FMSS: Lecture 8 handout

Order of a matrix

The order of a matrix is the number of rows $\times$ the number of columns.

$$\begin{pmatrix} 1 & 5 \\ -3 & 2 \\ 2 & 0 \end{pmatrix} \quad \text{is a} \quad 3 \times 2 \quad \text{matrix as it has 3 rows and 2 columns}$$

Only matrices with the **exact same order** can be added or subtracted to each other.

A **square matrix** (order $n \times n$) has the same number of rows and columns.

Matrix multiplication

The number of columns in the first matrix must match the number of rows in the second.

If this is satisfied, the result is is given by the remaining dimensions - the same number of rows as the first matrix and columns as the second matrix.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \times 5 + 2 \times 6 \\ 3 \times 5 + 4 \times 6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$$

2×2 $\qquad$ 2×1 $\qquad$ 2×1

Determinant

A **square matrix** A has a property called the determinant, denoted by $\det(A)$ or $|A|$.

Determinant of a 2×2 matrix:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Identity matrix

The $n \times n$ **identity matrix** has 1's on the diagonal entries and 0's elsewhere.

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

This is the only matrix which satisfies, for a matrix A of appropriate dimensions,

$$AI = IA = A$$

So it acts like a matrix version of the number "1" when it comes to multiplication.

Inverse matrix

For a **square** matrix A, there may exist an inverse matrix A^{-1} such that:

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

Inverse of a 2×2 matrix

For a general 2×2 square matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \text{or} \qquad A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If the determinant of a square matrix is **zero**, then that matrix has **no inverse!**

Determinant of a 3×3 matrix

Work across the **top row** and multiply each entry by the determinant of the corresponding 2×2 co-matrix of the rows and columns that the current entry is *not* in.

Then **change the sign of the middle entry**.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Eigenvalues and eigenvectors

To obtain the eigenvalue-eigenvector pairs of a matrix A, first find the eigenvalues by solving the **characteristic polynomial** for λ:

$$\det\big(A - \lambda I\big) = 0$$

Then for each eigenvalue $\lambda = \lambda_1, \lambda_2, \ldots$, we obtain a corresponding eigenvector $\underline{\mathbf{x}} = \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \ldots$

We can do this by substituting in the eigenvalue and solving:

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \qquad \text{for the column vector} \qquad \underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$$