FMSS: Lecture 9 handout

Calculating eigenvalues and eigenvectors

To obtain the eigenvalue-eigenvector pairs of a square matrix A:

1. First find the eigenvalues by solving the **characteristic polynomial** for λ:

   $$\det\big(A - \lambda I\big) = 0$$

   where I is the identity matrix with the same order as A.

2. Then for each eigenvalue $\lambda = \lambda_1, \lambda_2, \ldots$, we obtain a corresponding eigenvector $\underline{\mathbf{x}} = \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \ldots$

   We can do this by substituting in the eigenvalue and solving:

   $$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \qquad \text{for the column vector} \qquad \underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$$

Notes about eigenvalues and eigenvectors

- For an $n \times n$ square matrix A, there are n eigenvalues.
- Every eigenvalue has a family of **infinitely-many eigenvectors** associated with it. They all have the **same direction**, but can be of **any magnitude**.
- So if $\underline{\mathbf{x}}$ is an eigenvector of A with eigenvalue λ, then so is **any scalar multiple** of $\underline{\mathbf{x}}$.
- This means that when solving the set of equations to find the eigenvector $\underline{\mathbf{x}}$, there is **no one-and-only solution**. Instead, we can choose a value for one of the variables, and then use the equations to obtain the remaining variables.
- This also results in redundancy among the equations. In the 2×2 case, the two equations obtained will be the same.

Unit vectors

A **unit vector** has magnitude equal to one.

Given any vector, $\underline{\mathbf{v}}$ we can find the unit vector in the same direction by:

$$\hat{\underline{\mathbf{v}}} = \frac{\underline{\mathbf{v}}}{|\underline{\mathbf{v}}|}$$

In this case, the vertical lines denote the **magnitude** or absolute value of $\underline{\mathbf{v}}$, not the determinant.

Example 1: 2×2 Matrix

Determine the eigenvalues and eigenvectors of the following 2×2 matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

Example 2: 3×3 Matrix

Consider the following 3×3 matrix A.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

We will calculate the three eigenvalues and associated eigenvectors for this matrix.