FMSS: Lecture 10 handout

Obtaining the state variable description of an ODE problem

1. We define a new set of variables, x_i, called **state variables**.
2. Every variable, *apart from external inputs*, generates an additional state variable for each derivative.
3. Rearrange to a set of equations for the **derivative of each state variable in terms of the state variables and external inputs**.

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \dots, \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = \dots, \qquad \frac{\mathrm{d}x_3}{\mathrm{d}t} = \dots$$

4. Representing these as a single matrix equation:

$$\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}} + B\underline{\mathbf{u}}$$

The state variable description

- $\underline{\mathbf{x}}$ is the vector containing the state variables $x_1, x_2, \dots, x_n$.
- $\underline{\dot{\mathbf{x}}}$ is the vector containing the first derivative of each state variable $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n$.
- **External control inputs** are encoded in their own vector $B\underline{\mathbf{u}}$.
- If there is no control input for the problem, the state variable description is just:

$$\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}}$$

Output vector

- We may also choose to define some output measurements.
- An output vector $\underline{\mathbf{y}}$ can encode both the **control inputs** and the **measured outputs**, constructed from a linear combination of the state variable and control input vectors:

$$\underline{\mathbf{y}} = C\underline{\mathbf{x}} + D\underline{\mathbf{u}}$$

Companion form

A **companion matrix** A consists of 1's in the entries that are one above the diagonal, any real numbers in the bottom row entries, and zeros elsewhere.

This is a standard feature of A when $\underline{\dot{\mathbf{x}}} = A\underline{\mathbf{x}}$ is obtained from a single n^{th} order ODE.

Eigenmodes

For each eigenvalue λ_i and eigenvector $\underline{\mathbf{b}}_i$ pair for A, the corresponding eigenmode is:

$$\underline{\mathbf{x}}(t) = c_i\ \underline{\mathbf{b}}_i\,\mathrm{e}^{\lambda_i t} \qquad \text{where } c_i \text{ is any scalar.}$$

Example 1

Determine the state variable description of a circuit, whose currents are described by a pair of ODEs:

$$\begin{aligned}
\frac{\mathrm{d}i_1(t)}{\mathrm{d}t} &= -\frac{i_1(t)}{CR} - \frac{\mathrm{d}i_2(t)}{\mathrm{d}t} + \frac{1}{R}\frac{\mathrm{d}e(t)}{\mathrm{d}t} \\
\frac{\mathrm{d}^2 i_2(t)}{\mathrm{d}t^2} &= -\frac{R}{L}\frac{\mathrm{d}i_2(t)}{\mathrm{d}t} + \frac{i_1(t)}{LC}
\end{aligned}$$

L, C and R are positive constants and $e(t)$ is an **external input**.

Example 2

A control system is modelled by the following fourth-order ordinary differential equation:

$$\frac{\mathrm{d}^4 x(t)}{\mathrm{d}t^4} + a_3\frac{\mathrm{d}^3 x(t)}{\mathrm{d}t^3} + a_2\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + a_1\frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_0 x(t) = 0$$

where $x(t)$ is a scalar and the **output** of the system.

There is **no external control input** in this example.

Example 3

Determine the state variable description of a circuit, whose currents are described by:

$$e(t) = L\frac{\mathrm{d}i_1(t)}{\mathrm{d}t} + \frac{1}{C}\int_0^t \big(i_1(t) - i_2(t)\big)\mathrm{d}t$$

$$\frac{1}{C}\int_0^t \big(i_2(t) - i_1(t)\big)\mathrm{d}t + Ri_2(t) + L\frac{\mathrm{d}i_2(t)}{\mathrm{d}t} + \frac{1}{C}\int_0^t i_2(t)\mathrm{d}t = 0$$

$$v(t) = \frac{1}{C}\int_0^t i_2(t)\mathrm{d}t$$

$e(t)$ is an external control input, R, C, and L are constants, and the **output** is given by:

$$z = L\frac{\mathrm{d}i_2}{\mathrm{d}t}$$