FMSS: Lecture 11 handout

Solving ODE systems using eigenvalues and eigenvectors

An ODE system $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$ has a solution:

$$\underline{\mathbf{x}}(t) = \mathrm{e}^{At}\,\underline{\mathbf{x}}(0)$$

where e^{At} is the **state transition matrix**.

With initial conditions $\underline{\mathbf{x}}(0)$, we can use this formula to predict the state variables $\underline{\mathbf{x}}(t)$ at any future time.

Modal matrices

Given an $n \times n$ matrix A, the modal matrix T is constructed column-by-column using the eigenvectors of A:

$$T = (\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \ldots, \underline{\mathbf{e}}_n,)$$

where each column vector $\underline{\mathbf{e}}_i$ is the i^{th} eigenvector of A.

The actual ordering of eigenvectors is not important so long as **the ordering always matches with the corresponding eigenvalues**.

There are infinitely many modal matrices, since the order of the eigenvectors is interchangable, and the eigenvectors themselves are not unique.

Obtaining the state transition matrix

Given a system of n linear first-order ODEs formulated as $\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}$, where A is a square $n \times n$ matrix, we can obtain the state transition matrix using the following **diagonalisation process**:

1. Find **eigenvalues** $\lambda_1, \ldots, \lambda_n$ and **eigenvectors** $\underline{\mathbf{e}}_1, \ldots, \underline{\mathbf{e}}_n$ of A.

2. Construct the **diagonal matrix of eigenvalues** $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the $n \times n$ **modal matrix** T where the i^{th} column consists of the eigenvector of A corresponding to the eigenvalue in the i^{th} diagonal entry of D.

$$D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \quad \text{and} \quad T = (\underline{\mathbf{e}}_1, \ldots, \underline{\mathbf{e}}_n)$$

3. Construct the **diagonal matrix of exponentials** e^{Dt}

$$e^{Dt} = diag(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}) = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}$$

4. Perform two matrix multiplications to calculate e^{At}

$$e^{At} = T\, e^{Dt}\, T^{-1}$$

5. The solution is given by the matrix multiplication:

$$\underline{\mathbf{x}}(t) = e^{At}\, \underline{\mathbf{x}}(0)$$

Example 1

A simple continuous-time model of population dynamics for two species is given by:

$$\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}}, \quad \text{where} \quad A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \quad \text{with initial conditions } \underline{\mathbf{x}}(0).$$

The eigenvalue and eigenvector pairs of A are:

$$\lambda_1 = 1, \quad \underline{\mathbf{b}}_1 = \alpha \begin{pmatrix} 1 \\ -2 \end{pmatrix}; \quad \text{and} \quad \lambda_2 = 6, \quad \underline{\mathbf{b}}_2 = \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Determine the state transition matrix using the diagonalisation process.

Example 2

An electronic control system is described by the following set of state variable equations:

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = x_3, \qquad \frac{dx_3}{dt} = \frac{9}{2}x_1 - \frac{7}{2}x_3$$

Determine the state transition matrix and hence find solutions for $x_1(t), x_2(t), x_3(t)$.