FMSS: Lecture 3 handout

Laplace Transforms of Integrals and Derivatives

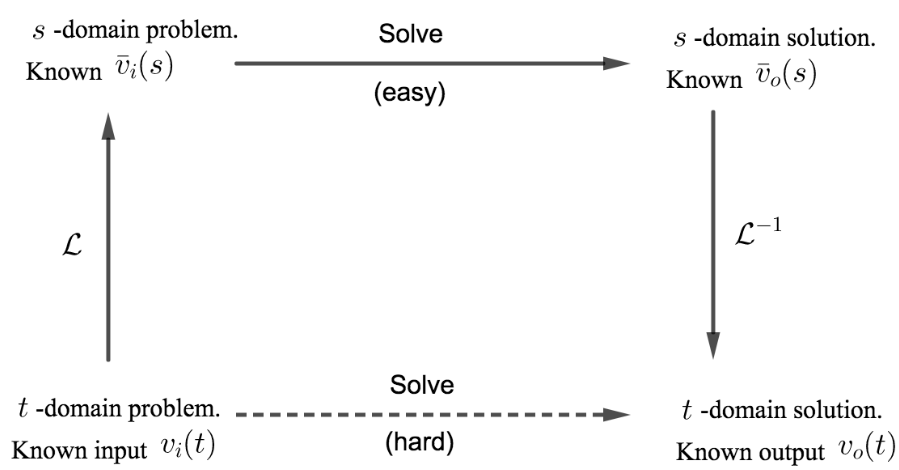
If $f(t)$ is some time-dependent function:

$$\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = s\bar{f}(s) - f(0)$$

$$\mathcal{L}\left\{\frac{\mathrm{d}^2f(t)}{\mathrm{d}t^2}\right\} = s^2\bar{f}(s) - sf(0) - \dot{f}(0)$$

$$\mathcal{L}\left\{\int_0^t f(t)\mathrm{d}t\right\} = \frac{\bar{f}(s)}{s}$$

Solving ODEs using Laplace transforms



Given a problem in the time-domain consisting of a set of equations with an input, output, and other time-dependent variables:

1. Take Laplace transforms of each equation to obtain the problem in the s-domain.
2. Rearrange and substitute the equations to obtain a formula for the transform of the ouput solely in terms of the transform of the input (all other s-dependent variables eliminated).
3. Incorporate a specific input function, if one is given.
4. Use the inverse Laplace transform to obtain the output in the time-domain.

Example 1

Solve the following first-order ordinary differential equation for $x(t)$ using Laplace transforms:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 3x(t) = U(t)$$

with the initial condition $x(0) = 0$

Example 2

The equations of a circuit are:

$$e(t) = Ri(t) + \frac{1}{C}\int_0^t i(t)\mathrm{d}t,$$

$$\text{where} \quad v(t) = \frac{1}{C}\int_0^t i(t)\mathrm{d}t \quad \text{is the p.d. across the capacitor.}$$

We wish to find the output $v(t)$, given an input voltage $e(t)$.

Example 3

Consider a system with input $f(t)$ and output $x(t)$, that is described by the following second-order nonhomogeneous ODE:

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 2x(t) = f(t)$$

with initial conditions

$$x(0) = 0 \quad \text{and} \quad \dot{x}(0) = 1$$

Using Laplace transforms, determine $x(t)$ when the input is:

$$f(t) = U(t-3)$$