Notes on the equations of motion

Applications of calculus to motion

In this part of the course we have been learning about how the displacement $s(t)$, velocity $v(t)$ and acceleration $a(t)$ at time t of a body are related to each other through calculus (i.e. through differentiation and integration). In particular, the key theory to remember is:

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \text{and} \quad v(t) = \frac{\mathrm{d}s}{\mathrm{d}t}, \quad \text{so also} \quad a(t) = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \tag{1}$$

in (other) words, velocity is the rate of change of displacement, and acceleration is the rate of change of velocity.

Going the other way using integrals instead:

$$s(t) = \int v(t)\,\mathrm{d}t \quad \text{and} \quad v(t) = \int a(t)\,\mathrm{d}t$$

What about "speed is distance over time"?

"But wait!", you may say, "isn't speed just distance divided by time, . . . and isn't acceleration just the change in velocity divided by the time passed?"

You may indeed be familiar with these ideas, expressed mathematically as:

$$v(t) = \frac{\Delta s}{\Delta t} \quad \text{and} \quad a(t) = \frac{\Delta v}{\Delta t} \tag{2}$$

where Δ (pronounced "delta") means "the change" of that particular quantity. Look at these formulae in (2) again. Then look at the new rules in (1), and back again. Do you notice anything? They actually look pretty similar except that instead of the Δ we have d in the new calculus rules! That's because the rules in (2) are just special cases of these more general rules that we are now learning. If velocity is **constant** then the rule $v = \Delta s/\Delta t$ is actually *the same* as $v = \mathrm{d}s/\mathrm{d}t$, while if velocity is not constant then it is only an approximation of the true relationship. Similarly, if acceleration is constant then $a = \Delta v/\Delta t$ is actually equivalent to $a = \mathrm{d}v/\mathrm{d}t$, while if acceleration is not constant it is only an approximation.

But what about the equations of motion?

These old rules in (2) are themselves just informal ways of stating some of the equations of motion that you may also have encountered. These include rules such as:

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

and a few others, where u is the initial speed, v is the final speed, a is the acceleration and s is the displacement during this time t.

These equations are still true, they haven't somehow become false, **but** once again what we need to recognise is that the equations of motion **only describe situations where acceleration is constant**. They do **not** apply to any other situation where the acceleration is not constant. That's why we learn this new application of differentiation and integration: these new rules in (1) supersede any previous ones you may have learned which were really just special cases of this more general set of rules!

To see this, let's consider that first equation of motion:

$$v = u + at$$

Here, the initial speed u is a constant, as is the acceleration a. So an example might be $v = 15 + 3t$ if $u = 15$ and $a = 3$. In other words, this is just a linear relationship (a straight line) between the final speed and the amount of time passed. Every second the speed increases at a constant rate of 3m/s^2. Now if we different this equation with respect to time, we get:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(u + at\big) = a$$

so even in the special case where these equations of motion apply, the more general truth still holds that acceleration is the derivative w.r.t time of velocity.

Conclusion

So what do you need to take away from this?

- It is **always** true that:

  $$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \text{and} \quad v(t) = \frac{\mathrm{d}s}{\mathrm{d}t}, \quad \text{so also} \quad a(t) = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

  and

  $$s(t) = \int v(t)\,\mathrm{d}t \quad \text{and} \quad v(t) = \int a(t)\,\mathrm{d}t$$

  In general, you should use these relationships when trying to solve problems about motion on this module.

- You can **only** use:

  $$v(t) = \frac{\Delta s}{\Delta t}$$

  when you know for a fact that velocity is **constant**. Otherwise this relationship is **not true!**

- You can **only** use:

  $$a(t) = \frac{\Delta v}{\Delta t}$$

  and the equations of motion when you know for a fact that acceleration (or deceleration) is **constant**. Otherwise this relationship is **not true!**