of the input:

$$\bar{v}_o(s) = G(s) \times \bar{v}_i(s), \quad \text{and so} \quad G(s) = \frac{\bar{v}_o(s)}{\bar{v}_i(s)}$$

It is independent of the input function, and describes the effect that the system has on it.

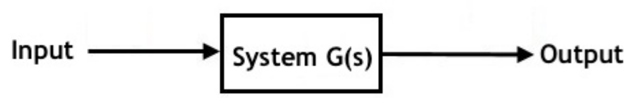


Figure 3: Transfer Function Block Diagram

- When the transfer function is in the form:

$$G(s) = \frac{P(s)}{Q(s)}$$

(i.e. as a simple fraction, so no fractions within the fraction!), then the **Characteristic Equation** of the system is given by:

$$Q(s) = 0$$

- A stable linear system is one that will remain at rest unless it is excited by an external source, and will return to rest if such external influences are removed. Therefore, in the absence of an input, the response function of a stable system will tend to zero as time approaches infinity. We could also say that any bounded input produces a bounded output if the system is stable.

- The real parts of the solutions to the characteristic equation of a system's transfer function determine the stability of the system (thus characterising the behaviour of the system). If there is a growing exponential, it indicates instability; while only decaying exponentials indicate stability. For stability, we require that the following condition is satisfied:

All solutions of $Q(s) = 0$ have negative real part $Re(s) < 0$.

- The order of the characteristic equation (i.e. the largest power of s, or equivalently, the number of solutions) is the "order" of the filter. This value has implications for the degree of complexity of behaviour that the system is capable of.

- Stable electronic systems can be used as a filter (a component which performs signal processing functions).

- Amplitude Bode Plots can be constructed from the transfer function and used to determine the type of filter.

- Say that for a circuit, we have obtained the transform of the output voltage: $\bar{v}_o(s) = G(s) \times \bar{v}_i(s)$. It is therefore the product of two transforms. We cannot just invert $G(s)$ and $\bar{v}_i(s)$ to obtain $v_o(t) = g(t) \times v_i(t)$, however instead we can obtain $v_o(t)$ using the **convolution integral** for a given input. This is denoted $v_o(t) = g(t) * v_i(t)$, where $*$ denotes the convolution operator and $g(t)$ is the inverse of the transfer function. The convolution integral is given by:

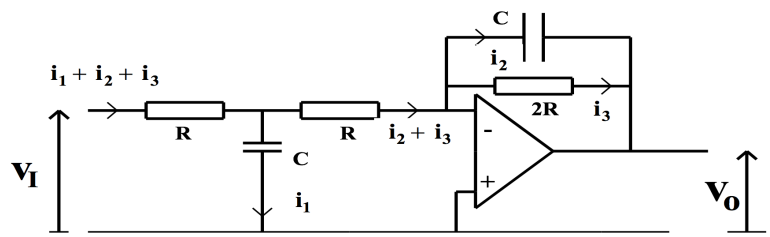
$$v_o(t) = \int_0^t g(t-z)v_i(z)dz = \int_0^t g(z)v_i(t-z)dz$$

  so $0 < z < t$.

- The **Response Function** $g(t)$ (or impulse response) is the inverse Laplace transform of the transfer function $G(s)$. It is the response of the system to an input $\delta(t)$ given by the delta function:
  Let $v_i(t) = \delta(t)$, then $\bar{v}_i(s) = 1$. Therefore $\bar{v}_o(s) = G(s) \times 1 = G(s)$, and so $g(t) = \mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}(\bar{v}_o(s)) = v_o(t)$. Hence the response function $g(t)$ is the output when the input is the delta function.

We will demonstrate how to apply these concepts in the following example:

Example 2.33. *Consider the circuit shown (based on an ideal "op-amp" (operational amplifier)). You may assume that $C, R > 0$.*



Analysing it using Kirchhoff's laws yields the following p.d. equations:

$$(a) \qquad v_i = R(i_1 + i_2 + i_3) + R(i_2 + i_3)$$

$$(b) \qquad \frac{1}{C}\int_0^t i_1 dt = R(i_2 + i_3)$$

$$(c) \qquad \frac{1}{C}\int_0^t i_2 dt = 2Ri_3$$

$$(d) \qquad v_o = -2Ri_3$$

Taking Laplace transforms of each yields:

$$(A) \qquad \bar{v}_i = R(\bar{i}_1 + 2\bar{i}_2 + 2\bar{i}_3)$$

$$(B) \qquad \frac{1}{sC}\bar{i}_1 = R(\bar{i}_2 + \bar{i}_3)$$

$$(C) \qquad \frac{1}{sC}\bar{i}_2 = 2R\bar{i}_3$$

$$(D) \qquad \bar{v}_o = -2R\bar{i}_3$$

We want to obtain a relationship between $\bar{v}_o$ and $\bar{v}_i$, so let's eliminate the other s-dependent variables $\bar{i}_1, \bar{i}_2, \bar{i}_3$. You should think carefully how to do this before you start, as you can end up going round in circles without a clear plan to obtain the variables you want. In this case, we will do the following:

1. Use (A) and (B) to obtain an equation with $\bar{v}_i$, $\bar{i}_2$ and $\bar{i}_3$.

2. Then use (C) to eliminate $\bar{i}_2$, leaving us with an equation in terms of $\bar{v}_i$ and $\bar{i}_3$.
3. Use (D) to substitute $\bar{i}_3$ for a function of $\bar{v}_o$.

Substituting (B) into (A) to remove $\bar{i}_1$:

$$\bar{v}_i = (\bar{i}_2 + \bar{i}_3)(2R + sCR^2)$$

Then eliminating $\bar{i}_2$ using (C):

$$\bar{v}_i = \bar{i}_3 R(2 + sCR)(1 + 2sCR)$$

Finally, substituting in (D) and rearranging removes $\bar{i}_3$ and gives us an equation involving both $\bar{v}_i$ and $\bar{v}_o$:

$$\bar{v}_o = \frac{-2}{(2 + sCR)(1 + 2sCR)}\bar{v}_i$$

Therefore the transfer function for this system is:

$$G(s) = \frac{-2}{(2 + sCR)(1 + 2sCR)}$$

and the characteristic equation given by setting the denominator equal to zero is:

$$Q(s) = (2 + sCR)(1 + 2sCR) = 0$$

which is a second-order equation in s and yields the two solutions $s_1 = -2/CR$ and $s_2 = -1/2CR$. These solutions determine the exponents that will appear in the time-domain solution for the output. That is, v_o will have terms that include factors $e^{-2t/CR}$ and $e^{-t/2CR}$. These are both decaying exponentials ($s_1, s_2 < 0$) and so the system is stable.

Finally, we use convolution to obtain the complete solution for $v_o(t)$, given the input function $v_i(t) = EU(t)$ (i.e. constant input E begins at $t = 0$). First, we invert the transfer function to obtain the response function.
Using partial fractions,

$$\frac{-2}{(2 + sCR)(1 + 2sCR)} = \frac{2}{3}\left(\frac{1}{2 + sCR}\right) - \frac{4}{3}\left(\frac{1}{1 + 2sCR}\right)$$

Therefore,

$$\begin{aligned}
g(t) &= \mathcal{L}^{-1}\left\{\frac{2}{3}\frac{1}{2+sCR} - \frac{4}{3}\frac{1}{1+2sCR}\right\} \\
&= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{1+s(CR/2)}\right\} - \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{1+s(2CR)}\right\} \\
&= \left(\frac{1}{3}\frac{1}{CR/2}e^{-2t/CR} - \frac{4}{3}\frac{1}{2CR}e^{-t/2CR}\right)U(t) \\
&= \frac{2}{3CR}U(t)(e^{-2t/CR} - e^{-t/2CR})
\end{aligned}$$

Then since $v_i(t) = EU(t)$, the convolution integral is:

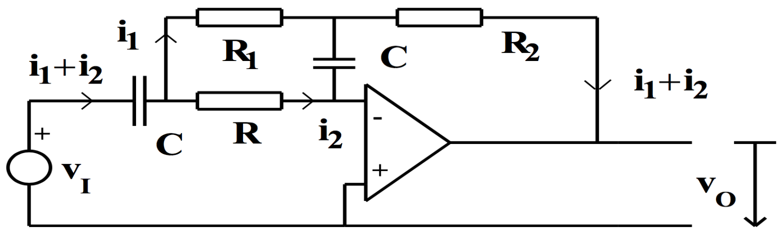
$$v_o(t) = \int_0^t g(t-z)v_i(z)dz = \int_0^t \frac{2}{3CR}U(t-z)(e^{-2(t-z)/CR} - e^{-(t-z)/2CR})EU(z)dz$$

For $t < 0$, $z < 0$ and so $U(z) = 0$. Therefore $v_o(t) = 0$ for all $t < 0$ and we can restrict our interest to $t > 0$. In this range, $0 < z < t$, so $t - z > 0$ and therefore both $U(t-z), U(z) = 1$. Thus for $t > 0$,

$$\begin{aligned}
v_o(t) &= \int_0^t \frac{2E}{3CR}\{e^{-2t/CR}e^{2z/CR} - e^{-t/2CR}e^{z/2CR}\}dz \\
&= \frac{2E}{3CR}\left\{e^{-2t/CR}\int_0^t e^{2z/CR}dz - e^{-t/2CR}\int_0^t e^{z/2CR}dz\right\} \\
&= \frac{2E}{3CR}\left\{e^{-2t/CR}\left[\frac{CR}{2}e^{2z/CR}\right]_0^t - e^{-t/2CR}\left[2CRe^{z/2CR}\right]_0^t\right\} \\
&= \frac{2E}{3CR}\left\{e^{-2t/CR}\frac{CR}{2}(e^{2t/CR} - 1) - 2CRe^{-t/2CR}(e^{t/2CR} - 1)\right\} \\
&= \frac{2E}{3CR}\left\{\frac{CR}{2}(1 - e^{-2t/CR}) - 2CR(1 - e^{-t/2CR})\right\} \\
&= \frac{E}{3}\{4e^{-t/2CR} - e^{-2t/CR} - 3\}U(t)
\end{aligned}$$

where we add the $U(t)$ rather than state "for $t > 0$ only".

Example 2.34. *Determine the stability of this filter, based on an ideal op-amp.*



It has the corresponding set of equations:

$$(a) \qquad v_i(t) = \frac{1}{C}\int_0^t (i_1 + i_2)dt + Ri_2$$

$$(b) \qquad R_1 i_1 = Ri_2 + \frac{1}{C}\int_0^t i_2 dt$$

$$(c) \qquad v_o(t) = \frac{1}{C}\int_0^t i_2 dt + R_2(i_1 + i_2)$$

where R, R_1, R_2 and C are poistive constants.

Begin by taking Laplace transforms,

$$(A) \qquad \bar{v}_i = \frac{1}{sC}(\bar{i}_1 + \bar{i}_2) + R\bar{i}_2$$

$$(B) \qquad R_1\bar{i}_1 = R\bar{i}_2 + \frac{1}{sC}\bar{i}_2$$

$$(C) \qquad \bar{v}_o = \frac{1}{sC}\bar{i}_2 + R_2(\bar{i}_1 + \bar{i}_2)$$

Plan for obtaining $\bar{v}_o$ in terms of $\bar{v}_i$ only (note that there are many ways to do this, so there is no uniquely correct set of steps. Whichever way you do it, you must demonstrate a clear plan for how you are going to obtain an equation with $\bar{v}_o$ and $\bar{v}_i$ as the only variables):

1. Use (B) to eliminate $\bar{i}_1$ from (A). Call this equation (D), it has variables $\bar{v}_i$ and $\bar{i}_2$.

2. Use (B) to eliminate $\bar{i}_1$ from (C). Call this equation (E), it has variables $\bar{v}_o$ and $\bar{i}_2$.

3. Then use (E) to substitute $\bar{i}_2$ for a function of $\bar{v}_o$ in (D).

Substituting equation (B) into (A) yields equation (D):

$$\bar{v}_i = \bar{i}_2\left(\frac{1}{sC} + R\right) + \frac{1}{sC}\left(\frac{R}{R_2}\bar{i}_2 + \frac{1}{sCR_1}\bar{i}_2\right) = \bar{i}_2\left(\frac{1}{Cs} + R + \frac{R}{sCR_1} + \frac{1}{C^2s^2R_1}\right)$$

Then substituting equation (B) into (C) yields equation (E):

$$\bar{v}_i = \bar{i}_2\left(\frac{1}{sC} + R_2\right) + R_2\left(\frac{R}{R_1}\bar{i}_2 + \frac{1}{sCR_1}\bar{i}_2\right) = \bar{i}_2\left(\frac{1}{sC} + R_2 + \frac{RR_2}{R_1} + \frac{R_2}{sCR_1}\right)$$

Combining (D) and (E) to eliminate $\bar{i}_2$*, we get:*

$$\bar{v}_o = \bar{v}i\left\{\frac{(1/sC) + R_2 + (RR_2/R_1) + (R_2/sCR_1)}{(1/sC) + R + (R/sCR_1) + (1/s^2C^2R_1)}\right\}$$

Tidying this up a bit will make it easier to obtain the solutions of the characteristic equation. Let's remove the subfractions by multiplying both the numerator and denominator by $s^2C^2R_1$*:*

$$\begin{aligned}\bar{v}_o &= \bar{v}_i\left\{\frac{sCR_1 + s^2C^2R_1R_2 + s^2C^2RR_2 + sCR_2}{sCR_1 + s^2C^2RR_1 + sCR + 1}\right\} \\ &= \bar{v}_i\left\{\frac{sC\big(s(CR_1R_2 + CRR_2) + (R_1 + R_2)\big)}{(sCR + 1)(sCR_1 + 1)}\right\}\end{aligned}$$

Therefore the transfer function for this system is

$$G(s) = \frac{sC\big(s(CR_1R_2 + CRR_2) + (R_1 + R_2)\big)}{(sCR + 1)(sCR_1 + 1)}$$

and the characteristic equation is

$$(sCR + 1)(sCR_1 + 1) = 0$$

which has solutions $s_1 = \frac{-1}{CR} < 0$ *and* $s_2 = \frac{-1}{CR_1} < 0$*. Hence* $Re(s) < 0$ *for all solutions, and so this second-order filter is stable.*

Example 2.35. *Consider example 3.1 again, however this time the input is given by*

$$v_i(t) = E\sin(\omega t)U(t)$$

Then the convolution integral in this case is:

$$\begin{aligned} v_o(t) &= \int_0^t g(t-z)v_i(z)dz \\ &= \frac{2E}{3CR}\int_0^t (e^{-2(t-z)/CR} - e^{-(t-z)/2CR})\sin(\omega z)dz \text{ for } t > 0 \text{ only} \\ &= \frac{2E}{3CR}\left\{e^{-2t/CR}\int_0^t e^{2z/CR}\sin(\omega z)dz - e^{-t/2CR}\int_0^t e^{z/2CR}\sin(\omega z)dz\right\} \end{aligned}$$

This integral can then be evaluated using the method of integration by parts.

2.9 Amplitude Bode Plots

When we have obtained the transfer function $G(s)$ for a linear system, and determined that it is stable, we can then sketch Bode plots in order to determine the nature of the filter.

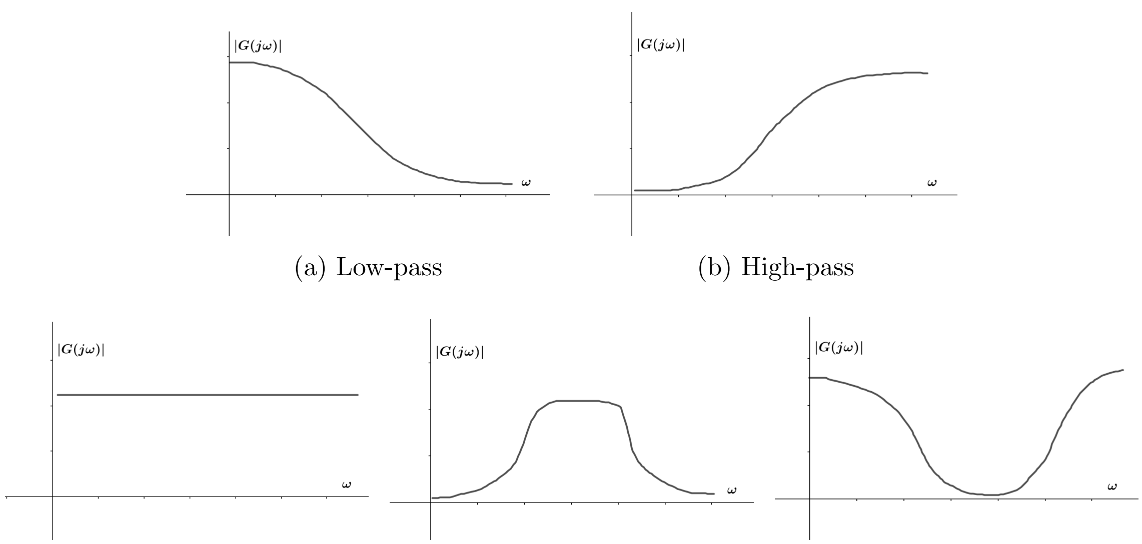
Once we have obtained the plot, we can determine if the filter is:

- High-pass (allows only high frequencies through)
- Low-pass (allows only frequencies below a certain value through)
- All-pass (allows all frequencies through)
- Band pass (allows only frequencies within a particular interval through)
- Band-eliminate/band-stop (allows all frequencies through except for a given interval).

This is determined simply by looking at where the plot is positive and negative. Remember that the transfer function $G(s)$ describes how the system changes the input signal to get the output signal:

$$\bar{v}_o(s) = G(s)\bar{v}_i(s)$$

so for example if the transfer function's amplitude approaches zero when the frequency ω of the input is high, that means the output signal will have close to zero amplitude, and so the system is in effect not allowing high frequency input signals to pass through.



(a) Low-pass (b) High-pass

(c) All-pass (d) Band-pass (e) Band-eliminate

Figure 4: Types of filter

We will be content to make some quick estimates of the behaviour of a simple filter by considering how the transfer function behaves when the frequency of the input signal is extremely small ($\omega << 1$) or extremely large ($\omega >> 1$). This can be done by considering which terms dominate the numerator and denominator of the transfer function at these frequencies.

Example 2.36 (First-order Low-pass)**.**

A system has the transfer function:

$$G(s) = \frac{-4}{1 + sRC}$$

We let $s = j\omega$, *then:*

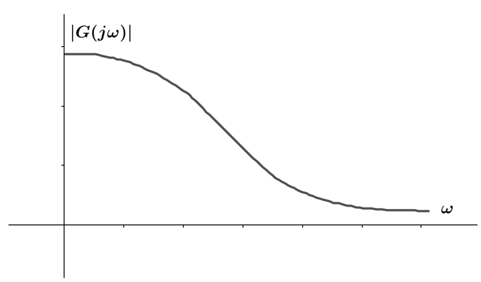
$$|G(j\omega)| = \left|\frac{-4}{1 + j\omega RC}\right| = \frac{|-4|}{|1 + j\omega RC|} = \frac{4}{|1 + j\omega RC|}$$

Analysing this function at very low and very high frequencies (ω)*:*

$$\text{When } \omega << 1: \quad |G(j\omega)| = \frac{4}{|1 + \textit{(smaller)}|} \approx \frac{4}{1} = 4$$

$$\text{When } \omega >> 1: \quad |G(j\omega)| = \frac{4}{|\textit{(smaller)} + j\omega RC|} \approx \frac{4}{|j\omega RC|} = \frac{4}{\omega RC} \to 0 \;\; \textit{as } \omega \to \infty$$

Hence this transfer function is positive for low frequencies and approaches zero for high frequencies, suggesting that this is likely to be a low-pass filter.



Example 2.37 (First-order High-pass).

A system has the transfer function:

$$G(s) = \frac{sCR}{1 + sCR}$$

We let $s = j\omega$, then:

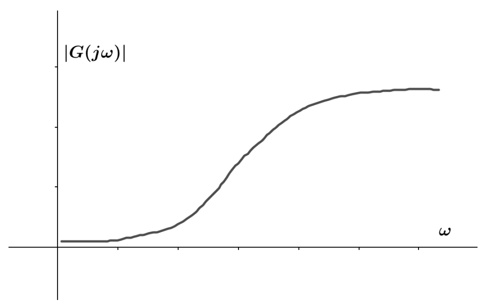
$$|G(j\omega)| = \left|\frac{j\omega CR}{1 + j\omega CR}\right| = \frac{|j\omega CR|}{|1 + j\omega CR|} = \frac{\omega CR}{|1 + j\omega CR|}$$

Approximating the behaviour at very low and very high frequencies (ω):

$$\text{When } \omega << 1: \quad |G(j\omega)| = \frac{\omega CR}{|1 + (\text{smaller})|} \approx \frac{\omega CR}{1} \to 0 \;\; \text{as } \omega \to 0$$

$$\text{When } \omega >> 1: \quad |G(j\omega)| = \frac{\omega CR}{|(\text{smaller}) + j\omega CR|} \approx \frac{\omega CR}{|j\omega CR|} = \frac{\omega CR}{\omega CR} = 1$$

Hence this transfer function approaches zero for low frequencies but is positive for high frequencies, and thus a high-pass filter.



Example 2.38 (Second-order Low-pass).

Consider the transfer function:

$$G(s) = \frac{1}{5Rs^2 + 18s + 1}$$

then let $s = j\omega$, *and*

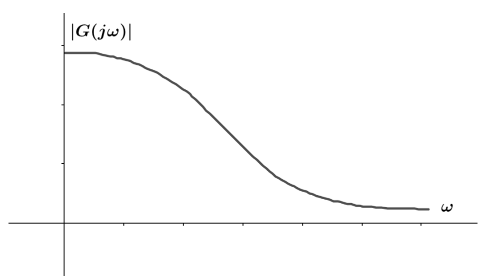
$$|G(j\omega)| = \frac{1}{|-5R\omega^2 + 18j\omega + 1|}$$

Then we consider which terms dominate for very small and very large ω*:*

$$\text{When } \omega << 1: \quad |G(j\omega)| = \frac{1}{|(\text{smaller}) + 1|} \approx 1$$

$$\text{When } \omega >> 1: \quad |G(j\omega)| = \frac{1}{|-5R\omega^2 + (\text{smaller})|} \approx \frac{1}{5R\omega^2} \to 0 \ \text{ as } \omega \to \infty$$

So this is probably a (second-order) low-pass filter.



3 Fourier Analysis

Named after Jean Baptiste Joseph Fourier (1768-1830). Also a French mathematician, he originally trained for the priesthood before teaching mathematics, and became involved in the French Revolution. He advised Napoleon during his invasion of Egypt, held a post at the *École Polytechnique* in Paris, was the Prefect of Grenoble and permanent secretary of the French Academy of Sciences.

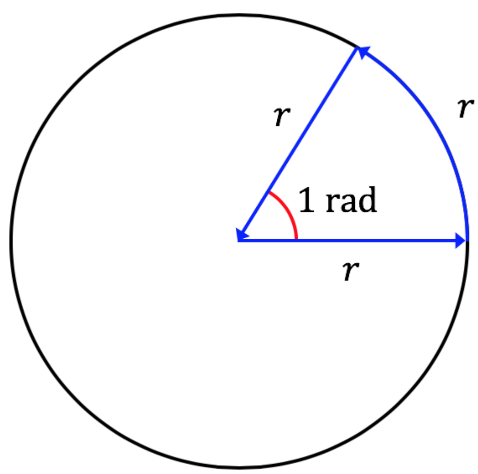
Fourier developed the idea of what we now call Fourier Series while working on the problem of heat conduction in solids. Other famous mathematicians including Leonhard Euler had used series like this in the past, but hadn't explored its applications in this way. Fourier's paper was rejected by reviewers including Laplace, but the academy subsequently made the problem of heat conduction the subject of its next Grand Prize. Fourier resubmitted some revised work and won the prize, but his paper was still not published in the academy journal for several years.

3.1 Trigonometry and Waveforms

Recall the functions sin and cos, and the trigonometric equations for right-angled triangles. We may also need some well-known trigonometric identities, such as

$$\sin^2(x) + \cos^2(x) = 1 \text{ for all values of } x$$

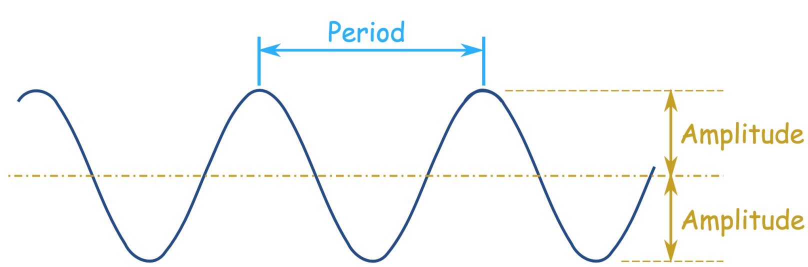
Rather than using degrees in a cycle from 0° to 360°, we prefer to use the unit **radians** for most scientific applications. These vary from 0 to 2π for a full cycle, and in particular, the radian is defined as the angle between two radii that create a circular arc with a length equal to one radius.



Consider a functions f and g of time that take the forms:

$$f(t) = A\sin(\omega t + \phi) + d \quad \text{and} \quad g(t) = B\cos(\omega t + \phi) + d$$

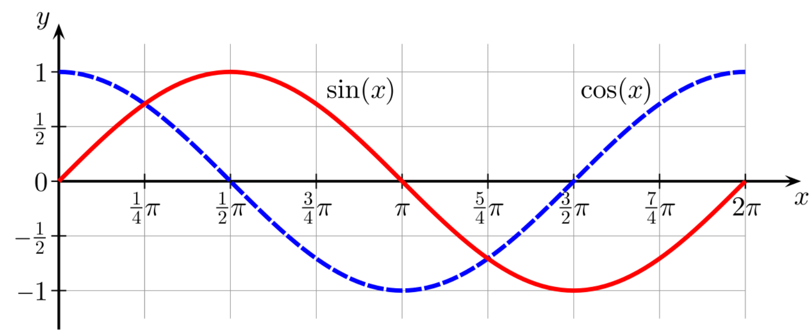
In this case, ω is the **angular frequency** of f and g, and ϕ is the **phase angle** which measures the lag or lead of the term from the pure sine or cosine function of the same frequency. Note that in this case, ϕ may also be called the "phase shift", which in engineering and physics is not the same as what is referred to as the "horizontal shift". Instead, the horizontal shift is ϕ/ω, obtained by factorising $f(t) = A\sin(\omega(t + \phi/\omega))$, but this terminology can vary slightly between disciplines.



Since sin and cos have unit **amplitude** (maximum absolute displacement of the oscillating term from equilibrium), the amplitude of functions $f(t)$ and $g(t)$ will be A and B respectively. The phase angle describes how the function is shifted to the left (for a positive value of ϕ) compared to the regular $\sin(\omega t)$ or $\cos(\omega t)$. Finally, d is the **vertical shift** and describes how much the final wave is shifted up or down. If d is positive, the wave is shifted up.

The cosine function is said to **lead** the sine function by $90°$ or $\frac{\pi}{2}$ radians, while the sine function is said to **lag** the cosine function by $90°$ or $\frac{\pi}{2}$ radians. This means that we can express cos as a sin function with a phase shift, or sin as cos with a phase shift. In particular,

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x) \quad \text{and} \quad \cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$



Sine and cosine are **periodic** functions. In general, a function $f(t)$ is periodic with period T if for all values of t in the domain, and for any integer m, we have:

$$f(t + mT) = f(t)$$

The time required for one complete cycle is called the **period** T of the function. Then the number of full cycles per unit of time (usually seconds) is called the frequency and given by $f = T^{-1}$. However, when using radians, it is usually more useful to talk about the **angular frequency** ω which is measured in radians (rather than complete cycles) per second. Hence,

$$\omega = \frac{2\pi}{T} \quad \text{or} \quad T = \frac{2\pi}{\omega}$$

As the period of oscillation T **increases**, the frequency (both angular and regular) **decreases** and vice versa.

3.2 Plotting Waveforms on a Computer

Throughout this chapter, you may find it helpful to visualise the signals you are analysing.

We can easily draw generalised trigonometric functions using a computer package. Geogebra is free and very easy to use on a web browser, or we can use MATLAB.

MATLAB has many different commands for seemingly-basic functions such as curve-plotting, that all work in slightly different ways and situations.

For visualing a simple function, such as a sine wave or a Heaviside function, we can declare it as a *symbolic* function and then use `fplot` to visualise it over $[-5, 5]$:

```
syms x
y = sin(x)
fplot(x,y)
```

For plotting lines from numeric data sets, or anything that isn't necessarily a pre-defined symbolic function, we can use the standard `plot` command. However, this requires two arguments: one vector containing all the x-coordinates of the points that you wish to join up, and a second vector of equal length containing the corresponding y-values.

For example, to plot the graph of $y = \sin(x)$ between $x = 0$ and $x = 5$:

```
x = linspace(0,5,1000)
```

This creates a vector of 1000 evenly spaced values between 0 and 5. These will be the x-coordinates of the points that `plot` will then join up.

```
y = sin(x)
plot(x,y)
```

Of course, you can include optional arguments in the `plot` command to customise the graph, such as changing the line thickness: `plot(x,y,'LineWidth',5)`
If you want to change the limits on the axes that are in view, follow-up with the command `xlim`. So if we wanted to zoom in on the graph between $x = 1$ and $x = 2$, we would use: `xlim([1 2])` and similarly for the y-axis using `ylim`. You can also use the command `hold on;` to plot multiple functions on the same image, as otherwise each new use of `plot` will replace the previous graph.

3.3 Introduction to Fourier Series

You may already have encountered the concept of representing a function as a series, for example using Maclaurin's theorem to obtain a power series expansion for a function. The core idea of Fourier analysis is that many functions can be approximated by a series (a summation) of sine and cosine functions of increasing angular frequency. It is usually applied to periodic functions. The process centres on calculating the amplitude of each sine and cosine term in the series that approximates our function, and this may be known as Fourier, harmonic, or spectral analysis.

Theorem 3.1. *Any finite periodic function $f(t)$ with period T can be represented by a* unique *series of sines and cosines. In particular, there exist two unique sequences $(a_n)_n$ and $(b_n)_n$ such that:*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

*where $\omega = \frac{2\pi}{T}$ is the angular frequency of f, and so the n^{th} term of the summation has angular frequency $n\omega$. This is the **Fourier Series** representation of $f(t)$.*

The terms in the series have certain names:

- $\frac{a_0}{2}$ is the **DC level** of $f(t)$.
- $a_1 \cos(\omega t) + b_1 \sin(\omega t)$ is the first harmonic, also called the **Fundamental mode**.
- $a_n \cos(n\omega t) + b_n \sin(n\omega t)$ is the $\mathbf{n^{th}}$ **Harmonic**. It has angular frequency $n\omega$, which is n times the angular frequency of the fundamental.

The standard method of obtaining these Fourier Coefficients is using the following integrals. This is **not** the method you will be expected to use but it is important that you are aware of it!

$$a_0 = \frac{2}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt$$

in each case, we are calculating $2\times$ the mean value of the integrand over the range of one period (so **the DC level is equivalent to the average value of $f(t)$ over one**

complete period). In particular, we used the first cycle $0 < t < T$, however any full period could be used. That is, we could integrate over the range $t_0 < t < t_0 + T$ for any value of t_0 and get the same results. Some textbooks will prefer the cycle $-T/2 < t < T/2$.

The N^{th} **Partial Sum** of the Fourier Series, say $f_N(t)$, for a function $f(t)$ is when it is truncated to a finite number of terms, in particular the first N. That is,

$$f_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N-1} a_n \cos(n\omega t) + \sum_{n=0}^{N-1} b_n \sin(n\omega t)$$

As we increase the number of terms used, the partial sum begins to approximate the function more accurately (in mathematical terms, the series converges to the original function). In practical terms, the lower n harmonics are lower frequencies, so taking a partial sum is equivalent to low frequency filtering (blocking higher frequencies) which is the same effect as when employing low-quality measuring equipment or transducers.

In the context of measuring equipment, if it is not even able to detect the fundamental (e.g. a moving coil voltmeter) only the DC level will be displayed. With more sophisticated measuring equipment, more of the signal will be detected (i.e. the fundamental frequency and possibly some harmonics) and so the "received" waveform will be closer to the actual waveform. The more harmonics that are detected, the closer we get to the actual waveform.

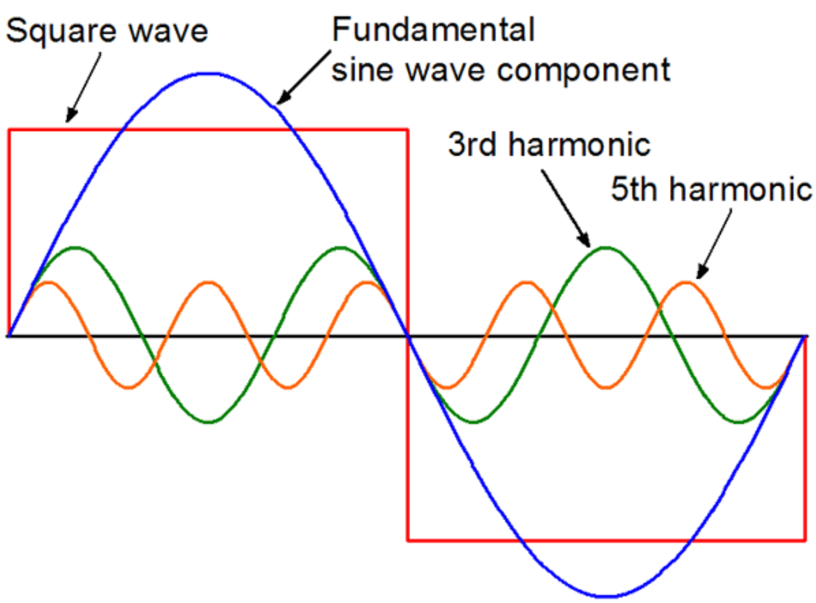


Figure 5: Square Wave approximation by Harmonics

3.3.1 Theory: Method of Determining the Fourier Series

1. Identify $f(t)$ and substitute it into the equations for a_n, b_n.

2. Perform the integrals. This often requires the use of the Integration By Parts method. If the function's behaviour differs qualitatively across a period ("piecewise"), we will need to split the integrals up.

3. Calculate a_0 separately, either by integration or from inspecting the graph of the function to determine the average values.

3.3.2 Basic Examples

Example 3.1 (Pulse Wave). *(a) Consider the pulse wave, used in digital switching circuits, with period T.*

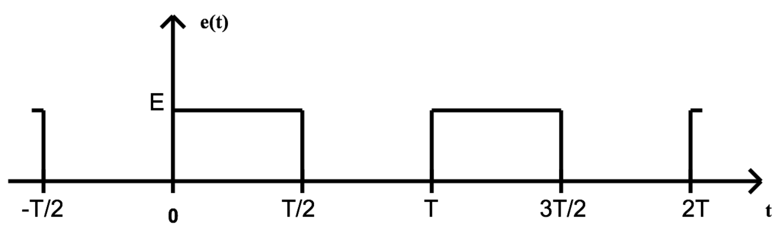


Figure 6: Pulse Wave

It is a piecewise function, defined in the following way:

$$f(t) = \begin{cases} E, & \text{for } nT < t < nT + \frac{T}{2} \\ 0, & \text{for } nT + \frac{T}{2} < t < (n+1)T \end{cases}$$

$\forall n \in \mathbb{Z}$

We will consider the behaviour in the first cycle $0 < t < T$, and will need to split the integrals for a_0, a_n, b_n over the two domains of t for which f displays different behaviour:

$$\begin{aligned} a_0 &= \frac{2}{T} \int_0^T f(t) dt \\ &= \frac{2}{T} \left(\int_0^{T/2} E dt + \int_{T/2}^{T} 0 dt \right) \\ &= \frac{2}{T} \big[Et \big]_0^{T/2} = E \end{aligned}$$

So the DC level is $\frac{a_0}{2} = \frac{E}{2}$*, and we could also see from the graph that this is the average value of* $f(t)$ *over a cycle.*

For $n \geq 1$*,*

$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt \\
&= \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi n}{T}\right)dt \\
&= \frac{2}{T}\left(\int_0^{T/2} E\cos\left(\frac{2n\pi}{T}t\right)dt + \int_{T/2}^{T} 0\times\cos\left(\frac{2n\pi}{T}t\right)dt\right) \\
&= \frac{2E}{T}\int_0^{T/2}\cos\left(\frac{2n\pi}{T}t\right)dt \\
&= \frac{2E}{T}\left[\frac{T}{2\pi n}\sin(\frac{2n\pi}{T}t)\right]_0^{T/2} \\
&= \frac{E}{\pi n}\Big(\sin(\pi n) - \sin(0)\Big) = 0
\end{aligned}$$

Note that this shows why we need to calculate a_0 *separately, as* $n = 0$ *in this formula returns* $0/0$ *which is undefined (although applying l'Hôpital's rule would also yield the correct answer).*

$$\begin{aligned}
b_n &= \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt \\
&= \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n}{T}\right)dt \\
&= \frac{2}{T}\left(\int_0^{T/2} E\sin\left(\frac{2n\pi}{T}t\right)dt + \int_{T/2}^{T} 0\times\sin\left(\frac{2n\pi}{T}t\right)dt\right) \\
&= \frac{2E}{T}\left[\frac{T}{-2\pi n}\cos\left(\frac{2n\pi}{T}t\right)\right]_0^{T/2} \\
&= \frac{-E}{\pi n}(\cos(\pi n) - \cos(0)) \\
&= \frac{-E}{\pi n}((-1)^n - 1) = \frac{E}{\pi n}((-1)^{n+1} + 1)
\end{aligned}$$

Hence the Fourier series is:

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \\ &= \frac{E}{2} + \sum_{n=1}^{\infty} 0 \times \cos(n\omega t) + \sum_{n=1}^{\infty} \frac{E}{\pi n}((-1)^{n+1} + 1) \sin(n\omega t) \\ &= \frac{E}{2} + \frac{E}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}((-1)^{n+1} + 1) \sin(n\omega t) \end{aligned}$$

(b) Consider this square wave $F(t)$*.*

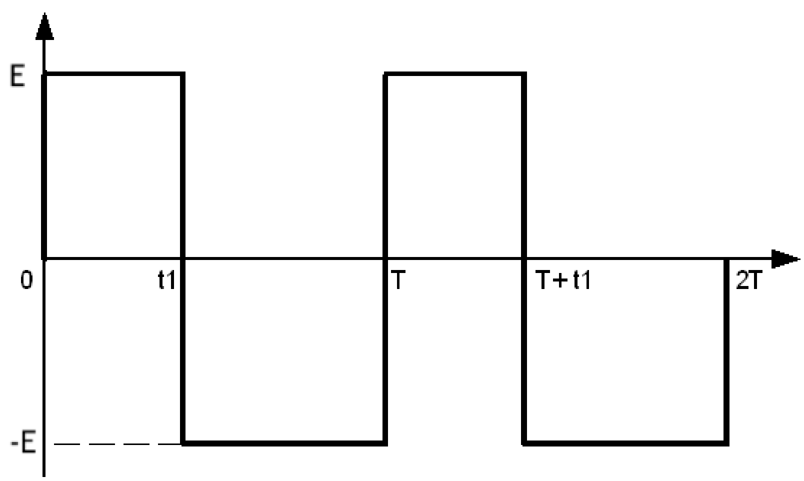


Figure 7: Square Wave

As an exercise, you should do this one from first principles of Fourier analysis yourself. However we will quickly verify the result by making a transformation from $f(t)$*. In particular, we can transform* $f(t)$ *to* $F(t)$ *by first making a translation (shift) down by* $E/2$ *in the* y*-axis, and then stretching by a factor of 2 in the* y*-axis. That is,* $F(t) = 2(f(t) - \frac{E}{2})$*. Therefore the same relationship exists between their respective Fourier representations:*

$$F(t) = 2(f(t) - \frac{E}{2}) = \frac{2E}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}((-1)^{n+1} + 1) \sin(n\omega t).$$

Note that there are sometimes several equivalent ways to write the Fourier series (although the series itself is unique, so these are just slightly different ways of *presenting the same result). Expanding the series above, we see:*

$$F(t) = \frac{2E}{\pi}\big(2\sin(\omega t) + 0 \times \sin(2\omega t) + 2\sin(3\omega t) + 0 \times \sin(4\omega t) + \dots\big)$$

That is, $((-1)^{n+1} + 1)$ *is equal to* 2 *for odd values of* n*, and zero for even values. Therefore, if we only account for the odd values of* n*, we can represent the wave by*

$$F(t) = \frac{4E}{\pi} \sum_{odd\ n \in \mathbb{N}} \frac{1}{n} \sin(n\omega t).$$

Alternatively we can build in to the function that we only count the odd natural numbers, so that the sum is from $n = 1, 2, 3, \dots$ *but that* n *maps to the* n^{th} *odd positive integer* $2n - 1$*:*

$$F(t) = \frac{4E}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)\omega t).$$

All three expressions of the series for $F(t)$ *are equivalent and correct.*

3.4 Special cases of Fourier Series

3.4.1 Odd and Even Functions

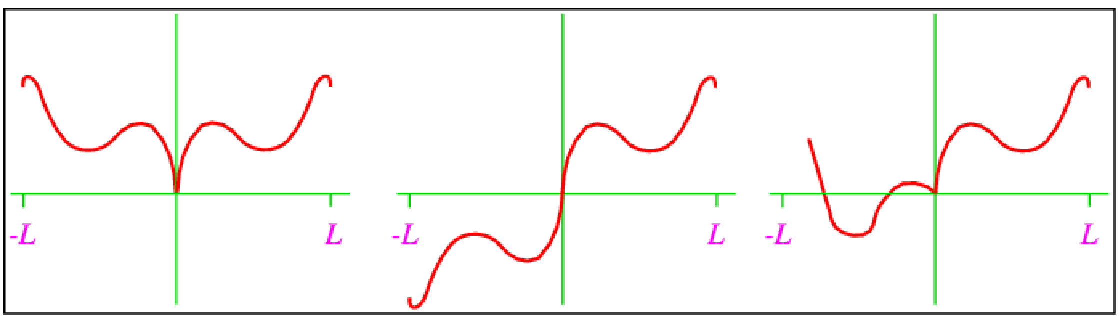
Some functions can be classified as **odd** or **even** (or, in some cases, both!).

Definition 3.1. *A function* $f(t)$ *is odd if it satisfies* $f(t_0) = -f(-t_0)$ *for all* $t_0 \in \mathbb{R}$.

Graphically, odd functions appear to have been *rotated* through 180° about the origin. Importantly, the *sine* function is odd.

Definition 3.2. *A function* $f(t)$ *is even if it satisfies* $f(t_0) = f(-t_0)$ *for all* $t_0 \in \mathbb{R}$.

Graphically, even functions appear *reflected* about the y-axis. The *cosine* function is even.



Consider the image. From left to right, the functions are even, odd, and neither.

Why do we mention this here? Because of the relationship between Fourier coefficients and *sine* and *cosine*, it turns out that:
(a) If $f(t)$ is an even function,

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt = 0 \quad \forall\, n \in \mathbb{N}$$

and,
(b) If $f(t)$ is an odd function,

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt = 0 \quad \forall\, n \in \mathbb{N}$$

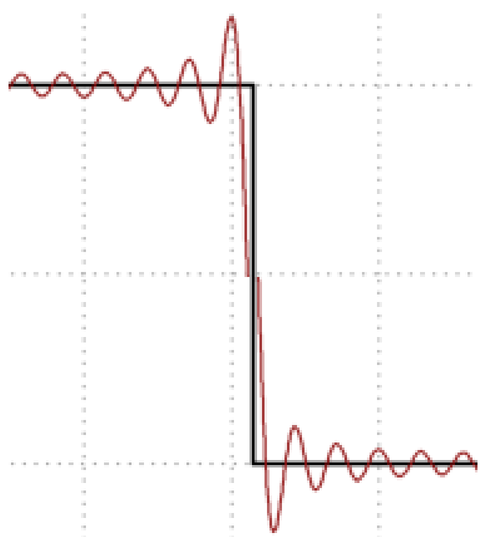
You can do some integration by parts to verify these results, which simply say that if the function $f(t)$ is even, then it is expressed as a sum of even functions (i.e. $\cos(n\omega t)$), while if $f(t)$ is odd then its Fourier series expansion will consist entirely of odd functions ($\sin(n\omega t)$). In applicable cases where we want to find the Fourier series of an odd or even function, we can use this as a shortcut in determining one half of the Fourier coefficients.

The key takeaway here is that when determining the regular Fourier coefficients of a periodic function $f(t)$:

(a) If $f(t)$ is an **even** function, then $b_n = 0 \quad \forall\, n \in \mathbb{N}$.
(b) If $f(t)$ is an **odd** function, then $a_n = 0 \quad \forall\, n \in \mathbb{N}$.

3.4.2 Discontinuities and the Gibbs Phenomenon

When using Fourier series to approximate a discontinuous function, the partial sums will always feature a spike when approaching the point of discontinuity t_0 from the either the left or the right sides. By adding sufficiently many terms, the domain (but not the magnitude) of the errors can be made arbitrarily small, but will always be present, leading to a sharper "spike" on each side of the discontinuity as $N \to \infty$. This is known as **Gibb's Phenomenon**.



3.5 Complex Form and Phasors

Complex number consist of a real part and an imaginary part that involves the **imaginary number**. The imaginary number is defined as the square root of -1, and is denoted i by mathematicians, and j by engineers. Using special properties of trigonometric functions, we can represent a complex number $A = a - jb$ in polar or exponential form:

$$A = R\big(\cos(\phi) - j\sin(\phi)\big) = R\,\mathrm{e}^{\phi j}$$

where

$$R = \sqrt{a^2 + b^2} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{-b}{a}\right)$$

Returning to Fourier series, using the exponential form we can construct a complex quantity from the harmonics called a "phasor", which is a vector in the complex plane that encodes information about the amplitude of the signal. Recall that the n^{th} harmonic for $f(t)$ is given by $a_n\cos(n\omega t) + b_n\sin(n\omega t)$. Then the phasor (the complex amplitude) for the n^{th} harmonic is:

$$A_n = a_n - jb_n = R_n\,\mathrm{e}^{j\phi_n}$$

where

$$R_n = \sqrt{a_n^2 + b_n^2} \quad \text{and} \quad \tan(\phi_n) = \frac{-b_n}{a_n}$$

Then we can use phasor representation to state the Fourier series for $f(t)$ in complex form:

$$f(t) = \frac{a_0}{2} + Re\left\{\sum_{n=1}^{\infty} A_n\,\mathrm{e}^{jn\omega t}\right\} \quad \text{where} \quad A_n = a_n - jb_n,\ \ \omega = \frac{2\pi}{T}$$

The phasor is the complex Fourier coefficient for the n^{th} harmonic. It can be obtained by:

$$\begin{aligned} A_n &= a_n - jb_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt - j\frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt \\ &= \frac{2}{T}\int_0^T f(t)(\cos(n\omega t) - j\sin(n\omega t))dt \\ &= \frac{2}{T}\int_0^T f(t)(\cos(-n\omega t) + j\sin(-n\omega t))dt \end{aligned}$$

and so we have:

$$A_n = \frac{2}{T}\int_0^T f(t)\,\mathrm{e}^{-jn\omega t}\,dt$$

However, we will not perform calculations using this method, as we can use the complex form to avoid using integration altogether!

3.6 Special Values

Due to the periodicity of trigonometric waves, there are certain exponential and polar forms to look out for, as they can be nicely simplified. In the exponential form, ϕ (called the argument) represents the angle (in radians) that we have rotated anticlockwise around the unit circle, starting from the positive real axis. You can also calculate these special values using polar form, as they are angles at which sine or cosine take a special value such as one, zero or negative one. Hence, we are looking for values at which the argument is an integer multiple of $\frac{\pi}{2}$:

- For any integer n,

$$\mathrm{e}^{2n\pi j} = \cos(2n\pi) + j\sin(2n\pi) = 1 + j \times 0 = 1$$

- For any integer n,

$$\begin{aligned} \mathrm{e}^{n\pi j} &= \cos(n\pi) + j\sin(n\pi) \\ &= \cos(n\pi) + j \times 0 = \cos(n\pi) = (-1)^n = \begin{cases} -1 & \text{for odd } n, \\ +1 & \text{for even } n. \end{cases} \end{aligned}$$

3.7 Method using Laplace transforms

Next we will apply what we learned about Laplace transforms to the complex form of Fourier series.

This is the method (Laplace transforms and phasors) of Fourier analysis that shall be assessed for this course.

Given a period function $f(t)$, define:

$$g(t) = \begin{cases} f(t) & \text{for } 0 < t < T, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} A_n &= a_n - jb_n = \frac{2}{T}\int_0^T f(t)\,\mathrm{e}^{-jn\omega t}\,dt \\ &= \frac{2}{T}\int_0^T g(t)\,\mathrm{e}^{-jn\omega t}\,dt \\ &= \frac{2}{T}\int_0^\infty g(t)\,\mathrm{e}^{-jn\omega t}\,dt \\ & \quad \text{since } g(t) = 0 \text{ for } t > T \quad \Longrightarrow \quad \frac{2}{T}\int_T^\infty g(t)\,\mathrm{e}^{-jn\omega t}\,dt = 0. \end{aligned}$$

Recognising that this involves the definition of a Laplace transform, we then obtain the important result:

$$A_n = \frac{2}{T}\bar{g}(jn\omega)$$

3.7.1 Method

Thus, our method for obtaining the Fourier series for a periodic signal $f(t)$ using the method of phasors and Laplace transforms is as follows:

1. Define

$$g(t) = \begin{cases} f(t) & \text{for } 0 < t < T, \\ 0 & \text{otherwise.} \end{cases}$$

2. Obtain the Laplace transform $\bar{g}(s)$.

3. Change the variable to obtain $\bar{g}(jn\omega)$.

4. The phasor of the n^{th} harmonic is then:

$$A_n = \frac{2}{T}\bar{g}(jn\omega)$$

5. Also find the DC level $\frac{a_0}{2}$ either from the average value of the graph, or using the integral

$$a_0 = \frac{2}{T}\int_0^T f(t)dt$$

6. State the complex form of the Fourier Series, given by:

$$f(t) = \frac{a_0}{2} + Re\left\{ \sum_{n=1}^{\infty} A_n \, \mathrm{e}^{jn\omega t} \right\}$$

7. (Optional) If you are asked to obtain the usual coefficients from the phasors, use:

$$a_n = Re\{A_n\} \quad \text{and} \quad b_n = -Im\{A_n\}$$

and then you can state the regular form of the Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

3.7.2 Examples

Example 3.2. *Consider again the sawtooth wave. We will now derive its Fourier Series using this method of phasors and Laplace transforms.*

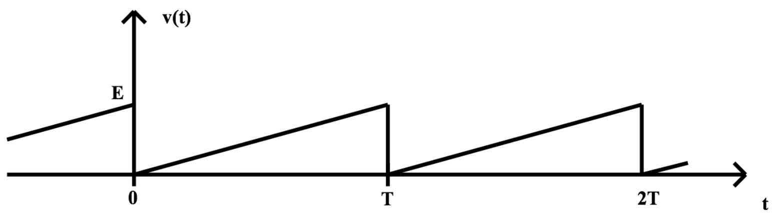


Figure 8: Sawtooth Wave

In this case, $f(t) = \frac{Et}{T}$ *during the first cycle, so we want* $g(t) = \frac{Et}{T}$ *for time* $0 < t < T$ *and zero otherwise. This can be achieved using a combination of step changes that will turn "on" at time* $t = 0$ *and "off" at* $t = T$*:*

$$g(t) = \frac{Et}{T}\big(U(t) - U(t-T)\big) = \frac{E}{T}\big(tU(t) - tU(t-T)\big)$$

We need to write the second term in delay form, so:

$$g(t) = \frac{E}{T}\Big(tU(t) - \big((t-T)+T\big)U(t-T)\Big)$$

Then the Laplace transfom is given by:

$$\begin{aligned}
\bar{g}(s) &= \mathcal{L}\left\{\frac{E}{T}\big(tU(t) - \big((t-T)+T\big)U(t-T)\big)\right\} \\
&= \frac{E}{T}\Big(\mathcal{L}\big\{tU(t)\big\} - \mathcal{L}\big\{\big((t-T)+T\big)U(t-T)\big\}\Big) \\
&= \frac{E}{T}\left(\frac{1}{s^2} - \mathcal{L}\big\{t+T\big\}\,\mathrm{e}^{-sT}\right) \\
&= \frac{E}{T}\left(\frac{1}{s^2} - \left[\frac{1}{s^2} + \frac{T}{s}\right]\mathrm{e}^{-sT}\right)
\end{aligned}$$

Changing the variable from s to $jn\omega$:

$$\begin{aligned}
\bar{g}(jn\omega) &= \frac{E}{T}\left(\frac{1}{(jn\omega)^2} - \left[\frac{1}{(jn\omega)^2} + \frac{T}{jn\omega}\right]\mathrm{e}^{-jn\omega T}\right) \\
&= \frac{E}{T}\left(\frac{-T^2}{4n^2\pi^2} - \left[\frac{-T^2}{4n^2\pi^2} - j\frac{T^2}{2\pi n}\right]\mathrm{e}^{-2jn\pi}\right) \\
&\qquad \textit{since } \ \omega = \frac{2\pi}{T},\ j^2 = -1,\ \frac{1}{j} = -j \\
&= \frac{E}{T}\left(\frac{-T^2}{4n^2\pi^2} - \frac{-T^2}{4n^2\pi^2} + j\frac{T^2}{2\pi n}\right) \\
&\qquad \textit{since } \ \mathrm{e}^{-2jn\pi} = \cos(2\pi n) - j\sin(2\pi n) = 1\ \forall n \in \mathbb{N} \\
&= j\frac{ET}{2\pi n}
\end{aligned}$$

Hence the phasor of the n^{th} harmonic is,

$$A_n = \frac{2}{T}j\frac{ET}{2\pi n} = j\frac{E}{\pi n}$$

From looking at the graph, the average value of $f(t)$ over one cycle is clearly $\frac{E}{2}$. Alternatively,

$$\begin{aligned}
a_0 &= \frac{2}{T}\int_0^T f(t)dt = \frac{2}{T}\int_0^T \frac{E}{T}tdt \\
&= \frac{2E}{T^2}\left[\frac{1}{2}t^2\right]_0^T = E
\end{aligned}$$

and so the DC level is $\frac{a_0}{2} = \frac{E}{2}$.

Thus the complex form of the Fourier series is:

$$\begin{aligned}
f(t) &= \frac{a_0}{2} + Re\left\{\sum_{n=1}^{\infty} A_n\,\mathrm{e}^{jn\omega t}\right\} \\
&= \frac{E}{2} + Re\left\{\sum_{n=1}^{\infty} j\frac{E}{\pi n}\,\mathrm{e}^{jn\omega t}\right\}
\end{aligned}$$

To obtain the regular Fourier coefficients:

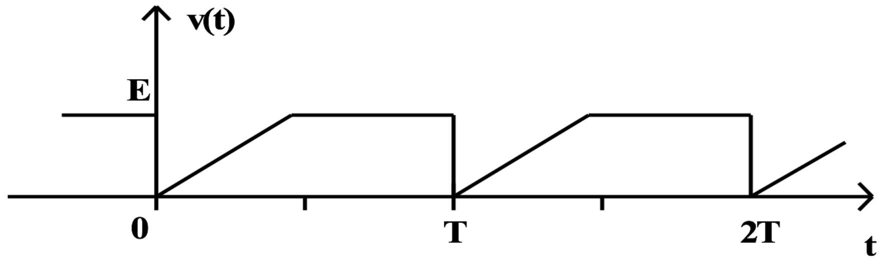
$$a_n = Re\{A_n\} = Re\left\{j\frac{E}{\pi n}\right\} = 0$$

$$b_n = -Im\{A_n\} = -Im\left\{j\frac{E}{\pi n}\right\} = \frac{-E}{\pi n}$$

So the Fourier series with real coefficients is:

$$f(t) = \frac{E}{2} - \frac{E}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2\pi nt}{T}\right)$$

Example 3.3. *Consider the clipped sawtooth waveform again.*



In the first cycle $0 < t < T$*, this wave is given by:*

$$f(t) = \begin{cases} \frac{2Et}{T} & \text{for } 0 < t < T/2, \\ \\ E & \text{for } T/2 < t < T. \end{cases}$$

Therefore we use step functions to define g*, with* $\frac{2Et}{T}$ *turning on at time* $t = 0$ *and off at* $t = T/2$*, and* E *turning on at* $t = T/2$ *and off at* $t = T$*.*

That is:

$$\begin{aligned} g(t) &= \frac{2Et}{T}\big\{U(t) - U(t - T/2)\big\} + E\big\{U(t - T/2) - U(t - T)\big\} \\ &= \frac{2Et}{T}U(t) - \frac{2Et}{T}U(t - T/2) + EU(t - T/2) - EU(t - T) \\ &= \frac{2Et}{T}U(t) + E\left(1 - \frac{2t}{T}\right)U(t - T/2) - EU(t - T) \end{aligned}$$

The first term has no time delay, and the final term is in delay form. Therefore we just need to put the middle term in delay form:

$$\begin{aligned} g(t) &= \frac{2Et}{T}U(t) + E\left(1 - \frac{2}{T}\big((t - \frac{T}{2}) + \frac{T}{2}\big)\right)U(t - T/2) - EU(t - T) \\ &= E\left\{\frac{2}{T}tU(t) + \left(1 - \frac{2}{T}\big(t - \frac{T}{2}\big) - 1\right)U(t - T/2) - U(t - T)\right\} \\ &= E\left\{\frac{2}{T}tU(t) - \frac{2}{T}(t - \frac{T}{2})U(t - T/2) - U(t - T)\right\} \end{aligned}$$

Taking Laplace transforms (using the delay theorem for the second and third terms):

$$\begin{aligned}
\bar{g}(s) &= \mathcal{L}\Big\{E\Big\{\frac{2}{T}tU(t) - \frac{2}{T}(t-\frac{T}{2})U(t-T/2) - U(t-T)\Big\}\Big\} \\
&= E\Big\{\frac{2}{T}\mathcal{L}\Big\{tU(t)\Big\} - \frac{2}{T}\mathcal{L}\Big\{(t-\frac{T}{2})U(t-T/2)\Big\} - \mathcal{L}\Big\{U(t-T)\Big\}\Big\} \\
&= E\Big\{\frac{2}{Ts^2} - \frac{2}{T}\mathcal{L}\{t\}\,\mathrm{e}^{-sT/2} - \mathcal{L}\{1\}\,\mathrm{e}^{-sT}\Big\} \\
&= E\Big\{\frac{2}{Ts^2} - \frac{2}{Ts^2}\,\mathrm{e}^{-sT/2} - \frac{1}{s}\,\mathrm{e}^{-sT}\Big\}
\end{aligned}$$

Then substituting s for $jn\omega$:

$$\begin{aligned}
\bar{g}(jn\omega) &= E\Big\{\frac{2}{T(jn\omega)^2}(1-\mathrm{e}^{-jn\omega T/2}) - \frac{1}{jn\omega}\,\mathrm{e}^{-jn\omega T}\Big\} \\
&= E\Big\{\frac{-2T^2}{4Tn^2\pi^2}(1-\mathrm{e}^{-jn\pi}) + \frac{jT}{2\pi n}\,\mathrm{e}^{-2\pi jn}\Big\} \\
&\qquad \text{since } \omega = \frac{2\pi}{T},\ j^2 = -1,\ \frac{1}{j} = -j \\
&= ET\Big\{\frac{1}{2\pi^2 n^2}(\mathrm{e}^{-jn\pi} - 1) + \frac{j}{2\pi n}\Big\} \\
&\qquad \text{since } \mathrm{e}^{-2jn\pi} = \cos(2\pi n) - j\sin(2\pi n) = 1\ \forall n \in \mathbb{N}
\end{aligned}$$

And since $\mathrm{e}^{-jn\pi} = \cos(-\pi n) + j\sin(-\pi n) = \cos(\pi n) = (-1)^n$*, we have:*

$$\bar{g}(jn\omega) = ET\Big\{\frac{1}{2\pi^2 n^2}\big((-1)^n - 1\big) + \frac{j}{2\pi n}\Big\} = \frac{ET}{2\pi^2 n^2}\Big\{\big((-1)^n - 1\big) + j\pi n\Big\}$$

Therefore we have the phasor of the n^{th} harmonic:

$$A_n = \frac{2}{T}\bar{g}(jn\omega) = \frac{E}{\pi^2 n^2}\Big\{(-1)^n - 1 + j\pi n\Big\}$$

From the graph, the average of $f(t)$ over one cycle is $\frac{1}{2} \times \frac{E}{2} + \frac{1}{2} \times E = \frac{3E}{4}$*. Alternatively* *we can use the integral:*

$$
\begin{aligned}
a_0 &= \frac{2}{T}\int_0^T f(t)dt = \frac{2}{T}\left\{\int_0^{T/2}\frac{2E}{T}tdt + \int_{T/2}^{T} Edt\right\} \\
&= \frac{2E}{T}\left\{\frac{2}{T}\left[\frac{1}{2}t^2\right]_0^{T/2} + \left[t\right]_{T/2}^{T}\right\} \\
&= \frac{2E}{T}\left\{\frac{2}{T}\frac{1}{2}\frac{T^2}{4} - 0 + T - \frac{T}{2}\right\} = \frac{3E}{2}
\end{aligned}
$$

And so again the DC level is $\frac{a_0}{2} = \frac{3E}{4}$.

Thus, the complex form of the Fourier Series representation of $f(t)$ *is:*

$$
\begin{aligned}
f(t) &= \frac{a_0}{2} + Re\left\{\sum_{n=1}^{\infty} A_n\, \mathrm{e}^{jn\omega t}\right\} \\
&= \frac{3E}{4} + Re\left\{\frac{E}{\pi^2}\sum_{n=1}^{\infty}\frac{\big((-1)^n - 1 + j\pi n\big)}{n^2}\,\mathrm{e}^{jn\omega t}\right\} \qquad \textit{where} \quad \omega = \frac{2\pi}{T}.
\end{aligned}
$$

To obtain the regular Fourier coefficients:

$$
a_n = \frac{-E}{\pi^2 n^2}\big(1 - \cos(\pi n)\big) = \begin{cases} \frac{-2E}{\pi^2 n^2} & \textit{for odd } n, \\ \\ 0 & \textit{for even } n, \end{cases}
$$

and

$$
b_n = -Im\{A_n\} = \frac{-E}{\pi n} \quad \forall n \in \mathbb{N}.
$$

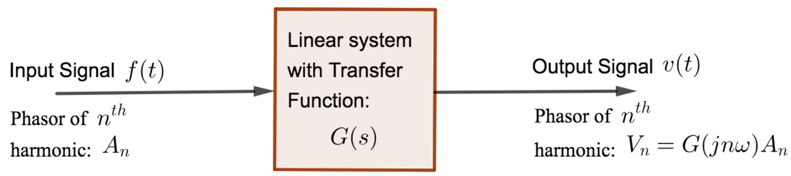
Then the real Fourier Series representation for $f(t)$ *is:*

$$
\begin{aligned}
f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \\
&= \frac{3E}{4} - \frac{2E}{\pi^2}\sum_{\textit{odd } n\in\mathbb{N}}\frac{1}{n^2}\cos\left(\frac{2\pi n t}{T}\right) - \frac{E}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2\pi n t}{T}\right)
\end{aligned}
$$

3.8 Use of Fourier Series in Linear Circuit Analysis

3.8.1 Theory

The complex form of Fourier series can be useful for analying the output voltage in linear circuits. Let $f(t)$ be a period input to a linear system. Then considering each harmonic individually, let's look at the effect the system has on the input, encoded in its transfer function $G(s)$. Then $G(jn\omega)$ is the frequency response function for the system, and can be obtained either from $G(s)$ or by analysing the system based on complex impedences.



For the n^{th} harmonic, let A_n be the input phasor to the system. Then the output phasor V_n is given by:

$$V_n = G(jn\omega)A_n$$

and so the output waveform is:

$$v(t) = G(0)\frac{a_0}{2} + Re\left\{\sum_{n=1}^{\infty} G(jn\omega)A_n\,\mathrm{e}^{jn\omega t}\right\}$$

A particular input phasor is modified by the frequency response function to produce the corresponding output phasor. Multiplying by $G(jn\omega)$ will multiply the modulus of A_n by the modulus of $G(jn\omega)$ and so produce the amplitude of the n^{th} harmonic of the output waveform. Furthermore, the phase angle (the argument) of $G(jn\omega)$ will be added to the phase angle of A_n to produce the phase angle of the n^{th} harmonic V_n of the output.

If the modulus of $G(jn\omega)$ is dependent on n then the amplitude of each harmonic may be scaled by a different factor (**amplitude distortion**). When the phase angle of $G(jn\omega)$ depends on n then each phase angle may be altered by a different amount (**phase distortion**). In most cases, both types of distortion will occur. The presence of either or both will result in the output waveform having a different shape to the input waveform.

Let,

$$A_n = r_n\,\mathrm{e}^{-j\phi_n}, \quad G(jn\omega) = g_n\,\mathrm{e}^{-j\Phi_n},$$

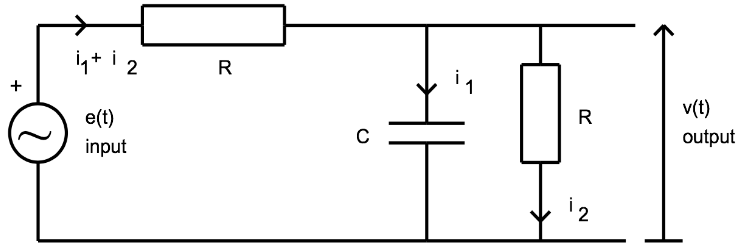
then

$$V_n = G(jn\omega)A_n = g_n r_n \,\mathrm{e}^{-j(\phi_n+\Phi_n)}$$

so the output phasor has amplitude $g_n r_n$ and phase angle $\phi_n + \Phi_n$.

3.8.2 Examples

Example 3.4. *Consider the circuit shown.*



Kirchoff's Laws yield the following equations:

$$e(t) = R(i_1(t) + i_2(t)) + Ri_2(t), \qquad \frac{1}{C}\int_0^t i_1(t)dt = Ri_2(t), \qquad v(t) = Ri_2(t)$$

where R and C are positive constants, $e(t)$ is the input signal and $v(t)$ is the output signal. Given a general input signal

$$e(t) = \frac{a_o}{2} + Re\left\{\sum_{n=1}^{\infty} A_n \,\mathrm{e}^{jn\omega t}\right\}$$

what will the Fourier Series of the corresponding output $v(t)$ be?

Taking Laplace transforms of each, we obtain:

$$\bar{e}(s) = R(\bar{i}_1 + 2\bar{i}_2), \qquad \frac{1}{sC}\bar{i}_1 = R\bar{i}_2, \qquad \bar{v}(s) = R\bar{i}_2$$

and so,

$$\begin{aligned} G(s) &= \frac{\bar{v}(s)}{\bar{e}(s)} = \frac{R\bar{i}_2}{R(\bar{i}_1 + 2\bar{i}_2)} = \frac{\bar{i}_2}{\bar{i}_1 + 2\bar{i}_2} \\ &= \frac{\bar{i}_2}{sCR\bar{i}_2 + 2\bar{i}_2} = \frac{1}{sCR + 2} \end{aligned}$$

Therefore the frequency response function is:

$$G(jn\omega) = \frac{1}{jn\omega CR + 2}.$$

Next we let

$$e_n(t) = Re\left\{A_n\, e^{jn\omega t}\right\}, \qquad v_n(t) = Re\left\{V_n\, e^{jn\omega t}\right\}$$

represent the n^{th} *harmonic of* $e(t), v(t)$ *respectively, so that* A_n, V_n *are phasors with angular frequency* $n\omega$ *satisfying:*

$$V_n = \left(\frac{1}{2 + jn\omega CR}\right) A_n$$

so that the frequency response function for the system is

$$G(jn\omega) = \frac{1}{2 + jn\omega CR}$$

For the DC level,

$$\frac{V_0}{2} = G(0)\frac{1}{2}A_0 = \frac{1}{4}a_0$$

Hence if the Fourier series of the input signal is

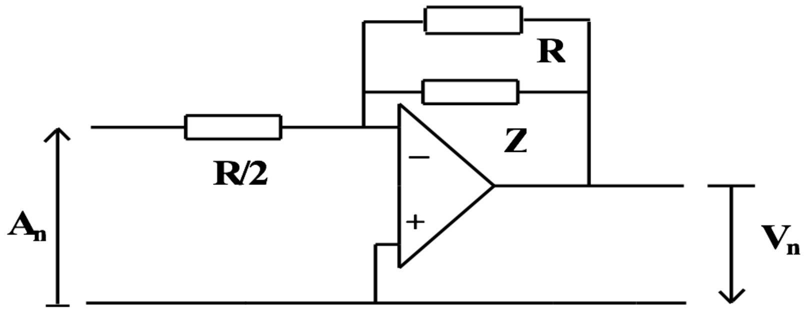
$$e(t) = \frac{a_0}{2} + Re\left\{\sum_{n=1}^{\infty} A_n\, e^{jn\omega t}\right\}$$

then we have output:

$$\begin{aligned} v(t) &= G(0)\frac{1}{2}a_0 + Re\left\{\sum_{n=1}^{\infty} G(jn\omega) A_n\, e^{jn\omega t}\right\} \\ &= \frac{a_0}{4} + Re\left\{\sum_{n=1}^{\infty} \frac{1}{2 + jn\omega CR} A_n\, e^{jn\omega t}\right\} \end{aligned}$$

Since $G(jn\omega)$ *is complex and frequency dependent we should expect both amplitude and phase distortion.*

Example 3.5. *Next consider the circuit shown.*



Analysing the complex impedences for an ideal op-amp on a dual rail supply, we would find that

$$G(jn\omega) = 2\left\{\frac{1 + jn\omega CR}{1 + 2jn\omega CR}\right\}$$

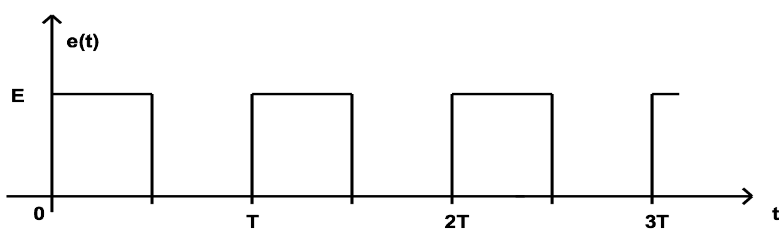
Therefore, given the input signal with Fourier series:

$$e(t) = \frac{a_0}{2} + Re\left\{\sum_{n=1}^{\infty} A_n\, \mathrm{e}^{jn\omega t}\right\}$$

the output from this system will be:

$$v(t) = a_0 + Re\left\{2\sum_{n=1}^{\infty} \frac{1 + jn\omega CR}{1 + 2jn\omega CR} A_n\, \mathrm{e}^{jn\omega t}\right\}.$$

For example, consider the pulse wave input.



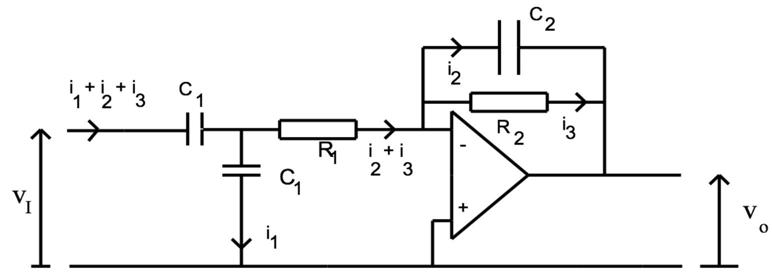
In this case,

$$A_n = a_n - jb_n = \frac{-jE}{\pi n}\big(1 - \cos(\pi n)\big) = \begin{cases} \frac{-2Ej}{\pi n} & \textit{for odd } n, \\ 0 & \textit{for even } n, \end{cases}$$

and $\frac{A_0}{2} = \frac{a_0}{2} \times G(0) = \frac{E}{2} \times 2 = E$. *So we have,*

$$v(t) = E + Re\left\{\sum_{\textit{odd } n\in\mathbb{N}} 2\left(\frac{1 + jn\omega CR}{1 + 2jn\omega CR}\right)\left(\frac{-2Ej}{\pi n}\right) \mathrm{e}^{jn\omega t}\right\}$$

Example 3.6. *Consider the filter shown.*



It is based on linear components and an ideal operational amplifier connected to a dual supply. The following equations describe the currents i_1, i_2, i_3, input voltage v_i and output voltage v_o:

$$\begin{aligned} v_i(t) &= \frac{1}{C_1}\int_0^t (i_1(t)+i_2(t)+i_3(t))dt + R_1(i_2(t)+i_3(t)) \\ \frac{1}{C_1}\int_0^t i_1(t)dt &= R_1(i_2(t)+i_3(t)) \\ R_2 i_3(t) &= \frac{1}{C_2}\int_0^t i_2(t)dt \\ v_o(t) &= -R_2 i_3(t) \end{aligned}$$

where R_1, R_2, C_1, C_2 are positive constants. Given a general input

$$v_i(t) = \frac{a_o}{2} + Re\left\{\sum_{n=1}^{\infty} A_n\, e^{jn\omega t}\right\}$$

determine the Fourier Series of the output $v_o(t)$.

Taking Laplace transforms of the circuit equations,

$$(1)\quad \bar{v}_i(s) = \frac{1}{sC_1}(\bar{i}_1+\bar{i}_2+\bar{i}_3) + R_1(\bar{i}_2+\bar{i}_3) = 0$$

$$(2)\quad \frac{1}{sC_1}\bar{i}_1 = R(\bar{i}_2+\bar{i}_3)$$

$$(3)\quad R_2\bar{i}_3 = \frac{1}{sC_2}\bar{i}_2$$

$$(4)\quad \bar{v}_o(s) = -R_2\bar{i}_3$$

Plan: i) Substitute (2) *into* (1) *to eliminate* $\bar{i}_1$*. ii) Then substitute in* (3) *to replace* $\bar{i}_2$ *with a function of* $\bar{i}_3$*. iii) Finally, use* (4) *to replace all occurrences of* $\bar{i}_3$ *with* $-\bar{v}_o/R_2$*.*

i) Using (2) *to eliminate* $\bar{i}_1$ *from* (1)*:*

$$\bar{v}_i = R_1(\bar{i}_2 + \bar{i}_3) + \frac{1}{sC_1}(\bar{i}_2 + \bar{i}_3) + R_1(\bar{i}_2 + \bar{i}_3)$$

$$\bar{v}_i = \frac{1}{sC_1}(\bar{i}_2 + \bar{i}_3) + 2R_1(\bar{i}_2 + \bar{i}_3)$$

$$\bar{v}_i = \left(\frac{1}{sC_1} + 2R_1\right)(\bar{i}_2 + \bar{i}_3)$$

$$\bar{v}_i = \frac{1}{sC_1}(1 + 2sC_1R_1)(\bar{i}_2 + \bar{i}_3) \quad (5)$$

ii) Next, rearranging (3)*, we obtain:*

$$\bar{i}_2 = sC_2R_2\bar{i}_3$$

Substituting this into (5) *to eliminate* $\bar{i}_2$*:*

$$\bar{v}_i = \frac{1}{sC_1}(1 + 2sC_1R_1)(sC_2R_2\bar{i}_3 + \bar{i}_3)$$

$$\bar{v}_i = \frac{1}{sC_1}(1 + 2sC_1R_1)(1 + sC_2R_2)\bar{i}_3 \quad (6)$$

iii) Finally, we rearrange (4) *to obtain:*

$$\bar{i}_3 = \frac{-1}{R_2}\bar{v}_o$$

And substitute this into (6)*, eliminating* $\bar{i}_3$ *and thus getting a relationship between* $\bar{v}_i$ *and* $\bar{v}_o$*:*

$$\bar{v}_i = \frac{-1}{sC_1R_2}(1 + 2sC_1R_1)(1 + sC_2R_2)\bar{v}_o$$

Rearranging, we obtain the transfer function of the system:

$$G(s) = \frac{\bar{v}_o}{\bar{v}_i} = \frac{-sC_1R_2}{(1 + 2sC_1R_1)(1 + sC_2R_2)}.$$

The characteristic equation $(1 + 2sC_1R_1)(1 + sC_2R_2) = 0$ *yields two solutions for* s*:*

$$s = \frac{-1}{2C_1R_1}, \quad \textit{and} \quad s = \frac{-1}{C_2R_2}$$

Thus for both solutions $Re\{s\} < 0$ and so this system is a stable second-order band-pass filter.

Then to obtain the frequency response function $G(jn\omega)$, simply replace s with $jn\omega$ in the transfer function $G(s)$.

$$G(jn\omega) = \frac{-jn\omega C_1 R_2}{(1 + 2jn\omega C_1 R_1)(1 + jn\omega C_2 R_2)}.$$

Therefore, if the Fourier series for the input wave is:

$$v_{in}(t) = \frac{a_0}{2} + Re\left\{ \sum_{n=1}^{\infty} A_n \, \mathrm{e}^{jn\omega t} \right\}, \qquad \omega = \frac{2\pi}{T},$$

then the corresponding Fourier series for the output is:

$$\begin{aligned} v_{out}(t) &= \frac{a_0}{2} G(0) + Re\left\{ \sum_{n=1}^{\infty} G(jn\omega) A_n \, \mathrm{e}^{jn\omega t} \right\} \\ &= Re\left\{ \sum_{n=1}^{\infty} \frac{-jn\omega C_1 R_2}{(1 + 2jn\omega C_1 R_1)(1 + jn\omega C_2 R_2)} A_n \, \mathrm{e}^{jn\omega t} \right\}, \qquad \omega = \frac{2\pi}{T}. \end{aligned}$$

Note that in this case, the DC response is $G(0) = 0$.

4 Matrices

4.1 Matrix Algebra

Let A be an $m \times n$ matrix. This means that A has m rows and n columns of entries.

4.1.1 Addition and Subtraction

In order to add or subtract two matrices, they *must* have exactly the same dimensions (both the same number of rows, and the same number of columns).

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & f+h \end{pmatrix}$$

4.1.2 Matrix Multiplication

Matrix multiplication is a **non-commutative** operation. This means that $A \times B$ is *not* equivalent to $B \times A$ and does not necessarily yield the same result.

Performing matrix multiplication involves the method of multiplying the entries of the rows of the first matrix by the entries in the columns of the second matrix. For this to be possible, the number of columns of the first matrix *must* equal the number of rows of the second matrix. The dimensions of both also tell you the dimensions of the resulting matrix. In particular, if A is an $m_1 \times n_1$ matrix, and B is an $m_2 \times n_2$ matrix, then we can perform $A \times B$ if and only if $n_1 = m_2$, and the result will be a $m_1 \times n_2$ matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

4.1.3 Scalar Multiplication

To multiply a matrix by a scalar, simply multiply each entry by that scalar.

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

4.1.4 Transpose

To obtain the transpose of a matrix A (denoted A^T), swap the rows and columns (or reflect all of the elements about the diagonal).

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

4.1.5 Determinants

Square matrices (with dimensions $n \times n$) have a property called the **determinant**. This represents the scaling factor when the matrix is used to transform an image. The determinant of matrix A can be denoted by $det(A)$ or $|A|$.

4.1.5.1 Determinant of a 2×2 Matrix

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is very simple to calculate by multiplying the diagonal entries:

$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 4.1. *Given the square matrix*

$$A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$$

The determinant is given by:

$$det(A) = 3 \times 2 - (-1) \times 4 = 6 + 4 = 10$$

4.1.5.2 Determinant of a 3×3 Matrix

For a 3×3 matrix, select a row or column (any will suffice, but we usually use the top row) and multiply each of its entries by the determinant of the corresponding 2×2 co-matrix consisting of the rows and columns that the current entry is *not* in, and then also multiply by a positive or negative sign according to the checkerboard pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}.$$

Therefore, given a 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, choosing the top row:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Example 4.2. *Find the determinant of*

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} det(A) &= \begin{vmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} \\ &= 3\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} - (0)\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 3\big(0 \times 1 - (-2) \times 1\big) - 0 + 2\big(2 \times 1 - 0 \times 0\big) \\ &= 3(0 + 2) + 2(2 - 0) \\ &= 3 \times 2 + 2 \times 2 \\ &= 10 \end{aligned}$$

4.1.6 The Identity Matrix

For each positive integer n, the $n \times n$ **identity matrix** consists of one's on the diagonal entries and zeroes elsewhere. That is,

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

and this is the only matrix which satisfies, for a matrix A of appropriate dimensions,

$$AI = IA = A$$

So the identity acts like a matrix version of "1" in the real numbers.

4.1.7 Inverse

If a square matrix A has *non-zero determinant*, then there exists a unique matrix A^{-1} with the same dimensions such that

$$AA^{-1} = A^{-1}A = I$$

4.1.7.1 Inverse of a 2×2 Matrix

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with non-zero determinant $(ad - bc \neq 0)$, the inverse is given by:

$$A^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example 4.3. *Given the square matrix*

$$A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$$

The inverse is given by:

$$A^{-1} = \frac{1}{3 \times 2 - (-1) \times 4} \begin{pmatrix} 2 & -(-1) \\ -4 & 3 \end{pmatrix} = \frac{1}{6+4} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1/5 & 1/10 \\ -2/5 & 3/10 \end{pmatrix}$$

We can check that we have obtained the correct answer by checking that $AA^{-1} = I$:

$$\begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix} \times \frac{1}{10} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 3 \times 2 + (-1) \times (-4) & 3 \times 1 + (-1) \times 3 \\ 4 \times 2 + 2 \times (-4) & 4 \times 1 + 2 \times 3 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and similarly $A^{-1}A = I$.

4.2 Eigenvalues and Eigenvectors

4.2.1 Motivation

In computer aided design (CAD), a graphical model of a physical object can be manipulated by applying a linear transformation to the co-ordinates of each mesh point in a wire-frame diagram. This can be expressed in general matrix form as:

$$\underline{\mathbf{y}} = A\underline{\mathbf{x}}$$

where $\underline{\mathbf{x}}$ is the initial co-ordinate of a point, $\underline{\mathbf{y}}$ is the co-ordinate it gets mapped to after the manipulation, and A encodes the action of the transformation. We use matrix multiplication to determine $\underline{\mathbf{y}}$.

Example 4.4. *Consider a two-dimensional graphical model to which the following transformation matrix is applied:*

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}.$$

Under the action of this transformation, calculate what happens to points with various co-ordinates:

$$i)\ \underline{\boldsymbol{x}}_1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \quad ii)\ \underline{\boldsymbol{x}}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad iii)\ \underline{\boldsymbol{x}}_3 = \begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix}$$

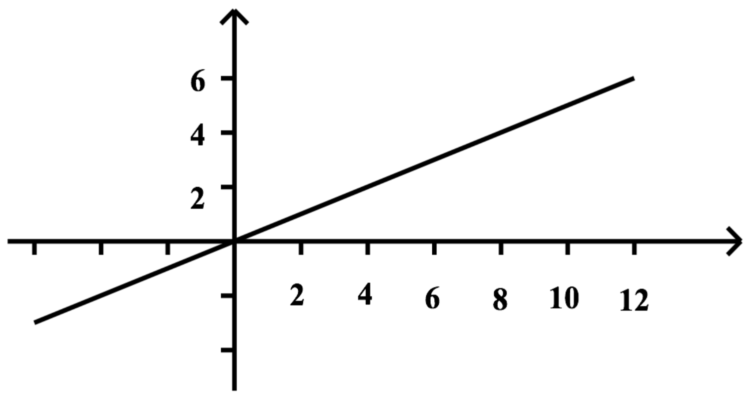
Solution:

$$i)\ \underline{\boldsymbol{y}}_1 = A\underline{\boldsymbol{x}}_1 = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 15-8 \\ 6-8 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}.$$

$$ii)\ \underline{\boldsymbol{y}}_2 = A\underline{\boldsymbol{x}}_2 = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10+2 \\ 4+2 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix} = 6\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$$iii)\ \underline{\boldsymbol{y}}_3 = A\underline{\boldsymbol{x}}_3 = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 10\alpha+2\alpha \\ 4\alpha+2\alpha \end{pmatrix} = \begin{pmatrix} 12\alpha \\ 6\alpha \end{pmatrix} = 6\begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix} = 6\alpha\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

In the second and third cases, the output vector has the same direction as the input vector. In particular, any vector with the same direction as these will retain its direction *after the action of A (i.e. these are vectors for which the action of A does* not *rotate them). The magnitude of the vector may change, but the direction is preserved.*



Graphically, any point which lies on the line shown will be mapped to a point on the same line after the transformation. This line could be regarded as a "natural direction" or "natural axis" of the transformation.

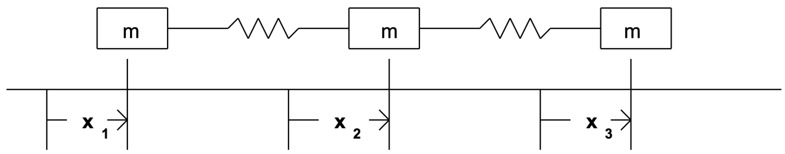
The question is, how many such axes are there for a general transformation matrix A, and how can we determine them systematically? Furthermore, for vectors that lie on these axes, how will their magnitude (for points, their distance from the origin) be affected by the action of A? This is the eigenvalue and eigenvector problem.

In particular, we want to find the scalar values λ (called **eigenvalues**) and associated column vectors $\underline{\mathbf{x}}$ (called **eigenvectors**) such that:

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}}$$

To see why these quantities are useful, we will now look at two different physical problems, where the solution can be found from the eigenvalues and eigenvectors of a matrix associated with the problem.

Example 4.5 (Harmonic Oscillators). *Consider three objects, each of mass m, coupled as shown by springs of stiffness (force constant) k. Let x_1, x_2, x_3 represent each object's displacement from equilibrium.*



It can be shown (by combining Hooke's law and Newton's Second Law) that the motion of these three objects can be modelled by the following set of second-order ordinary differential equations:

$$\begin{aligned} m\ddot{x}_1 &= k(x_2 - x_1) \\ m\ddot{x}_2 &= -k(x_2 - x_1) + k(x_3 - x_2) \\ m\ddot{x}_3 &= -k(x_3 - x_2) \end{aligned}$$

We want to represent this set of ODEs as a matrix problem. To see this, first rewrite the equations:

$$\begin{aligned} m\ddot{x}_1 &= -k\big(x_1 - x_2 + 0x_3\big) \\ m\ddot{x}_2 &= -k\big(-x_1 + 2x_2 - x_3\big) \\ m\ddot{x}_3 &= -k\big(0x_1 - x_2 + x_3\big) \end{aligned}$$

We can write these equations as the rows of matrices, and then separate the right-hand-side by constructing a matrix A of the coefficients of x_1, x_2, x_3:

$$\begin{pmatrix} m\ddot{x}_1 \\ m\ddot{x}_2 \\ m\ddot{x}_3 \end{pmatrix} = \begin{pmatrix} -k\big(x_1 - x_2 + 0x_3\big) \\ -k\big(-x_1 + 2x_2 - x_3\big) \\ -k\big(0x_1 - x_2 + x_3\big) \end{pmatrix} = -k \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Thus, we obtain the equivalent matrix equation for this problem:

$$m\ddot{\underline{\boldsymbol{x}}} = -kA\underline{\boldsymbol{x}}, \quad \textit{where} \quad \underline{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \ddot{\underline{\boldsymbol{x}}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} \quad \textit{and} \quad A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Suppose we seek a solution of the form $\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}\cos(\omega t)$ *with constants* ω *and* $\underline{\boldsymbol{b}} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

That is, $x_i = b_i \cos(\omega t)$ *for* $i = 1, 2, 3$, *so each of the masses oscillates with the same frequency but potentially with different amplitudes.*

Twice differentiating this equation for $\underline{\boldsymbol{x}}$, *we obtain:*

$$\dot{\underline{\boldsymbol{x}}} = -\omega \underline{\boldsymbol{b}} \sin(\omega t)$$

$$\ddot{\underline{\boldsymbol{x}}} = -\omega^2 \underline{\boldsymbol{b}} \cos(\omega t) = -\omega^2 \underline{\boldsymbol{x}}.$$

Using this relationship and the initial statement $m\ddot{\underline{\boldsymbol{x}}} = -kA\underline{\boldsymbol{x}}$, *we obtain:*

$$A\underline{\boldsymbol{x}} = \frac{-m}{k}\ddot{\underline{\boldsymbol{x}}} = \frac{m\omega^2}{k}\underline{\boldsymbol{x}},$$

and so

$$A\underline{\boldsymbol{b}}\cos(\omega t) = \frac{m\omega^2}{k}\underline{\boldsymbol{b}}\cos(\omega t)$$

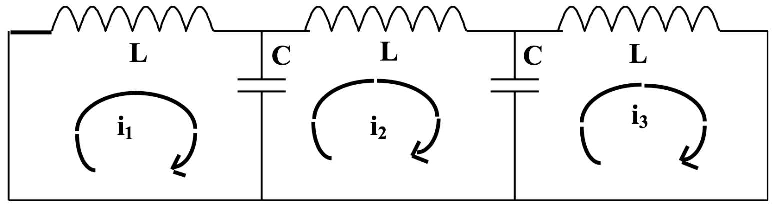
$$\therefore \quad A\underline{\boldsymbol{b}} = \left(\frac{m\omega^2}{k}\right)\underline{\boldsymbol{b}}, \quad \text{where} \quad A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Therefore the values of $\lambda = \frac{m\omega^2}{k}$ *which satisfy this equation are the eigenvalues of matrix* A. *They give the frequencies of oscillatory (harmonic) motion, and the corresponding eigenvectors give the amplitudes of this motion* b_1, b_2, b_3. *So if the problem was to determine the possible harmonic frequencies of this vibrating system, they could be found by:*

$$\omega = \sqrt{\frac{\lambda k}{m}}, \qquad \text{where } \lambda \text{ are the eigenvalues of matrix } A.$$

Solving the eigenvalues and eigenvectors of A *will thus give us* $\underline{\boldsymbol{b}}$ *and* λ *from which we can obtain* ω. *Therefore we will gain both pieces of information required to fully specify the solution* $\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}\cos(\omega t)$ *and thus fully-understand how this system behaves.*
There are three frequencies of oscillatory motion and hence three modes of vibration for this system. In a later section we will learn how to determine this.

Example 4.6 (Circuits)**.** *Consider the electronic circuit shown.*



It has the following equations associated with it, whcih define the three electric currents:

$$L\frac{di_1}{dt} + \frac{1}{C}\int_0^t (i_1 - i_2)dt + E_1 = 0$$

$$\frac{1}{C}\int_0^t (i_2 - i_1)dt - E_1 + L\frac{di_2}{dt} + \frac{1}{C}\int_0^t (i_2 - i_3)dt + E_2 = 0$$

$$\frac{1}{C}\int_0^t (i_3 - i_2)dt - E_2 + L\frac{di_3}{dt} = 0$$

where E_1 and E_2 are the initial potentials on the capacitors.

Differentiating each of these equations with respect to t, we obtain:

$$L\frac{d^2 i_1}{dt^2} + \frac{1}{C}(i_2 - i_1) = 0$$

$$L\frac{d^2 i_2}{dt^2} + \frac{1}{C}\big((i_2 - i_1) - (i_2 - i_3)\big) = 0$$

$$L\frac{d^2 i_3}{dt^2} + \frac{1}{C}(i_3 - i_2) = 0$$

and this can be represented in matrix form by:

$$\mathcal{L}\frac{d^2 \underline{\boldsymbol{i}}}{dt^2} = -\frac{1}{C}A\underline{\boldsymbol{i}}, \quad \text{where} \quad \underline{\boldsymbol{i}} = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

and L and C are scalar constants.

Letting $\underline{\boldsymbol{i}} = \underline{\boldsymbol{b}}e^{j\omega t}$, we can obtain $A\underline{\boldsymbol{b}} = \lambda\underline{\boldsymbol{b}}$ where $\lambda = \omega^2 LC$. This situation is therefore mathematically very similar to the previous example, however the practical interpretation of the eigenvalues is different.

4.2.2 Method: Evaluation of Eigenvalues and Eigenvectors

Given a square matrix, we will now consider how we go about finding these eigenvalues and eigenvectors.

Consider a square matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

When this matrix acts on a column vector $\underline{\mathbf{x}}$, we obtain a new vector $A\underline{\mathbf{x}}$ that may be stretched and rotated in some way. We want to find the solutions to

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}},$$

where λ is a scalar, so that matrix multiplication by A preserves the direction of the vector $\underline{\mathbf{x}}$.

For a 2×2 matrix A, there are two such eigenvalues $\lambda = \lambda_1, \lambda_2$ and their associated eigenvectors $\underline{\mathbf{x}} = \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2$. Note that if $\underline{\mathbf{e}}_i$ is an eigenvector of A with eigenvalue λ_i, then so is *any scalar multiple* of $\underline{\mathbf{e}}_i$, so we can obtain a direction for the eigenvector, and then a vector in that direction of any magnitude will suffice.

First, we re-arrange the equation to obtain $(A - \lambda I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$, where I is the **identity matrix** that has 1's on the diagonal entries and 0's elsewhere,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then in order to find the non-trivial solutions (i.e. excluding $\underline{\mathbf{x}} = \underline{\mathbf{0}}$) we determine the eigenvalues by calculating the determinant of $A - \lambda I$ and solving the values of λ for which this is zero[1]. That is, we solve:

Characteristic equation of matrix A: $\quad |A - \lambda I| = 0$

This will give us the **characteristic polynomial** (or characteristic equation) of A, and for a 2×2 matrix will be a quadratic equation. Solving this gives a pair of roots $\lambda = \lambda_1, \lambda_2$. For each of these, we can then obtain a corresponding non-zero eigenvector $\underline{\mathbf{x}} = \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2$ by solving

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \quad \text{or} \quad (A - \lambda I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

for $\underline{\mathbf{x}}$.

[1] This is because the existance of such a vector is the same as the matrix $A - \lambda I$ being "singular", or *not* invertible, which is equivalent to having determinant zero.

4.2.3 Examples of calculating Eigenvalues and Eigenvectors

Example 4.7 $(2 \times 2\,\text{matrix})$**.**

$$\textit{Let } A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix},$$

then

$$A - \lambda I = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{pmatrix}.$$

Therefore, we wish to solve

$$|A - \lambda I| = 0 \quad \Longrightarrow \quad \begin{vmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{vmatrix} = 0.$$

Since this is a 2×2 *matrix, we find the determinant by taking the difference of the product of the diagonals:*

$$(1-\lambda)(-4-\lambda) - (2)(3) = 0,$$

and so

$$\lambda^2 + 3\lambda - 10 = 0 \qquad \textit{(The characteristic polynomial of matrix A.)}$$

Solving this quadratic equation yields two distinct, real, integer roots:

$$\lambda_1 = -5, \quad \lambda_2 = 2. \qquad \textit{These are the eigenvalues of A.}$$

Next, we solve the eigenvectors one at a time. For the first eigenvalue, $\lambda_1 = -5$*, let's call the corresponding eigenvector* $\underline{\boldsymbol{e}}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$*. To find the values of the components* x *and* y*, we need to solve:*

$$A\underline{\boldsymbol{e}}_1 = -5\underline{\boldsymbol{e}}_1$$

which means

$$\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -5 \begin{pmatrix} x \\ y \end{pmatrix}$$

This yields a pair of simultaneous equations:

$$\begin{aligned} x + 2y &= -5x \\ 3x - 4y &= -5y \end{aligned}$$

These are linearly dependent (i.e. they are the same equation, just rearranged in different ways), and solving either of them gives $y = -3x$*. If we choose* $x = 1$ *(and we can, since recall that* any *scalar multiple of the eigenvector will work, so the important property to preserve is the* relative *values of the two components), then we will get* $y = -3$*, and so one eigenvector corresponding to* $\lambda_1 = -5$ *is:*

$$\underline{\boldsymbol{e}}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Similarly, the second eigenvector (corresponding to eigenvalue $\lambda_2 = 2$*) is*

$$\underline{\boldsymbol{e}}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Check this as an exercise.

Example 4.8 (3×3 matrix)**.** *Consider the 3×3 matrix A that has appeared Examples 4.5 and 4.6. We will calculate the three eigenvalues and associated eigenvectors for this matrix.*

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then $|A - \lambda I| = 0$ gives:

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

Recall the method of calculating determinants of 3×3 matrices. We select a row or column (any will suffice, but we usually use the top row) and multiply each of its entries by the determinant of the corresponding 2×2 co-matrix consisting of the rows and columns that the current entry is not *in, and then also multiply by a positive or negative sign according to the pattern:*

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}.$$

The resulting terms are summed to obtain the determinant.

Hence, in this case (using the top row) we have:

$$(1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} -1 & 2-\lambda \\ 0 & -1 \end{vmatrix} = 0$$

$$(1-\lambda)\big((2-\lambda)(1-\lambda) - (-1)(-1)\big) + \big((-1)(1-\lambda) - (-1)(0)\big) = 0$$

$$(1-\lambda)\big((2-\lambda)(1-\lambda) - 1\big) - (1-\lambda) = 0$$

$$(1-\lambda)\big((2-\lambda)(1-\lambda) - 2\big) = 0$$

$$(1-\lambda)(\lambda^2 - 3\lambda + 2 - 2) = 0$$

$$(1-\lambda)(\lambda)(\lambda - 3) = 0$$

Hence there are three eigenvalues: $\lambda = 0, 1, 3$.

> ***Note:*** *In this case, we have successfully factorised by keeping out the common factor of* $(\lambda - 1)$*. In general, you are not expected to solve cubic equations. Therefore, you may be given the eigenvalues and asked to* ***verify*** *them. This means that you must obtain the characteristic polynomial, and then show that substituting in the proposed value of the eigenvalue* λ *satisfies the equation.*
>
> *e.g. If we had multiplied out the characteristic polynomial to obtain* $\lambda^3 - 4\lambda^2 + 3\lambda = 0$*, we could verify that* $\lambda = 3$ *is an eigenvalue in the following way:*
>
> $$(3)^3 - 4(3)^2 + 3(3) = 27 - 4 \times 9 + 9 = 27 - 36 + 9 = 0$$

Now we must obtain the eigenvectors which correspond to each of these. For a general value of λ *and a corresponding eigenvector* $\underline{\boldsymbol{x}}$*, the equation* $(A - \lambda I)\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ *gives:*

$$\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From the components of this we obtain the following three simultaneous equations:

$$(1-\lambda)x_1 - x_2 = 0$$

$$-x_1 + (2-\lambda)x_2 - x_3 = 0$$

$$-x_2 + (1-\lambda)x_3 = 0$$

Hence,

i) For the first eigenvalue $\lambda_1 = 0$*:*

$$x_1 - x_2 = 0 \quad (E_1)$$

$$-x_1 + 2x_2 - x_3 = 0 \quad (E_2)$$

$$-x_2 + x_3 = 0 \quad (E_3)$$

From (E_1) *we have* $x_1 = x_2$*, and from* (E_3) *we obtain* $x_3 = x_2$*. This is all we need to do in this case, but we can check the consistency of the equations by using* $(E_4 = E_2 + E_1)$ *to eliminate* x_1*:*

$$x_2 - x_3 = 0 \quad (E_4)$$

and so we see that (E_4) *is just the same as* (E_3)*.*
Then we let $x_2 = \alpha$*, where* α *is just an arbitrary constant, and use the previous results to obtain both other co-ordinates in terms solely of* α*:*

$$x_1 = x_2 = \alpha, \quad \text{and} \quad x_3 = x_2 = \alpha.$$

Hence we have the eigenvector

$$\underline{\boldsymbol{x}}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

As before, this represents an infinite set of eigenvectors that all have the same direction but can be of any non-zero magnitude.

ii) $\lambda_2 = 1$*:*

$$\begin{aligned} -x_2 \qquad &= 0 \quad (E_1) \\ -x_1 + x_2 - x_3 &= 0 \quad (E_2) \\ -x_2 \qquad &= 0 \quad (E_3) \end{aligned}$$

Clearly (E_1) *and* (E_3) *are identical and give* $x_2 = 0$*.*
Substituting this result into (E_2) *then yields* $x_3 = -x_1$ *or* $x_1 = -x_3$*.*
Therefore let $x_1 = \beta$ *(an arbitrary constant) to obtain:*

$$\underline{\boldsymbol{x}}_2 = \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

iii) $\lambda_3 = 3$*:*

$$\begin{aligned} -2x_1 - x_2 \qquad &= 0 \quad (E_1) \\ -x_1 - x_2 - x_3 &= 0 \quad (E_2) \\ - x_2 - 2x_3 &= 0 \quad (E_3) \end{aligned}$$

Eliminate x_1 from (E_2) using $(E_4) = 2(E_2) - (E_1)$:

$$-x_2 - 2x_3 = 0 \quad (E_4)$$

and then eliminate x_2 from (E_3) using $(E_5) = (E_3) - (E_4)$. As expected, this results in the tautology:

$$0 = 0 \quad (E_5)$$

Then from (E_4): $x_2 = -2x_3$, and from (E_1): $x_1 = -\frac{1}{2}x_2 = x_3$. Therefore let $x_3 = \gamma$, and we obtain $x_2 = -2\gamma$ and $x_1 = \gamma$, so that the eigenvector is:

$$\underline{\boldsymbol{e}}_3 = \gamma \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The complete solution to the problem is therefore:

$$\lambda_1 = 0, \ \ \underline{\boldsymbol{e}}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \qquad \lambda_2 = 1, \ \ \underline{\boldsymbol{e}}_2 = \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \qquad \lambda_3 = 3, \ \ \underline{\boldsymbol{e}}_3 = \gamma \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

You can verify the solutions by calculating the matrix multiplications $A\underline{\boldsymbol{e}}_1$, $A\underline{\boldsymbol{e}}_2$, $A\underline{\boldsymbol{e}}_3$ and checking that we get the product of the corresponding eigenvalue and eigenvector each time.

4.2.4 Calculating using MATLAB

MATLAB can automatically determine the eigenvalue and eigenvector pairs of a square matrix for you, using the Symbolic Math Toolbox:

```
A = [1 2; 3 -4];

B = sym(A);

[M,S] = eig(B);
```

4.2.5 Unit Vectors

It is sometimes useful to consider "normalised" or "unit" vectors. These have magnitude (size) equal to 1. To normalise a vector, we find its magnitude and then divide the vector by this scalar value. The direction is unchanged, so the components in each direction will still have the same ratio.

A unit vector is one that has magnitude equal to one. Given any vector, $\underline{\mathbf{v}}$ we can find a unit vector in the same direction by:

$$\hat{\underline{\mathbf{v}}} = \frac{\underline{\mathbf{v}}}{|\underline{\mathbf{v}}|}$$

Example 4.9. *Consider the eigenvectors we found in Example 4.7. The first eigenvector is:*

$$\underline{\boldsymbol{e}}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Therefore it has magnitude:

$$|\underline{\boldsymbol{e}}_1| = \sqrt{(1)^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$$

So a unit vector in the same direction as $\underline{\boldsymbol{e}}_1$, *which we denote by* $\hat{\underline{\mathbf{e}}}_1$ *is:*

$$\hat{\underline{\boldsymbol{e}}}_1 = \frac{\underline{\boldsymbol{e}}_1}{|\underline{\boldsymbol{e}}_1|} = \frac{1}{\sqrt{(1)^2 + (-3)^2}} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Similarly, the second normalised eigenvector is

$$\hat{\underline{\boldsymbol{e}}}_2 = \frac{\underline{\boldsymbol{e}}_2}{|\underline{\boldsymbol{e}}_2|} = \frac{1}{\sqrt{(2)^2 + (1)^2}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

4.3 State Variable Description

Next we look at representing the control equations for a system in a form to which we can then apply the above matrix and eigenvalue analysis. In particular, we use a technique known as state-space representation.

4.3.1 Theory

- The goal here is to: **reduce ordinary differential equation (ODE) initial value problems of order n to a set of n first-order ODEs.**
- We accomplish this by defining a new set of variables called **state variables**. Every variable in the original problem, *apart from external inputs*, generates an additional state variable for each derivative. For example, a variable whose second-derivative occurs in the equations will generate two state variables.
- We then rearrange to obtain a set of equations for the derivative of each state variable, in terms only of the state variables and the external inputs.
  This is the **state variable description** of the system:

  $$\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}} + B\underline{\mathbf{u}}$$

  where $\underline{\dot{\mathbf{x}}}$ is a vector containing the time-derivatives of the state variables, $\underline{\mathbf{x}}$ is a vector containing the state variables themselves, and the external control inputs are encoded in their own vector $B\underline{\mathbf{u}}$.
- If there is no control input for the problem, the state variable description is just:

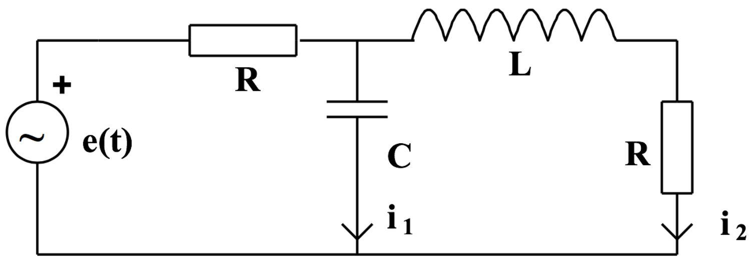
  $$\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}}$$

- We may also choose to define some output measurements, and then define an output vector $\underline{\mathbf{y}}$ which would usually encode both the control inputs and the measured outputs, constructed from a linear combination of the state variable and control input vectors in a similar manner to the state variable description:

  $$\underline{\mathbf{y}} = C\underline{\mathbf{x}} + D\underline{\mathbf{u}}$$

4.3.2 Examples

Example 4.10. *Determine the state variable description of the circuit shown.*



The currents in this circuit are determined by solving a pair of ODEs:

$$\begin{aligned}\frac{di_1}{dt} &= -\frac{i_1}{CR} - \frac{di_2}{dt} + \frac{1}{R}\frac{de}{dt} \\ \frac{d^2 i_2}{dt^2} &= -\frac{R}{L}\frac{di_2}{dt} + \frac{i_1}{LC}\end{aligned}$$

where L, C and R are positive constants and $e(t)$ is an external input.

This description involves both first and second derivatives, and is in effect a third-order system. We can introduce new dependent variables and reformulate the equations, so that the circuit can be modelled using first derivatives only.

Dependent variable i_1 is differentiated once in the above equations, so it generates one state variable.
Dependent variable i_2 is differentiated twice, and so generates two state variables.

Therefore we define:

$$x_1 \equiv i_1, \qquad x_2 \equiv i_2, \qquad x_3 \equiv \frac{di_2}{dt}$$

and differentiating these:

$$\frac{dx_1}{dt} = \frac{di_1}{dt}, \qquad \frac{dx_2}{dt} = \frac{di_2}{dt}, \qquad \frac{dx_3}{dt} = \frac{d^2 i_2}{dt^2}.$$

Then we want to obtain a set of equations for the derivative of each state variable, in terms only of the state variables, and not featuring any

derivatives.

Using the original equations, we obtain:

$$\begin{aligned}
\frac{dx_1}{dt} &= -\frac{x_1}{CR} - x_3 + \frac{1}{R}\frac{de}{dt} \\
\frac{dx_2}{dt} &= x_3 \\
\frac{dx_3}{dt} &= -\frac{R}{L}x_3 + \frac{x_1}{LC}
\end{aligned}$$

Any set of linear ODEs can be manipulated into this form, called ***canonical form****, of first order ODEs written explicity as*

$$\frac{dx_i}{dt} = f(x_1, x_2, x_3)$$

To see how we can represent this set of equations in matrix form, and obtain the decomposition in terms of a state variable vector and a control vector, we need to rewrite them with the coefficients of all state variables (including zeros) in the correct order (x_1, then x_2, then x_3):

$$\begin{aligned}
\frac{dx_1}{dt} &= -\frac{1}{CR}x_1 + 0x_2 - x_3 + 1 \times \frac{1}{R}\frac{de}{dt} \\
\frac{dx_2}{dt} &= 0x_1 + 0x_2 + 1x_3 + 0 \times \frac{1}{R}\frac{de}{dt} \\
\frac{dx_3}{dt} &= \frac{x_1}{LC} + 0x_2 - \frac{R}{L}x_3 + 0 \times \frac{1}{R}\frac{de}{dt}
\end{aligned}$$

Then putting these equations as the rows of a matrix and decomposing:

$$\begin{aligned}
\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} &= \begin{pmatrix} -\frac{1}{CR}x_1 + 0x_2 - x_3 + 1 \times \frac{1}{R}\frac{de}{dt} \\ 0x_1 + 0x_2 + 1x_3 + 0 \times \frac{1}{R}\frac{de}{dt} \\ \frac{x_1}{LC} + 0x_2 - \frac{R}{L}x_3 + 0 \times \frac{1}{R}\frac{de}{dt} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{CR}x_1 + 0x_2 - x_3 \\ 0x_1 + 0x_2 + 1x_3 \\ \frac{x_1}{LC} + 0x_2 - \frac{R}{L}x_3 \end{pmatrix} + \begin{pmatrix} 1 \times \frac{1}{R}\frac{de}{dt} \\ 0 \times \frac{1}{R}\frac{de}{dt} \\ 0 \times \frac{1}{R}\frac{de}{dt} \end{pmatrix} \\
&= \begin{pmatrix} \frac{-1}{CR} & 0 & -1 \\ 0 & 0 & 1 \\ \frac{1}{LC} & 0 & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \left(\frac{1}{R}\frac{de}{dt}\right)
\end{aligned}$$

Which can be stated as:

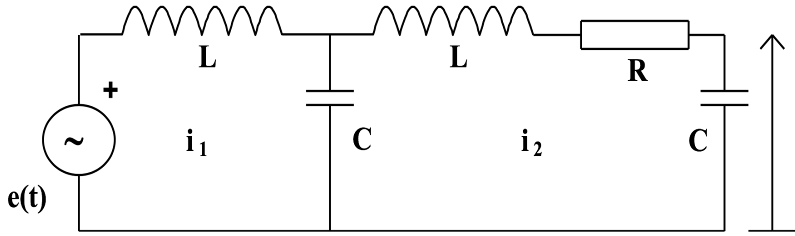
$$\dot{\underline{\boldsymbol{x}}} = A\underline{\boldsymbol{x}} + B\underline{\boldsymbol{u}}$$

where

$$\underline{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} -\frac{1}{CR} & 0 & -1 \\ 0 & 0 & 1 \\ \frac{1}{LC} & 0 & -\frac{R}{L} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{\boldsymbol{u}} = \begin{pmatrix} \frac{1}{R}\frac{de}{dt} \end{pmatrix}$$

The column vector $\underline{\boldsymbol{x}}$ *is the vector of* ***state variables****, and* $\dot{\underline{\boldsymbol{x}}} = A\underline{\boldsymbol{x}} + B\underline{\boldsymbol{u}}$ *is the* ***state variable description*** *or* ***state variable equations*** *of the electronic system. In particular,* $A\underline{\boldsymbol{x}}$ *can represent the internal feedback (or internal control) of the system, while* $B\underline{\boldsymbol{u}}$ *encodes the single external "control".*

Example 4.11. *Determine the state variable description of the following circuit:*



It has the following set of equations. R, C, and L are constants, whilst $e(t)$, $v(t)$, $i_1(t)$ and $i_2(t)$ are time-dependent variables.

$$e(t) = L\frac{di_1(t)}{dt} + \frac{1}{C}\int_0^t (i_1(t) - i_2(t))dt$$

$$\frac{1}{C}\int_0^t \big(i_2(t) - i_1(t)\big)dt + Ri_2(t) + L\frac{di_2(t)}{dt} + \frac{1}{C}\int_0^t i_2(t)dt = 0$$

$$v(t) = \frac{1}{C}\int_0^t i_2(t)dt$$

and the output is given by:

$$z = L\frac{di_2}{dt}$$

Before introducing the state variables, differentiate these equations to remove the integrals. This yields:

$$\frac{de}{dt} = L\frac{d^2 i_1}{dt^2} + \frac{1}{C}(i_1 - i_2)$$

$$\frac{1}{C}(i_2 - i_1) + R\frac{di_2}{dt} + L\frac{d^2 i_2}{dt^2} + \frac{1}{C}i_2 = 0$$

$$\frac{dv}{dt} = \frac{1}{C}i_2$$

Then rearranging,

$$\begin{aligned}
\frac{d^2 i_1}{dt^2} &= -\frac{1}{LC}i_1 + \frac{1}{LC}i_2 + \frac{1}{L}\frac{de}{dt} \\
\frac{d^2 i_2}{dt^2} &= \frac{1}{LC}i_1 - \frac{2}{LC}i_2 - \frac{R}{L}\frac{di_2}{dt} \\
\frac{dv}{dt} &= \frac{1}{C}i_2
\end{aligned}$$

The dependent variables i_1 and i_2 are each differentiated twice, and so generate two state variables each. v is differentiated only once, so it generates an additional one state variable. Therefore, we define the five state variables $x_1, \ldots, x_5$:

$$x_1 \equiv i_1, \qquad x_2 \equiv \frac{di_1}{dt}, \qquad x_3 \equiv i_2, \qquad x_4 \equiv \frac{di_2}{dt}, \qquad x_5 \equiv v.$$

We differentiate each of these equations,

$$\frac{dx_1}{dt} = \frac{di_1}{dt}, \qquad \frac{dx_2}{dt} = \frac{d^2 i_1}{dt^2}, \qquad \frac{dx_3}{dt} = \frac{di_2}{dt}, \qquad \frac{dx_4}{dt} = \frac{d^2 i_2}{dt^2}, \qquad \frac{dx_5}{dt} = \frac{dv}{dt}$$

and then substitute in the original equations to obtain equations for $\frac{dx_i}{dt}$ in terms of x_i:

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= -\frac{1}{LC}x_1 + \frac{1}{LC}x_3 + \frac{1}{L}\frac{de}{dt} \\
\frac{dx_3}{dt} &= x_4 \\
\frac{dx_4}{dt} &= \frac{1}{LC}x_1 - \frac{2}{LC}x_3 - \frac{R}{L}x_4 \\
\frac{dx_5}{dt} &= \frac{1}{C}x_3
\end{aligned}$$

and we have $z = Lx_4$ as a measured output.

The state variable description is therefore:

$$\dot{\underline{\boldsymbol{x}}} = A\underline{\boldsymbol{x}} + B\underline{\boldsymbol{u}}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{-1}{LC} & 0 & \frac{1}{LC} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{LC} & 0 & \frac{-2}{LC} & \frac{-R}{L} & 0 \\ 0 & 0 & \frac{1}{C} & 0 & 0 \end{pmatrix}, \quad \underline{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{\boldsymbol{u}} = \left(\frac{1}{L}\frac{de}{dt} \right)$$

The outputs can also be assembled into a vector.

There is one true output $z = Lx_4$ *and the control input is* $\frac{1}{L}\frac{de}{dt}$*. Therefore the output vector, which encodes both of these, is:*

$$\begin{aligned}
\underline{\boldsymbol{y}} &= \begin{pmatrix} z \\ \frac{1}{L}\frac{de}{dt} \end{pmatrix} = \begin{pmatrix} Lx_4 \\ \frac{1}{L}\frac{de}{dt} \end{pmatrix} = \begin{pmatrix} Lx_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{L}\frac{de}{dt} \end{pmatrix} \\
&= \begin{pmatrix} 0x_1 + 0x_2 + 0x_3 + Lx_4 + 0x_5 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 \end{pmatrix} + \begin{pmatrix} 0 \times \frac{1}{L}\frac{de}{dt} \\ \frac{1}{L}\frac{de}{dt} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(\frac{1}{L}\frac{de}{dt} \right)
\end{aligned}$$

or

$$\underline{\boldsymbol{y}} = C\underline{\boldsymbol{x}} + D\underline{\boldsymbol{u}}$$

where

$$C = \begin{pmatrix} 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \underline{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \underline{\boldsymbol{u}} = \left(\frac{1}{L}\frac{de}{dt} \right)$$

Example 4.12. *A control system is modelled by the following fourth-order ordinary differential equation:*

$$\frac{d^4x}{dt^4} + a_3\frac{d^3x}{dt^3} + a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x = 0$$

where $x(t)$ is a scalar and is the output of the system. Note that there is no external control input in this example.

As before, we seek to define the state variables and obtain the state variable description, as well as determining a suitable output vector $\underline{\boldsymbol{y}}$. In the case of a system described by a single n^{th} order ordinary differential equation without external control, the state equations will be of the form $\underline{\dot{\boldsymbol{x}}} = A\underline{\boldsymbol{x}}$ and A will have special properties.

The dependent variable x is differentiated four times, so it generates four state variables:

$$x_1 \equiv x, \qquad x_2 \equiv \frac{dx}{dt}, \qquad x_3 \equiv \frac{d^2x}{dt^2}, \qquad x_4 \equiv \frac{d^3x}{dt^3}$$

Differentiating,

$$\frac{dx_1}{dt} = \frac{dx}{dt}, \qquad \frac{dx_2}{dt} = \frac{d^2x}{dt^2}, \qquad \frac{dx_3}{dt} = \frac{d^3x}{dt^3}, \qquad \frac{dx_4}{dt} = \frac{dx^4}{dt^4}$$

and so we obtain equations for $\frac{dx_i}{dt}$ in terms of x_i:

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_3 \\
\frac{dx_3}{dt} &= x_4 \\
\frac{dx_4}{dt} &= -a_0x_1 - a_1x_2 - a_2x_3 - a_3x_4
\end{aligned}$$

This set of equations is the state variable description of the control system, and may be *written as* $\underline{\dot{\boldsymbol{x}}} = A\underline{\boldsymbol{x}}$ *(as there no control input, there is therefore no* $D\underline{\boldsymbol{u}}$ *term), where:*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{pmatrix}, \quad \underline{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

This form of matrix A *is called* ***companion form****, that is* A *is a* ***companion matrix****.*

A ***companion matrix*** A *consists of* 1*'s in the entries that are one above the diagonal, any real numbers in the bottom row entries, and zeros elsewhere. This is a standard feature of* A *when* $\underline{\dot{\boldsymbol{x}}} = A\underline{\boldsymbol{x}}$ *is obtained from a single* n^{th} *order ODE.*

The output, as specified, is the original variable x *which is identical to the state variable* x_1*. There is no input, and so the vector output is:*

$$\underline{\boldsymbol{y}} = \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} x_1 \end{pmatrix} = \begin{pmatrix} 1x_1 + 0x_2 + 0x_3 + 0x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = C\underline{\boldsymbol{x}}$$

$$\text{where} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \underline{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

As with the state variable description, the absence of control input means there is no $D\underline{\boldsymbol{u}}$ *term in the output.*

4.3.3 Solution of State Variable Equations (without external control)

In some examples in the previous section, we considered equations of the form $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$, and sought a solution of the $\underline{\mathbf{x}} = \mathrm{e}^{kt}\,\underline{\mathbf{b}}$ where k is an unspecified scalar constant and $\underline{\mathbf{b}}$ is an unspecified vector constant. (Note: in some examples we sought solutions in the trigonometric form $\underline{\mathbf{x}} = \underline{\mathbf{b}}\cos(kt)$ or $\underline{\mathbf{x}} = \underline{\mathbf{b}}\sin(kt)$. Using the exponential form is a more general version of the same approach.)

Substituting this solution into the differential equation leads to the result that k must be an eigenvalue of A and $\underline{\mathbf{b}}$ is the corresponding eigenvector.

From each eigenvalue and eigenvector pair, we can find one of the modes (eigenmodes):

$$\underline{\mathbf{x}}(t) = c_i\;\underline{\mathbf{b}}_i\,\mathrm{e}^{\lambda_i t} \qquad \text{where } c_i \text{ is any scalar.}$$

These time-dependent functions describe natural vibrations of the system where all parts move at the same frequency.

"Solving" a system means obtaining a function for the state variables $\underline{\mathbf{x}}$, so that if we know some initial conditions (where the system starts) then we can find the value of all of the state variables at any given time. Adding together the eigenmodes of the state variable matrix, scaled by some particular constants, gives the full solution to the system without accounting for external controls (i.e. the solution to $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$ without a control vector $\underline{\mathbf{u}}$). However, to actually find the choice of constants necessary, and thus truly solve the ODE system, we will need to use the diagonalisation technique in the next section.

$$\underline{\mathbf{x}}(t) = \sum_{i=1}^{n} c_i\;\underline{\mathbf{b}}_i\,\mathrm{e}^{\lambda_i t} \qquad \text{but we don't know what the } c_i\text{'s should be!}$$

Example 4.13. *The state variables for a certain electronic system are given by:*

$$\dot{\underline{\boldsymbol{x}}} = A\underline{\boldsymbol{x}}, \quad \textit{where} \quad A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}.$$

To determine the modes of this system, recall from a previous example that the eigenvalue and eigenvector pairs for A are:

$$\lambda_1 = 1, \quad \underline{\boldsymbol{b}}_1 = \alpha \begin{pmatrix} 1 \\ -2 \end{pmatrix}; \qquad \lambda_2 = 6, \quad \underline{\boldsymbol{b}}_2 = \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore there are two modes:

$$\underline{\boldsymbol{x}}_1 = \mathrm{e}^{t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}; \quad \underline{\boldsymbol{x}}_2 = \mathrm{e}^{6t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

or any scalar multiples of each of these.

4.4 Solving systems of linear first order ODEs

Consider the first-order linear ODE

$$\frac{dx}{dt} = kx$$

where k is a constant and the initial condition $x(0) = x_0$ is known. The solution to this initial value problem (e.g. obtained using Laplace Transforms) is

$$x(t) = x_0\,\mathrm{e}^{kt}$$

Now consider a system of multiple such ODEs:

$$\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}}$$

From our experience with the scalar problem above, we extend that result to specify the solution to this vector problem formally as:

$$\underline{\mathbf{x}}(t) = \mathrm{e}^{At}\,\underline{\mathbf{x}}(0)$$

However, in this case e^{At} is a matrix (known as the **exponential matrix** or **state transition matrix**) and cannot be obtained as a single application of the exponential function.

The problem then, is how to determine this state transition matrix e^{At}. It is obtained using a **diagonalisation process** involving a modal matrix T of A.

Given an $n \times n$ matrix A, the **modal matrix** T is constructed column-by-column using the eigenvectors of A:

$$T = \left(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \ldots, \underline{\mathbf{e}}_n,\right)$$

where each vector $\underline{\mathbf{e}}_i$ is a single column of n values and is the i^{th} eigenvector of A. The actual ordering of eigenvectors is not important so long as **the ordering always matches with the corresponding eigenvalues**.

Note that there are infinitely many choices of modal matrix T, since in addition to the order of the eigenvectors being interchangable, the eigenvectors themselves are not unique as we have seen. This means that each column of a modal matrix of A could be scaled separately (e.g. multiply every element of the first column by 5, and every element of the second column by -3.1 etc...) and we would obtain another matrix that is still a modal matrix of A.

4.4.1 Method: Solving systems of ODEs using Diagonalisation

Given a system of n linear first-order ODEs formulated as $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$, where A is a square $n \times n$ matrix:

1. Obtain the eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $\underline{\mathbf{e}}_1, \ldots, \underline{\mathbf{e}}_n$ of A.

2. Construct the diagonal matrix of eigenvalues $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the $n \times n$ modal matrix T where the i^{th} column consists of the eigenvector of A corresponding to the eigenvalue in the i^{th} diagonal entry of D. Thus,

$$D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \quad \text{and} \quad T = \big(\underline{\mathbf{e}}_1, \ldots, \underline{\mathbf{e}}_n\big)$$

3. Construct the diagonal matrix of exponentials $\mathrm{e}^{Dt} = diag(\mathrm{e}^{\lambda_1 t}, \mathrm{e}^{\lambda_2 t}, \ldots, \mathrm{e}^{\lambda_n t})$. So,

$$\mathrm{e}^{Dt} = \begin{pmatrix} \mathrm{e}^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & \mathrm{e}^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathrm{e}^{\lambda_n t} \end{pmatrix}$$

4. Perform two matrix multiplications to calculate:

$$\mathrm{e}^{At} = T\,\mathrm{e}^{Dt}\,T^{-1}$$

5. The solution is given by the matrix multiplication:

$$\underline{\mathbf{x}}(t) = \mathrm{e}^{At}\,\underline{\mathbf{x}}(0)$$

4.4.2 Examples: Solving systems of ODEs using Diagonalisation

Example 4.14 (Second-order)**.** *A simple continuous-time model of population dynamics for two species is given by:*

$$\underline{\dot{\boldsymbol{x}}} = A\underline{\boldsymbol{x}}, \quad \textit{where} \quad A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \quad \textit{with initial conditions } \underline{\boldsymbol{x}}(0).$$

The eigenvalue and eigenvector pairs of A are:

$$\lambda_1 = 1, \quad \underline{\boldsymbol{b}}_1 = \alpha \begin{pmatrix} 1 \\ -2 \end{pmatrix}; \quad \textit{and} \quad \lambda_2 = 6, \quad \underline{\boldsymbol{b}}_2 = \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Thus, we take the modal matrix of A to be: $T = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, and set $D = diag(1, 6)$. Therefore,

$$\mathrm{e}^{Dt} = diag(\mathrm{e}^{t}, \mathrm{e}^{6t}) = \begin{pmatrix} \mathrm{e}^{t} & 0 \\ 0 & \mathrm{e}^{6t} \end{pmatrix}$$

Inverting, we find that

$$T^{-1} = \frac{1}{(1)(1) - (2)(-2)} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

Then performing matrix multiplication,

$$T\,\mathrm{e}^{Dt} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{t} & 0 \\ 0 & \mathrm{e}^{6t} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{t} & 2\,\mathrm{e}^{6t} \\ -2\,\mathrm{e}^{t} & \mathrm{e}^{6t} \end{pmatrix}$$

Therefore,

$$\begin{aligned} \mathrm{e}^{At} &= (T\,\mathrm{e}^{Dt})T^{-1} = \frac{1}{5} \begin{pmatrix} \mathrm{e}^{t} & 2\,\mathrm{e}^{6t} \\ -2\,\mathrm{e}^{t} & \mathrm{e}^{6t} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} (\mathrm{e}^{t} + 4\,\mathrm{e}^{6t}) & (-2\,\mathrm{e}^{t} + 2\,\mathrm{e}^{6t}) \\ (-2\,\mathrm{e}^{t} + 2\,\mathrm{e}^{6t}) & (4\,\mathrm{e}^{t} + \mathrm{e}^{6t}) \end{pmatrix} \end{aligned}$$

Now, $\underline{\boldsymbol{x}}(t) = \mathrm{e}^{At}\,\underline{\boldsymbol{x}}(0)$, and so:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{1}{5} \begin{pmatrix} (\mathrm{e}^{t} + 4\,\mathrm{e}^{6t}) & (-2\,\mathrm{e}^{t} + 2\,\mathrm{e}^{6t}) \\ (-2\,\mathrm{e}^{t} + 2\,\mathrm{e}^{6t}) & (4\,\mathrm{e}^{t} + \mathrm{e}^{6t}) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

That is,

$$x_1(t) = \frac{1}{5}\left\{\left(\mathrm{e}^t + 4\,\mathrm{e}^{6t}\right)x_1(0) + \left(-2\,\mathrm{e}^t + 2\,\mathrm{e}^{6t}\right)x_2(0)\right\}$$

$$x_2(t) = \frac{1}{5}\left\{\left(-2\,\mathrm{e}^t + 2\,\mathrm{e}^{6t}\right)x_1(0) + \left(4\,\mathrm{e}^t + \mathrm{e}^{6t}\right)x_2(0)\right\}$$

Example 4.15 (Third-order)**.** *An electronic control system is described by the following set of state variable equations:*

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = x_3, \quad \frac{dx_3}{dt} = \frac{9}{2}x_1 - \frac{7}{2}x_3$$

We use the process of diagonalisation to obtain the state transition matrix and hence obtain solutions for x_1, x_2, x_3.

First, note that we may write these three first-order ODEs in the form $\underline{\dot{\boldsymbol{x}}} = A\underline{\boldsymbol{x}}$, where:

$$\underline{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{9}{2} & 0 & -\frac{7}{2} \end{pmatrix}$$

and we can obtain the eigenvalues and eigenvectors of A:

$$\lambda_1 = 1, \ \underline{\boldsymbol{b}}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad \lambda_2 = -3, \ \underline{\boldsymbol{b}}_2 = \beta \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}; \quad \lambda_3 = -\frac{3}{2}, \ \underline{\boldsymbol{b}}_3 = \gamma \begin{pmatrix} 4 \\ -6 \\ 9 \end{pmatrix};$$

Hence, define the modal matrix of A and calculate its inverse:

$$T = \begin{pmatrix} 1 & 1 & 4 \\ 1 & -3 & -6 \\ 1 & 9 & 9 \end{pmatrix}, \quad \Longrightarrow \quad T^{-1} = \frac{1}{60}\begin{pmatrix} 27 & 27 & 6 \\ -15 & 5 & 10 \\ 12 & -8 & -4 \end{pmatrix}$$

We also define the diagonal matrices

$$D = diag(1, -3, -3/2) \quad and \quad \mathrm{e}^{Dt} = diag(\mathrm{e}^t, \mathrm{e}^{-3t}, \mathrm{e}^{-3t/2})$$

Twice performing matrix multiplication,

$$
\begin{aligned}
e^{At} &= T\,e^{Dt}\,T^{-1} \\
&= \frac{1}{60}\begin{pmatrix} 1 & 1 & 4 \\ 1 & -3 & -6 \\ 1 & 9 & 9 \end{pmatrix}\begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-3t/2} \end{pmatrix}\begin{pmatrix} 27 & 27 & 6 \\ -15 & 5 & 10 \\ 12 & -8 & -4 \end{pmatrix} \\
&= \frac{1}{60}\begin{pmatrix} \left(27\,e^{t} - 15\,e^{-3t} + 48\,e^{-3t/2}\right) & \left(27\,e^{t} + 5\,e^{-3t} - 32\,e^{-3t/2}\right) & \left(6\,e^{t} + 10\,e^{-3t} - 16\,e^{-3t/2}\right) \\ \left(27\,e^{t} + 45\,e^{-3t} - 72\,e^{-3t/2}\right) & \left(27\,e^{t} - 15\,e^{-3t} + 48\,e^{-3t/2}\right) & \left(6\,e^{t} - 30\,e^{-3t} + 24\,e^{-3t/2}\right) \\ \left(27\,e^{t} - 135\,e^{-3t} + 108\,e^{-3t/2}\right) & \left(27\,e^{t} + 45\,e^{-3t} - 72\,e^{-3t/2}\right) & \left(6\,e^{t} + 90\,e^{-3t} - 36\,e^{-3t/2}\right) \end{pmatrix}
\end{aligned}
$$

Then the solution is $\underline{\boldsymbol{x}}(t) = e^{At}\,\underline{\boldsymbol{x}}(0)$*, where* $\underline{\boldsymbol{x}}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}$*, and so finally we obtain:*

$$
\begin{aligned}
x_1(t) = \frac{1}{60}\Big\{ & \left(27\,e^{t} - 15\,e^{-3t} + 48\,e^{-3t/2}\right) x_1(0) + \\
& \left(27\,e^{t} + 5\,e^{-3t} - 32\,e^{-3t/2}\right) x_2(0) + \left(6\,e^{t} + 10\,e^{-3t} - 16\,e^{-3t/2}\right) x_3(0) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
x_2(t) = \frac{1}{60}\Big\{ & \left(27\,e^{t} + 45\,e^{-3t} - 72\,e^{-3t/2}\right) x_1(0) + \\
& \left(27\,e^{t} - 15\,e^{-3t} + 48\,e^{-3t/2}\right) x_2(0) + \left(6\,e^{t} - 30\,e^{-3t} + 24\,e^{-3t/2}\right) x_3(0) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
x_3(t) = \frac{1}{60}\Big\{ & \left(27\,e^{t} - 135\,e^{-3t} + 108\,e^{-3t/2}\right) x_1(0) + \\
& \left(27\,e^{t} + 45\,e^{-3t} - 72\,e^{-3t/2}\right) x_2(0) + \left(6\,e^{t} + 90\,e^{-3t} - 36\,e^{-3t/2}\right) x_3(0) \Big\}
\end{aligned}
$$

5 What do I need to be able to do?

5.1 Laplace Transforms

- Confidently use tables to find Laplace transforms and inverse transform.
- Understand how to construct piecewise functions by combining step functions.
- Putting functions into Delay Form and taking Laplace transforms of these using the Delay Theorem:

$$\mathcal{L}\{g(t-T)U(t-T)\} = e^{-sT}\mathcal{L}\{g(t)\}$$

- Inverting Laplace transforms involving a delay.
- Solving differential and integral equations (that may include discontinuities) using Laplace transforms.
- Obtain the Transfer Function for a linear system, given a set of ODEs:

$$G(s) = \frac{\bar{v}_o(s)}{\bar{v}_i(s)}$$

- Identify the Characteristic Equation, determine the order, and interpreting the stability of the system. It is stable if *all* solutions of the characteristic equation $Q(s) = 0$ have negative real part $Re(s) < 0$.
- Determine nature/type of a filter by estimating the low and high frequency behaviour and sketching the Amplitude Bode Plot.
- Understand how convolution integrals can be used to obtain solutions for the output signal.

5.2 Fourier Analysis

- Generalised sine and cosine functions, angular frequency and phase angle.
- Definition of Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

- Understand DC level, fundamental, harmonics:

  - $\frac{a_0}{2}$ is the **DC level** of $f(t)$. It is the average value of the signal over one full cycle.
  - $a_1\cos(\omega t) + b_1\sin(\omega t)$ is the first harmonic, also called the **Fundamental mode**.
  - $a_n\cos(n\omega t) + b_n\sin(n\omega t)$ is the $\mathbf{n^{th}}$ **Harmonic**. It has angular frequency $n\omega$, which is n times the angular frequency of the fundamental.

- Understand the core idea of Fourier Series: approximating a periodic function by the partial sums of harmonics. The more terms used, the better the approximation.

- Understand the concept of Odd and Even functions, and be able to use their relevance to Fourier Series:

  (a) If $f(t)$ is an even function, then $b_n = 0 \quad \forall\, n \in \mathbb{N}$.
  (b) If $f(t)$ is an odd function, then $a_n = 0 \quad \forall\, n \in \mathbb{N}$.

- The complex form of Fourier series:

$$f(t) = \frac{a_0}{2} + Re\left\{\sum_{n=1}^{\infty} A_n\, \mathrm{e}^{jn\omega t}\right\}$$

- Determining the complex Fourier coefficient (phasor) using the Laplace transform method:

$$A_n = \frac{2}{T}\bar{g}(jn\omega)$$

  where

$$g(t) = \begin{cases} f(t) & \text{for } 0 < t < T, \\ 0 & \text{otherwise.} \end{cases}$$

- Using the frequency response function $G(jn\omega)$ of a linear system to calculate phasors of the output given a particular input signal:

$$V_n = G(jn\omega)A_n$$

- Also be able to find the DC level of the output signal, given the frequency response function of the system and the DC level of the input signal:

$$G(0) \times \frac{a_0}{2}$$

- Determine amplitude and phase distortion.

- Know how the real/regular Fourier coefficients a_n and b_n can be obtained from the phasor of the harmonics. Since $A_n = a_n - jb_n$,

$$a_n = Re\{A_n\}, \quad \text{and} \quad b_n = -Im\{A_n\}$$

5.3 Matrix Algebra

- Know what the identity matrix I is.
- Determine the characteristic equation of a square matrix A:

  $$\det(A - \lambda I) = 0$$

- Calculating eigenvalues and eigenvectors of a 2×2 and 3×3 square matrix.
- Verifying solutions to cubic polynomials to confirm that a number is an eigenvalue given the characteristic equation.
- Define the state variables for a set of mixed-order ODEs.
- Derive the state variable equations in the form:

  $$\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}} + B\underline{\mathbf{u}},$$

  and (when appropriate) find a suitable output vector in the form:

  $$\underline{\mathbf{y}} = C\underline{\mathbf{x}} + D\underline{\mathbf{u}}$$

- Determine the natural modes (eigenmodes) of a system from the eigenvalues and eigenvectors.
- Understand the definition of modal matrix, and be able to construct one from the eigenvectors of a square matrix.
- Use the diagonalisation process to obtain the state transition matrix. Know and use the result:

  $$\mathrm{e}^{At} = T\,\mathrm{e}^{Dt}\,T^{-1}$$

- Obtain individual state variables using:

  $$\underline{\mathbf{x}} = \mathrm{e}^{At}\,\underline{\mathbf{x}}(0)$$

5.4 General notes on what will not be tested:

- Circuit analysis. Many of the examples are given in the context of analysing the potential differences of electronic circuits. This is just one possible application of these methods, and is shown to help you see how we would use the method. Circuits themselves are not part of this section of the module, which is about mathematical methods, and so if such a question should appear in assessment then the equations for analysis will be provided.

- EXCEL and MATLAB programming.

- Obtaining Fourier coefficients by the integration method (although it is good to be aware of this as it is the most common method):

$$a_0 = \frac{2}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt$$

6 Appendix 1: Revision

6.1 Polynomial Functions

A polynomial function of x consists of a linear combination of terms that only include non-negative integer powers of x. Therefore they take the form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

where the coefficents $a_0, a_1, \ldots$ can take any real scalar value.

6.1.1 Quadratic functions

A quadratic function is a second-order polynomial, $f(x) = ax^2 + bx + c$. When we consider potentially complex solutions, f has precisely two *roots*, which are the values of x such that $f(x) = 0$. They are found using the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $b^2 - 4ac$ is the *discriminant*. Its value determines the nature of the solutions:

$$b^2 - 4ac \begin{cases} = 0 & \implies \text{One real, repeated root.} \\ < 0 & \implies \text{No real roots. The solutions are a complex conjugate pair.} \\ > 0 & \implies \text{Two real, distinct roots.} \end{cases}$$

6.1.1.1 Completing the Square Sometimes a quadratic expression cannot be factorised in the usual way. However, we can re-write it as a multiple of the form $(x+\alpha)^2 + \beta$, and it may then be useful in some circumstances to write it as $(x+\alpha)^2 + \gamma^2$ if $\beta > 0$. In order to do this for $f(x) = ax^2 + bx + c$, first factor out the coefficient of x^2, then α will be half of the coefficent of x, and since we will have added an additional term equal to the square of α, we need to account for that by removing it.

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right\} \\ &= a\left\{\left(x + \frac{b}{2a}\right)^2 + \left(\sqrt{\frac{c}{a} - \left(\frac{b}{2a}\right)^2}\right)^2\right\} \end{aligned}$$

Example 6.1. *Complete the square for the quadratic expression $x^2 + 6x + 34$:*

$$\begin{aligned}
x^2 + 6x + 34 &= \left(x + \frac{6}{2}\right)^2 + 34 - \left(\frac{6}{2}\right)^2 \\
&= (x+3)^2 + 34 - 9 \\
&= (x+3)^2 + 25 \\
&= (x+3)^2 + (5)^2
\end{aligned}$$

6.2 Partial Fractions

This is a method of expanding fractions that have a polynomial on the denominator that can be factorised, and obtaining a linear combination of simpler fractions that are easier to work with. The form of the expansion depends on the nature of the fraction, but there are three main possibilities:

- All linear terms in the denominator:
$$\frac{7x-1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$$

- Repeated term in the denominator:
$$\frac{2x^2+5x-13}{(x+5)(x-4)^2} = \frac{A}{x+5} + \frac{B}{x-4} + \frac{C}{(x-4)^2}$$

- x^2 in the denominator (make sure to check for common factors on the numerator and denominator first):
$$\frac{4x^2-7x+17}{(x-5)(x^2+7)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+7}$$

When we have determined the appropriate layout of the expansion, multiply both sides of the equation by *all* of the denominator terms.

Finally, we obtain the values of the constants from the resulting equation either by choosing specific values of x so that the results drop out, or by equating the coefficients of x^0, x^1, x^2, etc.

6.3 Integration by Parts

$$\int u\frac{dv}{dt}dt = uv - \int v\frac{du}{dt}$$

Integration by parts is particularly useful for integrals involving the product of a polynomial function and an exponential or trigonometic function. Sometimes multiple applications will be necessary. A key point is in the selection of u and $\frac{dv}{dt}$. In particular, if there is a polynomial (non-negative exponents), always choose it as u, since repeated differentiation will eventually "get rid" of it.

6.4 Complex Exponentials

Recall that j is the imaginary number, defined as:

$$j = \sqrt{-1}$$

Hence $j^2 = -1$, and more generally $j^{2n} = -1$ and $j^{2n+1} = (-1)^n j$.

6.5 Drawing Waveforms by hand

You should also be able to draw a general sine or cosine function by hand. This involves (and demonstrates) an understanding of the role that the amplitude, angular frequency, phase angle and vertical shift play. Given the function $A\sin(\omega t+\phi)+d$ (or a similar cosine function), we can use the following systematic method:

1. Identify the amplitude A.
2. Identify the period $T = 2\pi/\omega$.
3. Draw the sine (or cosine) wave rescaled by A and T.
4. Shift the wave *left* by $\phi/(2\pi)$ fraction of a cycle.
5. Shift the wave up by d.

It is important that point 2. and 3. are carried out before points 4. and 5. in this procedure. Consider the following example:

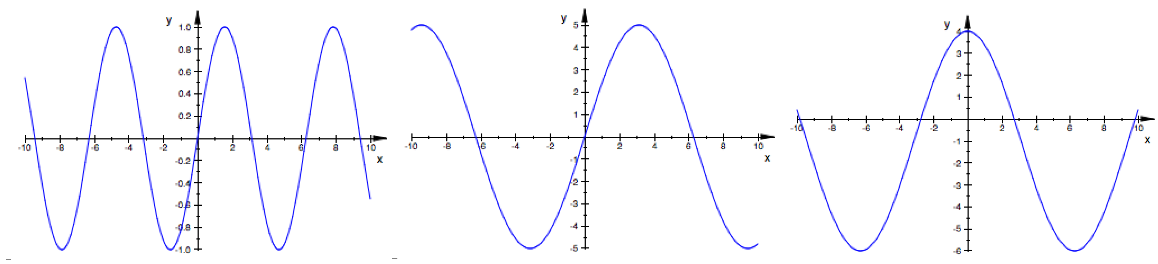
Example 6.2. *Draw* $y = 5\sin\left(\frac{t}{2} + \frac{\pi}{2}\right) - 1$.

1. *Comparing this with* $A\sin(\omega t + \phi) + d$, *the amplitude* $A = 5$ *and so the initial wave (ignoring the vertical shift) will oscillate between* -5 *and* $+5$.
2. $\omega = \frac{1}{2}$, *so the period is* $T = (2\pi)/(1/2) = 4\pi$.

3. *Hence, for the intermediate plot, we draw a sine wave rescaled to oscillate between* -5 *and* $+5$ *and with period* 4π.

4. $\phi = \frac{\pi}{2}$, *and so* $\phi/(2\pi) = \frac{\pi}{2} \div 2\pi = \frac{1}{4}$. *Hence, the waveform must now be shifted to the left by one quarter of a cycle.*

5. *Finally,* $d = -1$, *so the wave is shifted vertically down by 1. This final form is shown in the last figure.*



(a) $y = \sin(x)$ (b) $y = 5\sin(x/2)$ (c) $y = 5\sin\big(x/2 + \pi/2\big) - 1$

Figure 9: Stages of drawing $y = 5\sin\big(x/2 + \pi/2\big) - 1$

We can check this by plotting the graph in MATLAB.

7 Appendix 2: Extra Material

7.1 Chapter 1: Laplace Transforms

7.1.1 Additional Example from First Principles

An additional example of deriving the Laplace transform of a function from the integral defintion.

Example 7.1. *Derive the Laplace transform of*

$$f(t) = \begin{cases} 0 & \text{for } t < 0; \\ t & \text{for } t > 0. \end{cases}$$

$$\begin{aligned}
\bar{f}(s) &= \mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)dt = \int_0^\infty t\,\mathrm{e}^{-st}\,dt \\
&= \left[\frac{-1}{s}\mathrm{e}^{-st}\,t\right]_0^\infty - \int_0^\infty \left(\frac{-1}{s}\right)\mathrm{e}^{-st}\,1dt \quad \textit{by Integration by Parts, with } u = t,\ \frac{dv}{dt} = \mathrm{e}^{-st} \\
&= \left[\frac{-1}{s}\mathrm{e}^{-st}\,t\right]_0^\infty - \left[\frac{1}{s^2}\mathrm{e}^{-st}\right]_0^\infty \\
&= \left(\left\{\frac{-1}{s}0\right\} - \left\{\frac{-1}{s}0\right\}\right) - \left(\left\{\frac{1}{s^2}0\right\} - \left\{\frac{1}{s^2}1\right\}\right) \quad \textit{provided } \ s > 0 \\
\bar{f}(s) &= \frac{1}{s^2} \quad \textit{for } \ s > 0
\end{aligned}$$

Example 7.2. *Consider the Laplace transform of*

$$f(t) = \begin{cases} 0 & \text{for } t < 0; \\ \mathrm{e}^{t^2} & \text{for } t > 0. \end{cases}$$

In this case,

$$\bar{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)dt = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{e}^{t^2}\,dt$$

From previous examples, we know that to obtain a well-defined solution, we require the decaying exponential to dominate the integrand expression. That is $\bar{f}(s)$ will be defined only for finite values of s such that $\mathrm{e}^{-st}\,\mathrm{e}^{t^2} = \mathrm{e}^{-st+t^2} = \mathrm{e}^{-t(s-t)}$ is a decaying integral. Hence we require s such that $t(s-t) > 0$, so since $t > 0$ we need $s - t > 0$ and thus $s > t$ is required for all values of t. However, t ranges between zero and infinity, so no finite value of s will satisfy this condition.

7.1.2 Existence of the Laplace Transform

The previous example illustrates that the Laplace transform is not well-defined for all functions. In particular, a sufficient condition to ensure that a function $f(t)$ has a Laplace transform is that there exists a constant α and some positive constants t_0 and M, such that:

$$|f(t)| < M\,\mathrm{e}^{\alpha t} \qquad \text{for all} \quad t > t_0$$

This describes a function of "exponential order", which means that its rate of growth is not faster than that of exponential functions. Technically, we may also require that there are only finitely many points of discontinuity in any finite interval of the domain, but this condition will be satisfied by any function you may practically be asked to consider.

7.1.3 Proof of the Linearity of Laplace Transforms

Proving that the Laplace transform is a linear operator is very simple, and relies on the fact that integration is also linear (i.e. the integral of a linear combination of terms is equal to the same combination of their individual integrals).

Proof. Let f, g be time-dependent functions and let a, b be scalar constants.

$$\begin{aligned}
\mathcal{L}\{af(t)+bg(t)\} &= \int_0^\infty \mathrm{e}^{-st}(af(t)+bg(t))dt = \int_0^\infty a\,\mathrm{e}^{-st}\,f(t) + b\,\mathrm{e}^{-st}\,g(t)dt \\
&= \int_0^\infty a\,\mathrm{e}^{-st}\,f(t)dt + \int_0^\infty b\,\mathrm{e}^{-st}\,g(t)dt \\
&= a\int_0^\infty \mathrm{e}^{-st}\,f(t)dt + b\int_0^\infty \mathrm{e}^{-st}\,g(t)dt \\
&= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}
\end{aligned}$$

□

7.1.4 Amplitude Bode Plots

In this course, we calculate very rough approximations to the amplitude bode plot. The full method of constructing one is quite involved, and an example is shown here. Given a transfer function $G(s)$, it will typically consist of a ratio of polynomials, which we can factorise. For example, consider the following:

$$G(s) = \frac{4\times 10^3(5+s)}{s(100+s)(20+s)}$$

We can see that both the numerator and the denominator are factorised polynomials. Next, we need to express this in standard form, which means that each factor needs to be rewritten as $(1 + \tau s)$:

$$G(s) = \frac{10(1 + 0.2s)}{s(1 + 0.01s)(1 + 0.05s)}$$

Then make the substitution $s = j\omega$:

$$G(j\omega) = \frac{10(1 + 0.2j\omega)}{j\omega(1 + 0.01j\omega)(1 + 0.05j\omega)}$$

Engineers typically use amplitude gain in decibels, so take 20 times the logarithm to base 10:

$$20 \log |G(j\omega)| = 20 \log \left| \frac{10(1 + 0.2j\omega)}{j\omega(1 + 0.01j\omega)(1 + 0.05j\omega)} \right|$$

The advantage of logs is that this function can be expressed by adding and subtracting each of these zeros (values of ω that are roots of the numerator) and poles (values of ω that are roots of the denominator).

$$20 \log |G(j\omega)| = 20 \log(10) + 20 \log |1 + j0.2\omega| - 20 \log |j\omega| - 20 \log |1 + 0.01j\omega| - 20 \log |1 + 0.05j\omega|$$

We can plot the amplitude gain from each of these terms separately, and then construct the final plot by adding them all together. There are five kinds of term involving simple (multiplicity one) poles and zeros:

(a) For a simple gain k, a plot of $20 \log(k)$ is just a horizontal straight line. It is above 0dB if $k > 1$, and below otherwise.

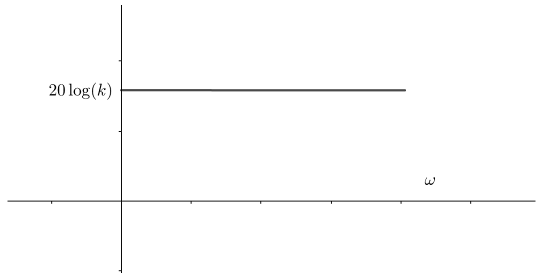
(b) For a simple zero at the origin, a plot of $20 \log(\omega)$ is a straight line with slope 20dB/decade that intersects 0dB at $\omega = 1$.

(c) For a simple pole at the origin, a plot of $-20 \log(\omega)$ is a straight line with slope -20dB/decade that intersects 0dB at $\omega = 1$.

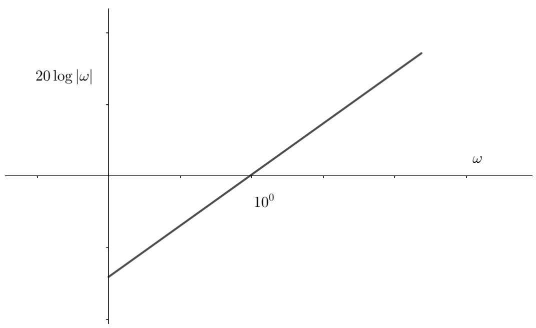
(d) For a simple zero not at the origin, a plot of $20 \log(1 + j\omega\tau)$ can be approximated by two parts. Zero for $\omega < 1/\tau$ and a straight line with slope 20dB/decade that intersects 0dB at $\omega = 1/\tau$ for $\omega > 1/\tau$.

(e) For a simple pole not at the origin, a plot of $-20 \log(1 + j\omega\tau)$ can be approximated by two parts. Zero for $\omega < 1/\tau$ and a straight line with slope -20dB/decade that intersects 0dB at $\omega = 1/\tau$ for $\omega > 1/\tau$.

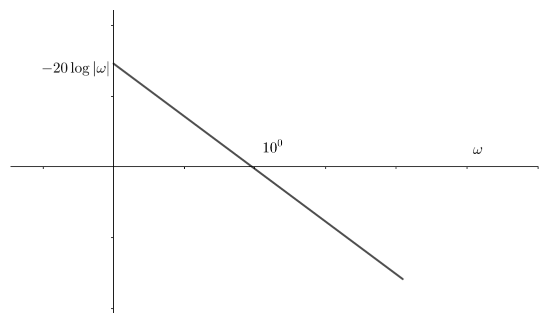
These are shown in the figure below. Note that the absolute value of the slopes are the same in all cases.



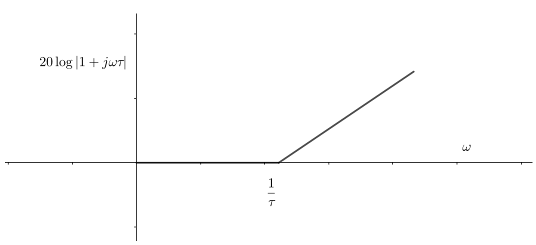
(a) Constant gain



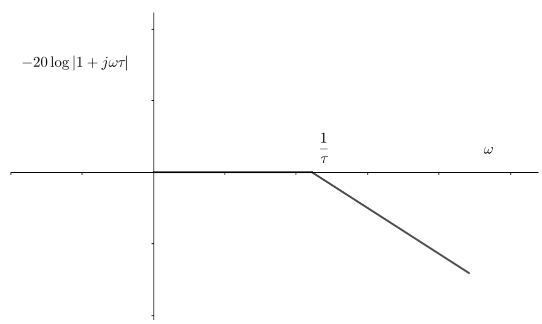
(b) Zero at the origin



(c) Pole at the origin



(d) Zero not at the origin



(e) Pole not at the origin

Figure 10: Approximate amplitude bode plot contributions from simple poles and zeros

Let's return to our example. In this case, we have a constant gain of 20dB, a simple zero with breakpoint $\omega = 1/0.2 = 5rad\ s^{-1}$, and three simple poles - one at the origin, and two others with breakpoints at the following frequencies respectively: $\omega = 100rad\ s^{-1}$ and $\omega = 20rad\ s^{-1}$. We plot these five contributions on the same diagram (dotted lines), and then the approximate final amplitude is obtained by the summation of the values along the plot.

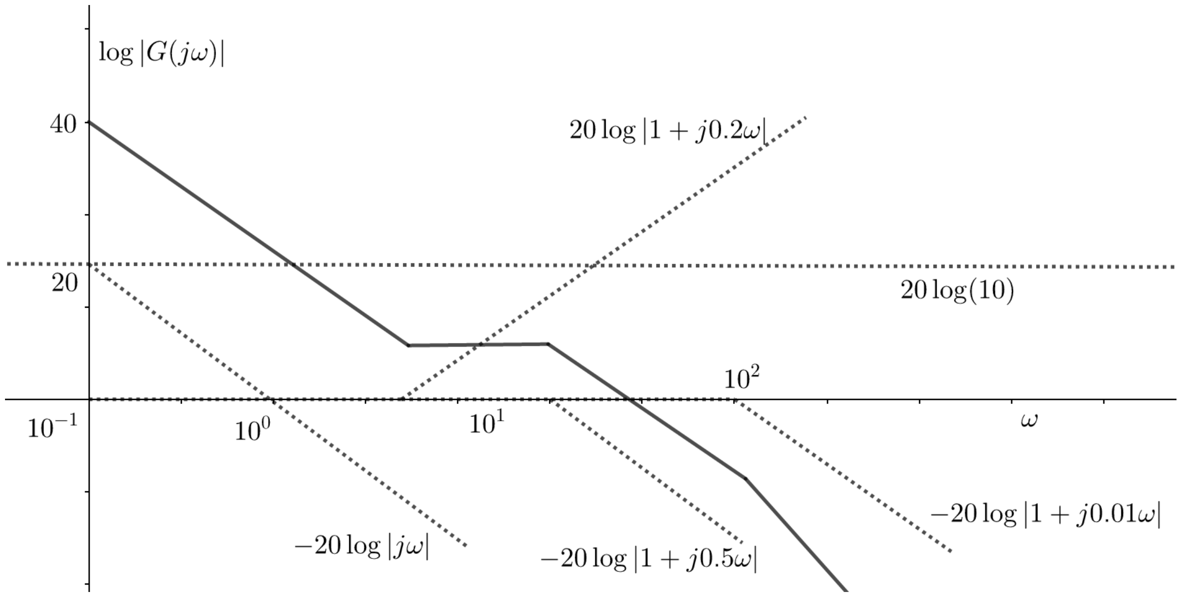


Figure 11: Approximate amplitude bode plot contributions from simple poles and zeros

So this example is a third order (three poles) low-pass filter.

In some cases, we can also determine the type of filter directly from the transfer function if it *exactly* matches certain known forms. For a first-order system (k is some constant):

$G(s) = \frac{1}{1+ks}$ is a low-pass filter.

$G(s) = \frac{ks}{1+ks}$ is a high-pass filter.

$G(s) = \frac{1-ks}{1+ks}$ is an all-pass filter.

For second-order systems where a, b, c, d are constants:

$G(s) = \frac{c}{s^2+as+b}$ is a low-pass filter.

$G(s) = \frac{cs^2}{s^2+as+b}$ is a high-pass filter.

$G(s) = \frac{cs}{s^2+as+b}$ is a band-pass filter.

$G(s) = \frac{cs^2+d}{s^2+as+b}$ is a band-eliminate filter.

$G(s) = \frac{s^2-cs+d}{s^2+as+b}$ is an all-pass filter.

7.2 Chapter 2: Fourier Series

7.2.1 Orthogonality

How do we know how to calculate the Fourier coefficients a_n and b_n by integration?

The key idea that these results rely on on is the *orthogonality* of sines and cosines of different frequencies, which means':

$$\int_{-T/2}^{T/2} \cos(n\omega t)\sin(n\omega t)dt = 0 \quad \forall m, n \in \mathbb{Z}$$

$$\int_{-T/2}^{T/2} \sin(n\omega t)\sin(n\omega t)dt = 0 \quad \forall m, n \in \mathbb{Z}$$

$$\int_{-T/2}^{T/2} \cos(n\omega t)\cos(n\omega t)dt = 0 \quad \text{if} \quad m \neq n$$

7.2.2 Convergence of the Fourier Series to the actual function

Fourier originally believed that his series could be used to approximate *any* periodic function. However, Gustav and Dirichlet showed that some conditions are required, and were able to construct pathological functions for which the Fourier series does not converge to the function. In particular, Dirichlet showed that provided a periodic function $f(t)$ has, within a single period:

- A finite number of Maxima and Minima.
- A finite number of points of (finite) discontinuity.

then the corresponding Fourier series expansion of $f(t)$ will converge to the correct result at all points where $f(t)$ is continuous, and to the average of the right-hand and left-hand limits of $f(t)$ at points of discontinuity.

These are known as the Dirichlet Conditions, and will be satisfied by any function we consider in this course.

7.2.3 Additional examples using the standard method of integration to obtain the Fourier Coefficents

Example 7.3 (Sawtooth Wave)**.** *Consider the sawtooth wave shown.*

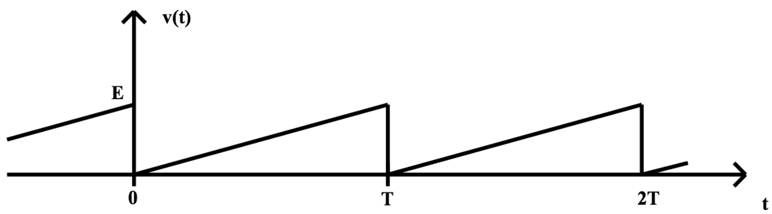


Figure 12: Sawtooth Wave

It satisfies $f(t) = \frac{Et}{T}$ *throughout the first cycle* $0 < t < T$*. Therefore the integrals for the coefficients are:*

$$a_n = \frac{2}{T}\int_0^T \frac{Et}{T}\cos\left(\frac{2\pi nt}{T}\right)dt, \qquad b_n = \frac{2}{T}\int_0^T \frac{Et}{T}\sin\left(\frac{2\pi nt}{T}\right)dt$$

which, through application of Integration By Parts, yield $a_n = 0$ *and* $b_n = \frac{-E}{n\pi}$ *for all* $1 \leq n$*.*
Clearly the average value of f *is* $\frac{E}{2}$*, so the DC level* $\frac{a_0}{2} = \frac{E}{2}$*, and therefore the Fourier series is given by:*

$$f(t) = \frac{E}{2} - \frac{E}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2\pi nt}{T}\right)$$

If the DC level is not obvious, it can be found simply by calculating the area under the graph for a full period and then normalising (divide by T), or else explicitly using the formula for a_0 as in example 1.

Example 7.4 (Triangular Wave)**.** *Consider the triangular wave $f(t)$ shown.*

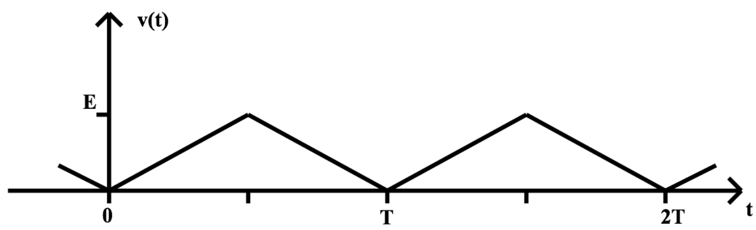


Figure 13: Triangular Wave

For this wave,

$$f(t) = \begin{cases} \frac{E}{T/2}t = \frac{2Et}{T}, & \text{for } 0 < t < \frac{T}{2} \\ \\ \frac{-2E}{T}(t-T) = 2E\big(1-\frac{t}{T}\big), & \text{for } \frac{T}{2} < t < T \end{cases}$$

Therefore for $n \geq 1$,

$$a_n = \frac{2}{T}\Big\{\int_0^{T/2} \frac{2Et}{T}\cos\big(\frac{2n\pi t}{T}\big)dt + \int_{T/2}^{T} 2E(1-\frac{t}{T})\cos\big(\frac{2n\pi t}{T}\big)dt\Big\}$$

Using integration by parts (choosing the polynomial part for differentiation, and the trigonometric part for integration) gives:

$$\begin{aligned} a_n \;&=\; \frac{2}{T}\Big\{\Big[\frac{2Et}{T}\frac{T}{2n\pi}\sin\big(\frac{2n\pi t}{T}\big)\Big]_0^{T/2} - \frac{2E}{T}\frac{T}{2n\pi}\int_0^{T/2}\sin\big(\frac{2n\pi t}{T}\big)dt \\ &\quad + \Big[2E(1-\frac{t}{T})\frac{T}{2n\pi}\sin\big(\frac{2n\pi t}{T}\big)\Big]_{T/2}^{T} - \frac{T}{2n\pi}\int_{T/2}^{T} 2E\frac{-1}{T}\sin\big(\frac{2n\pi t}{T}\big)\Big\} \end{aligned}$$

which eventually results in

$$a_n = \frac{-2E}{n^2\pi^2}(1-\cos(n\pi)) = \begin{cases} \frac{-4E}{n^2\pi^2} & \text{for } n \text{ odd,} \\ \\ 0 & \text{for } n \text{ even.} \end{cases}$$

For the remaining coefficients,

$$b_n = \frac{2}{T}\Big\{\int_0^{T/2} \frac{2Et}{T}\sin\big(\frac{2n\pi t}{T}\big)dt + \int_{T/2}^{T} 2E(1-\frac{t}{T})\sin\big(\frac{2n\pi t}{T}\big)dt\Big\}$$

which eventually will collapse to $b_n = 0$.

Finally, it should be clear from the graph that the DC level is $E/2$. *Therefore,*

$$\frac{a_0}{2} = \frac{E}{2}$$

Hence we have the Fourier series:

$$\begin{aligned} f(t) &= \frac{E}{2} - \frac{4E}{\pi^2} \sum_{odd\ n \in \mathbb{N}} \frac{1}{n^2} \cos\left(\frac{2n\pi t}{T}\right) \\ &= \frac{E}{2} - \frac{4E}{\pi^2} \left\{ \cos\left(\frac{2\pi t}{T}\right) + \frac{1}{9} \cos\left(\frac{6\pi t}{T}\right) + \frac{1}{25} \cos\left(\frac{10\pi t}{T}\right) + \ldots \right\} \end{aligned}$$

7.2.4 Derivation of the Phasors

How do we know the relationship between the phasors and the harmonics?

Recall the trigonometric expansion $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$. Then consider the following general cosine wave with a phase shift:

$$\begin{aligned} R_n \cos(n\omega t + \phi_n) &= R_n \cos(\phi_n) \cos(n\omega t) - R_n \sin(\phi_n) \sin(n\omega t) \\ &= a_n \cos(n\omega t) + b_n \sin(n\omega t) \end{aligned}$$

where $a_n = R_n \cos(\phi_n)$ and $b_n = -R_n \sin(\phi_n)$.
So we can write the n^{th} harmonic as $R_n \cos(n\omega t + \phi_n)$, where:

$$a_n^2 + b_n^2 = R_n^2(\cos^2(\phi_n) + \sin^2(\phi_n)) = R_n^2 \quad \text{so} \quad R_n = \sqrt{a_n^2 + b_n^2},$$

and

$$\frac{-b_n}{a_n} = \frac{-(-R_n \sin(\phi_n))}{R_n \cos(\phi_n)} = \tan(\phi_n).$$

Next, recall the Euler formula $\mathrm{e}^{jx} = \cos(x) + j\sin(x)$ where $x \in \mathbb{R}$ and $j = \sqrt{-1}$ is the imaginary number. Therefore,

$$R_n\, \mathrm{e}^{j(n\omega t + \phi_n)} = R_n \cos(n\omega t + \phi_n) + jR_n \sin(n\omega t + \phi_n)$$

so taking the real parts only and then splitting the exponential, the n^{th} harmonic is:

$$\begin{aligned} R_n \cos(n\omega t + \phi_n) &= Re\left\{R_n\, \mathrm{e}^{j(n\omega t + \phi_n)}\right\} = Re\left\{R_n\, \mathrm{e}^{j\phi_n}\, \mathrm{e}^{jn\omega t}\right\} \\ &= Re\left\{(a_n - jb_n)\, \mathrm{e}^{jn\omega t}\right\} \end{aligned}$$

since

$$R_n\,\mathrm{e}^{j\phi_n} = R_n(\cos(\phi_n) + j\sin(\phi_n)) = a_n - jb_n, \text{ which we set to } A_n.$$

As an alternative derivation, consider the following:

$$\begin{aligned}(a_n - jb_n)\,\mathrm{e}^{jn\omega t} &= (a_n - jb_n)(\cos(n\omega t) + j\sin(n\omega t)) \\ &= (a_n\cos(n\omega t) + b_n\sin(n\omega t)) + j(a_n\sin(n\omega t) - b_n\cos(n\omega t))\end{aligned}$$

so considering the real and imaginary parts, the n^{th} harmonic is equivalent to $Re\big\{(a_n - jb_n)\,\mathrm{e}^{jn\omega t}\big\}$.

7.2.5 Examples of obtaining the Complex Form by Integration

Here are some examples where we calculate the complex form of the Fourier series, but still using the integration method rather than the Laplace Transforms method. As we shall see, one advantage of this over the trigonometric form (i.e obtaining the regular coefficients by integration) is that we only have to evaluate one integral instead of two.

Example 7.5. *Consider again the sawtooth wave.*

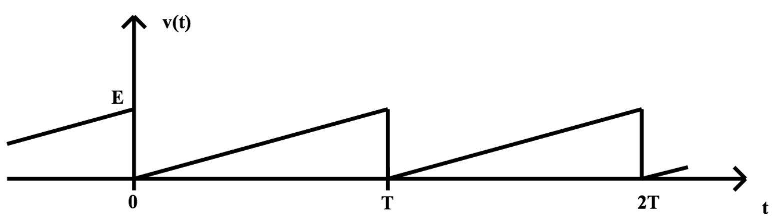


Figure 14: Sawtooth Wave

In the first cycle, $f(t) = \frac{Et}{T}$. Therefore the phasors are found by:

$$\begin{aligned}A_n &= \frac{2E}{T^2}\int_0^T t\,\mathrm{e}^{-jn\omega t}\,dt = \frac{2E}{T^2}\left\{\left[\frac{-1}{jn\omega}t\,\mathrm{e}^{-jn\omega t}\right]_0^T + \frac{1}{jn\omega}\int_0^T \mathrm{e}^{-jn\omega t}\,dt\right\} \\ &= \frac{2E}{T^2}\left\{\frac{-T}{jn\omega}\,\mathrm{e}^{-jnT\omega} - \frac{1}{j^2n^2\omega^2}\left[\mathrm{e}^{-jn\omega t}\right]_0^T\right\} \\ &= \frac{2E}{T^2}\frac{1}{j^2n^2\omega^2}\left\{\mathrm{e}^{-jnT\omega}(-jnT\omega - 1) + 1\right\}\end{aligned}$$

Substituting in $\omega = \frac{2\pi}{T}$ *and simplifying,*

$$\begin{aligned} A_n &= \frac{2E}{T^2}\frac{T^2}{j^2n^2(2\pi)^2}\Big\{ \mathrm{e}^{-2\pi jn}(-2\pi jnT - 1) + 1\Big\} \\ &= \frac{E}{2\pi^2 n^2 j^2}(-2\pi nj - 1 + 1) = \frac{-2\pi njE}{2\pi^2n^2j^2} \\ &= \frac{-E}{\pi nj} = \frac{Ej}{\pi n} \end{aligned}$$

since for any integer n*,*

$$\mathrm{e}^{-2\pi jn} = \cos(-2\pi jn) + j\sin(-2\pi jn) = 1 + 0 = 1$$

Then $A_n = a_n - jb_n$*, so*

$$a_n = Re\big\{A_n\big\} = 0, \quad \textit{and} \quad b_n = -Im\big\{A_n\big\} = \frac{-E}{\pi n}$$

Whether from the graph or the formula, the DC level $\frac{a_0}{2} = \frac{E}{2}$*. Hence,*

$$f(t) = \frac{E}{2} - \frac{E}{\pi}\sum_{n=0}^{\infty}\frac{1}{n}\sin\big(\frac{2\pi nt}{T}\big)$$

Example 7.6. *Next, consider this clipped sawtooth waveform.*

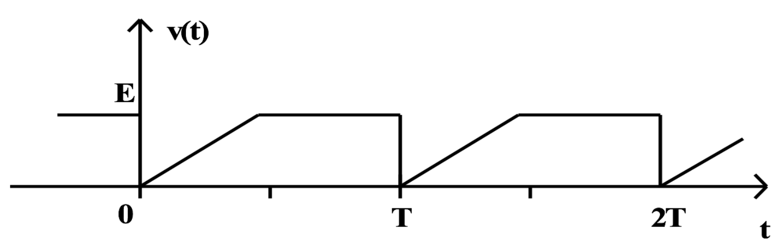


Figure 15: Clipped Sawtooth Wave

In this case, during the first cycle:

$$f(t) = \begin{cases} \frac{2Et}{T} & \textit{for } 0 < t < \frac{T}{2}, \\ \\ E & \textit{for } \frac{T}{2} < t < T \end{cases}$$

Therefore,

$$\begin{aligned}
A_n &= \frac{2}{T}\int_0^T f(t)\,\mathrm{e}^{-jn\omega t}\,dt = \frac{2E}{T}\left\{\frac{2}{T}\int_0^{T/2} t\,\mathrm{e}^{-jn\omega t}\,dt + \int_{T/2}^{T}\mathrm{e}^{-jn\omega t}\,dt\right\}\\
&= \frac{2E}{T}\left\{\frac{2}{T}\left(\left[\frac{-t}{jn\omega}\,\mathrm{e}^{-jn\omega t}\right]_0^{T/2} + \frac{1}{jn\omega}\int_0^{T/2}\mathrm{e}^{-jn\omega t}\,dt\right) + \left[\frac{-1}{jn\omega}\,\mathrm{e}^{-jn\omega t}\right]_{T/2}^{T}\right\}\\
&= \frac{2E}{T}\left\{\frac{2}{T}\left(\left(\frac{-T}{2jn\omega}\,\mathrm{e}^{-jn\pi\omega T/2} - 0\right) - \frac{1}{j^2n^2\omega^2}\left[\,\mathrm{e}^{-jn\omega t}\,\right]_0^{T/2}\right) - \frac{1}{jn\omega}\left(\,\mathrm{e}^{-jn\omega T} - \mathrm{e}^{-jn\omega T/2}\,\right)\right\}\\
&= \frac{2E}{T}\left\{\frac{-1}{jn\omega}\,\mathrm{e}^{-jn\omega T/2} - \frac{2}{Tj^2n^2\omega^2}\left(\,\mathrm{e}^{-jn\omega T/2} - 1\right) - \frac{1}{jn\omega}\,\mathrm{e}^{-jn\omega T} + \frac{1}{jn\omega}\,\mathrm{e}^{-jn\omega T/2}\right\}\\
&= \frac{-2E}{j^2n^2\omega^2T^2}\left\{2(\mathrm{e}^{-jn\omega T/2} - 1) + jn\omega T\,\mathrm{e}^{-jn\omega T}\right\}
\end{aligned}$$

Then using $\omega = \frac{2\pi}{T}$*,*

$$A_n = \frac{E}{2n^2\pi^2}\left\{2(\mathrm{e}^{-jn\pi} - 1) + 2\pi jn\,\mathrm{e}^{-2jn\pi}\right\}$$

Now, as before $\mathrm{e}^{-2jn\pi} = 1$*. However, we also have* $\mathrm{e}^{-jn\pi} = \cos(n\pi) = (-1)^n$*, so the phasor for the* n^{th} *harmonic of* $f(t)$ *is given by:*

$$A_n = \frac{E}{n^2\pi^2}\left\{\cos(n\pi) - 1 + j\pi n\right\} = \left(\frac{E\big((-1)^2 - 1\big)}{n^2\pi^2}\right) + j\left(\frac{E}{n\pi}\right)$$

Considering the real and imaginary parts,

$$a_n = Re\{A_n\} = \begin{cases} \frac{-2E}{\pi^2n^2} & \textit{for } n \textit{ odd,}\\[2em] 0 & \textit{for } n \textit{ even}\end{cases}$$

and

$$b_n = -Im\{A_n\} = \frac{-E}{\pi n} \quad \textit{for all } n.$$

From the graph or formula, the DC level is $\frac{a_0}{2} = \frac{3E}{4}$*. Therefore the Fourier series is:*

$$f(t) = \frac{3E}{4} - \frac{2E}{\pi^2}\sum_{odd\ \ n\in\mathbb{N}}\frac{1}{n^2}\cos\left(\frac{2\pi nt}{T}\right) - \frac{E}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2\pi nt}{T}\right)$$

7.3 Chapter 3: Matrix Algebra

7.3.1 Inverse of a 3×3 Matrix

There are four basic steps to this method for determining the inverse of a 3×3 matrix (if you wish, you can use other methods such as Gaussian elimination):

1. Calculate the "matrix of minors".
2. Create the Co-factor Matrix.
3. Determine the Adjunct Matrix.
4. Finally, multiply the Adjunct Matrix by $1/Determinant$.

Example 7.7. *Find the inverse of*

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

1. *Calculate the "matrix of minors":*

   To do this, for each element of the matrix: ignore the values on the current row and column, and calculate the determinant of the remaining values. Then put these determinants into a matrix.

$$\begin{pmatrix} \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \times 1 - (-2) \times 1 & 2 \times 1 - (-2) \times 0 & 2 \times 1 - 0 \times 0 \\ 0 \times 1 - 2 \times 1 & 3 \times 1 - 2 \times 0 & 3 \times 1 - 0 \times 0 \\ 0 \times (-2) - 2 \times 0 & 3 \times (-2) - 2 \times 2 & 3 \times 0 - 0 \times 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{pmatrix}$$

2. *Create the Co-factor Matrix:*

To turn the matrix of minors into the co-factor matrix, apply a checkerboard pattern of minus signs on alternate entries.

$$\begin{pmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{pmatrix} \quad \textit{Applying the sign-switching pattern:} \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\textit{Results in the co-factor matrix:} \begin{pmatrix} 2 & -2 & 2 \\ 2 & 3 & -3 \\ 0 & 10 & 0 \end{pmatrix}$$

3. *Determine the Adjunct Matrix (also known as the adjugate or adjoint matrix):*

The adjunct matrix is the transpose of the co-factor matrix, which means we reflect the matrix across the diagonal:

$$\begin{pmatrix} 2 & -2 & 2 \\ 2 & 3 & -3 \\ 0 & 10 & 0 \end{pmatrix}^T = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{pmatrix}$$

4. *Finally, multiply the Adjunct Matrix by* $1/Determinant$*:*

In Step 1, we already obtained most of the information required to calculate the determinant. Going across the top row of A and multiplying each entry by the corresponding co-factor:

$$det(A) = 3(2) + 0(2) + 2(2) = 6 + 4 = 10$$

Thus, the inverse of A is given by:

$$A^{-1} = \frac{1}{10} \begin{pmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1/5 & 1/5 & 0 \\ -1/5 & 3/10 & 1 \\ 1/5 & -3/10 & 0 \end{pmatrix}$$

5. *We can check that we have obtained the correct answer by checking that* $AA^{-1} = I$*:*

$$\begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \times \frac{1}{10} \begin{pmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 3\times 2+0\times(-2)+2\times 2 & 3\times 2+0\times 3+2\times(-3) & 3\times 0+0\times 10+2\times 0 \\ 2\times 2+0\times(-2)+(-2)\times 2 & 2\times 2+0\times 3+(-2)\times(-3) & 2\times 0+0\times 10+(-2)\times 0 \\ 0\times 2+1\times(-2)+1\times 2 & 0\times 2+1\times 3+1\times -3 & 0\times 0+1\times 10+1\times 0 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly we could check that $A^{-1}A = I$*.*

7.3.2 General Properties of Eigenvalues

Theorem 7.1. *The determinant of an invertible square matrix is equal to the product of it's eigenvalues. That is, for an invertible* $n \times n$ *matrix* A *with eigenvalues* $\lambda_1, \ldots, \lambda_n$*:*

$$|A| = \lambda_1 \lambda_2 \ldots \lambda_n$$

This is because the characteristic polynomial of A can be factorised in the following way:

$$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\ldots(\lambda_n - \lambda)$$

and so setting $\lambda = 0$ results in $|A| = \lambda_1 \lambda_2 \ldots \lambda_n$.

Theorem 7.2. *The sum of the eigenvalues of a square matrix is equal to the "trace" of the matrix, that is, the sum of its diagonal elements.*

$$tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

These two theorems can be used to obtain two eigenvalues of an $n \times n$ matrix A from a pair of simultaneous equations, given that the other $(n-2)$ are already known.

Theorem 7.3. *Let* A *be a real, symmetric* $n \times n$ *matrix. Then,*
i) Every eigenvalue of A *is real and has a corresponding real eigenvector.*

ii) Eigenvectors corresponding to distinct eigenvalues are orthogonal (perpendicular).
iii) Let the eigenvalues of A be the entries on a diagonal matrix D, and let U be a matrix whose columns are the corresponding eigenvectors. Then, $A = UDU^T$,
where U^T is the transpose of U (swap the rows and columns).

This is known as the Spectral Theorem. Basically, it says that for real, symmetric matrices there is a way of converting them to a related diagonal matrix, which can be useful in certain contexts.

There are many other properties of eigenvalues and eigenvectors for different classes of matrices, but they are not within the scope of this course. Matrix algebra is fundamental to one of the largest and most active areas of pure mathematics, as well as its many physical applications.

7.3.3 An application of Matrix algebra: Linear stability analysis of systems of ODEs

Eigenvalues are an extremely important concept in algebra, and have many uses when related to matrices that model physical problems. For systems of linear first-order ODEs in the form $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$, there is an equilibrium at the origin since if $\underline{\mathbf{x}} = \underline{0}$ then $\dot{\underline{\mathbf{x}}} = 0$ which is the definition of an equilibrium point. We can then determine the stability of the system at the origin by calculating the eigenvalues of the matrix of coefficients A. For equilibrium points not located at the origin, a system in the stated form can be obtained by linearising about that point.

- If all of the eigenvalues of A have negative real part, the system is linearly stable, which means that after a small perturbation the system will tend to return to the equilibrium point (think of a ball initially located at the bottom of a basin).
- If at least one of the eigenvalues has positive real part, the system is unstable and a small perturbation will be magnified (think of a ball positioned at the top of a hill).
- If all of the eigenvalues have non-positive real part, and at least one eigenvalue has real part exactly zero, the system is called marginally stable. This situation requires a bit more work to analyse.

This is an important tool in the study of dynamical systems, which can be used to model many different real-world scenarios including weather patterns, population biology, and chemical reactions. For example, if the set of ODEs represents a fixed point in a population biology system, knowing whether or not it is linearly stable will help us understand the effect of a small change to the system (for example, a small temporary increase in one species's population) - if it is stable, the effect will be contained and the system will recover back to the equilibrium. But if it is unstable, a small change could be disasterous.

7.3.4 Obtaining the State Transition Matrix

This is the full process used to derive e^{At} using a method based on a diagonalisation process. Suppose we have the state variable description $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$ where A is an $n \times n$ matrix.

Let $\underline{\mathbf{x}} = T\underline{\mathbf{y}}$, where T is a constant square matrix of appropriate dimensions.

$$\begin{aligned} \therefore \dot{\underline{\mathbf{x}}} &= T\dot{\underline{\mathbf{y}}} \\ \implies T\dot{\underline{\mathbf{y}}} &= \dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}} = AT\underline{\mathbf{y}} \end{aligned}$$

and so we have

$$\dot{\underline{\mathbf{y}}} = B\underline{\mathbf{y}}, \quad \text{where} \quad B = T^{-1}AT.$$

The relationship between matrices A and B ($B = T^{-1}AT$) is known as a **similarity transformation**. They have the same set of eigenvalues, and their sets of eigenvectors are related but not (necessarily) the same. To prove these results, let λ and $\underline{\mathbf{x}}$ be an eigenvalue-eigenvector pair for matrix A. By definition $A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}}$, so then $AT\underline{\mathbf{y}} = \lambda T\underline{\mathbf{y}}$, and so we obtain $T^{-1}AT\underline{\mathbf{y}} = \lambda\underline{\mathbf{y}}$ or $B\underline{\mathbf{y}} = \lambda\underline{\mathbf{y}}$. Hence, λ is also an eigenvalue of B, and has the corresponding eigenvector $\underline{\mathbf{y}} = T^{-1}\underline{\mathbf{x}}$.

Next, choose T to be the modal matrix of A.
Now,

$$\begin{aligned} AT &= A\left(\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \ldots, \underline{\mathbf{x}}_n,\right) \\ &= \left(A\underline{\mathbf{x}}_1, A\underline{\mathbf{x}}_2, \ldots, A\underline{\mathbf{x}}_n,\right) \\ &= \left(\lambda_1\underline{\mathbf{x}}_1, \lambda_2\underline{\mathbf{x}}_2, \ldots, \lambda_n\underline{\mathbf{x}}_n,\right) \\ &= \left(\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \ldots, \underline{\mathbf{x}}_n,\right) \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \end{aligned}$$

$\therefore AT = TD$, where $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. That is, D is the matrix with the eigenvalues of A on the digonal and zeros elsewhere. Note that the ordering of eigenvalues in D *must* match the ordering of the eigenvectors in T. Hence,

$$A = TDT^{-1} \quad \text{and} \quad D = T^{-1}AT$$

Therefore if T is chosen as the modal matrix of A, then matrix B from before is equal to this diagonal matrix D, and so we can obtain the n scalar equations in a simple form as follows:

$$\underline{\dot{\mathbf{y}}} = D\underline{\mathbf{y}}, \quad \text{or} \quad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

So the n scalar first-order ODEs are:

$$\dot{y}_1 = \lambda_1 y_1, \dot{y}_2 = \lambda_2 y_2, \dots, \dot{y}_n = \lambda_n y_n$$

and these can be easily solved using the method for *separable* linear first-order ODEs (or using Laplace transforms), yielding:

$$y_i(t) = \mathrm{e}^{\lambda_i t}\, y_i(0) \quad i = 1, 2, \dots, n$$

Putting these solutions back into matrix form:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & \mathrm{e}^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathrm{e}^{\lambda_n t} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \\ \vdots \\ y_n(0) \end{pmatrix}$$

and we write this as

$$\underline{\mathbf{y}}(t) = \mathrm{e}^{Dt}\, \underline{\mathbf{y}}(0), \quad \text{where} \quad D = diag(\lambda_1, \lambda_2, \dots, \lambda_n) \quad \text{and} \quad \mathrm{e}^{Dt} = diag(\mathrm{e}^{\lambda_1 t}, \mathrm{e}^{\lambda_2 t}, \dots, \mathrm{e}^{\lambda_n t})$$

Finally, let's perform some substitutions in order to get the solution back in terms of the original variable $\underline{\mathbf{x}}$. Recall that $\underline{\mathbf{x}}(t) = T\underline{\mathbf{y}}(t)$ and so $\underline{\mathbf{x}}(0) = T\underline{\mathbf{y}}(0)$ in particular. Therefore, $\underline{\mathbf{y}}(t) = T^{-1}\underline{\mathbf{x}}(t)$ and $\underline{\mathbf{y}}(0) = T^{-1}\underline{\mathbf{x}}(0)$. Hence,

$$\begin{aligned} & \underline{\mathbf{y}}(t) = \mathrm{e}^{Dt}\, \underline{\mathbf{y}}(0) \\ \Longrightarrow \quad & T^{-1}\underline{\mathbf{x}}(t) = \mathrm{e}^{Dt}\, T^{-1}\underline{\mathbf{x}}(0) \\ \Longrightarrow \quad & \underline{\mathbf{x}}(t) = T\, \mathrm{e}^{Dt}\, T^{-1}\underline{\mathbf{x}}(0) \end{aligned}$$

and so

$$\underline{\mathbf{x}}(t) = \mathrm{e}^{At}\, \underline{\mathbf{x}}(0), \quad \text{where} \quad \mathrm{e}^{At} = T\, \mathrm{e}^{Dt}\, T^{-1}.$$

Since we know how to obtain T and D from analysing the eigenvalues and eigenvectors of A, this then tells us how to obtain the state transition matrix e^{At} and thus solve the original ODE problem $\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}}$.