Introduction to Differentiation

Core topics in Mathematics

Lecture 11

Learning Outcomes

- State what is meant by the gradient of a curve at a point.
- Differentiate simple functions to obtain their derivatives.

Introduction

Differentiation allows us to calculate the **gradient** of a curve or, more specifically, a **rate of change** of one variable with respect to another variable.

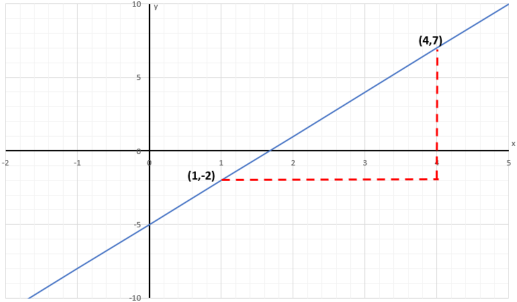
slope $\equiv$ **gradient** $\equiv$ **derivative** $\equiv$ **rate of change**

Examples of rates of change:

- velocity (rate of change of displacement with respect to time)
- acceleration (rate of change of velocity w.r.t. time)
- power (rate of low of energy w.r.t. time)

Calculating Gradients

We have already seen how to calculate the gradient of linear functions (straight lines):



Here the gradient is:

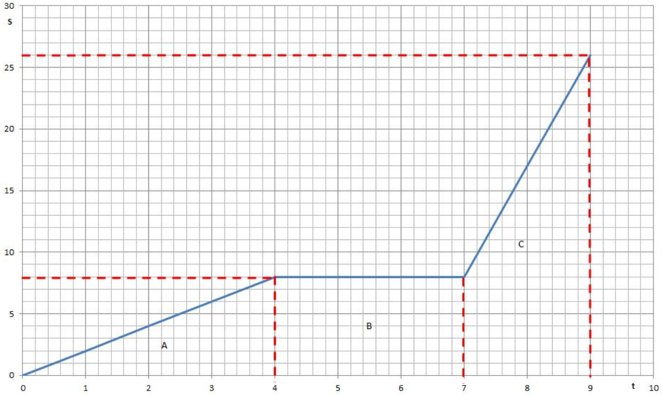
$$m = \frac{\Delta y}{\Delta x} = \frac{7-(-2)}{4-1} = \frac{9}{3} = 3$$

where Δ is the change, or difference in, y or x.

For a linear function the gradient is constant throughout, i.e. there is no dependence on x.

Physical Example

Consider this piece-wise graph which illustrates the displacement S (m) of an object over time t (s).



Physical Example

From the graph we can see that the object is moving in regions A and C and is stationary in region B.

Calculating the gradient in region A:

$$m = \frac{\Delta S}{\Delta t} = \frac{8}{4} = 2$$

Consider the units:

$$\frac{\Delta S}{\Delta t} = \frac{\text{m}}{\text{s}} \quad \text{which is m/s.}$$

So the gradient of a displacement-time curve gives a velocity (**rate of change** of displacement w.r.t. time). As the graph is a straight line, in A the object has a constant velocity of 2 m/s.

Physical Example

Calculating the gradient in region B:

$$m = \frac{\Delta S}{\Delta t} = \frac{0 \text{ m}}{3 \text{ s}} = 0 \text{ m/s}$$

This indicates that the object is travelling at 0 m/s, i.e. it is stationary.

Looking back at the graph: at 4 seconds the object is at the 8 metre mark and at 7 seconds the object is still at the 8 metre mark, so cannot be moving.

Calculating Gradients

The physical example has provided us with two important results:

1) Consider the equation of the line in region A: it has form $y = mx + c$ and specifically $y = 2x$ (think of x and y rather than t and S). The gradient here was simply 2. If the equation of the line was $y = 5x$, then the gradient would be 5, etc.

Therefore, if:

$$y = ax, \quad \text{then gradient:} \quad m = \frac{\Delta y}{\Delta x} = \frac{\text{difference in } y}{\text{difference in } x} = \frac{\mathrm{d}y}{\mathrm{d}x} = a$$

Calculating Gradients

Similarly for region B the equation of the line is of the form $y = mx + c$ and specifically $y = 8$.

2) The gradient here was simply 0. If the equation of the line was $y = 9$, then the gradient would also be 0, as it is a straight horizontal line and has no steepness.

Therefore, if:

$$y = a, \quad \text{then gradient:} \quad m = \frac{\Delta y}{\Delta x} = \frac{\text{difference in } y}{\text{difference in } x} = \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Gradients of linear or constant functions

y	$\dfrac{\mathrm{d}y}{\mathrm{d}x}$
a (any constant)	0
ax	a

The functions in the right-hand column of this table are known as the derivatives of the functions in the left-hand column.

We obtain the derivative of a function by differentiation.

Notation

$\frac{\mathrm{d}y}{\mathrm{d}x}$ represents the gradient/derivative of a curve $y = f(x)$.

y' and $f'(x)$ are common alternatives to the symbol $\frac{\mathrm{d}y}{\mathrm{d}x}$. They all mean gradient/derivative/rate of change, where $f(x)$ is another way of writing that y is a function f of x.

$\dot{y}$ is also another way to represent the derivative, but in the case specifically w.r.t. **time**, i.e. $\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t}$

If, instead of $y = f(x)$, we have, say, $r = f(t)$ then the derivative of r is written as $\frac{\mathrm{d}r}{\mathrm{d}t}$

Exercise:

Determine the gradients of the following lines:

1) $y = 6x$

Exercise:

Determine the gradients of the following lines:

1) $y = 6x$ $\qquad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 6$

2) $x = 9.7t$

Exercise:

Determine the gradients of the following lines:

1) $y = 6x$ $\qquad \frac{\mathrm{d}y}{\mathrm{d}x} = 6$

2) $x = 9.7t$ $\qquad \frac{\mathrm{d}x}{\mathrm{d}t} = 9.7$

3) $r = \frac{3}{5}\theta$

Exercise:

Determine the gradients of the following lines:

1) $y = 6x$ $\qquad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 6$

2) $x = 9.7t$ $\qquad \dfrac{\mathrm{d}x}{\mathrm{d}t} = 9.7$

3) $r = \dfrac{3}{5}\theta$ $\qquad \dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \dfrac{3}{5}$

4) $y = -12$

Exercise:

Determine the gradients of the following lines:

1) $y = 6x$ $\qquad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 6$

2) $x = 9.7t$ $\qquad \dfrac{\mathrm{d}x}{\mathrm{d}t} = 9.7$

3) $r = \dfrac{3}{5}\theta$ $\qquad \dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \dfrac{3}{5}$

4) $y = -12$ $\qquad y' = 0$

5) $P = \dfrac{7}{8}$

Exercise:

Determine the gradients of the following lines:

1) $y = 6x$ $\qquad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 6$

2) $x = 9.7t$ $\qquad \dfrac{\mathrm{d}x}{\mathrm{d}t} = 9.7$

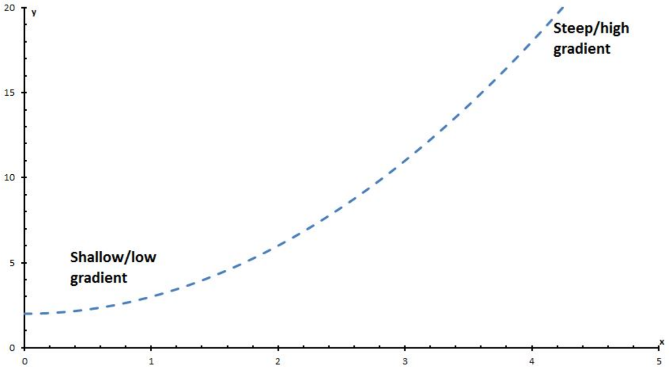
3) $r = \dfrac{3}{5}\theta$ $\qquad \dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \dfrac{3}{5}$

4) $y = -12$ $\qquad y' = 0$

5) $P = \dfrac{7}{8}$ $\qquad P' = 0$

Differentiation

Calculating the gradient of a **curve** is harder as the gradient varies along the curve, i.e. it depends on x.



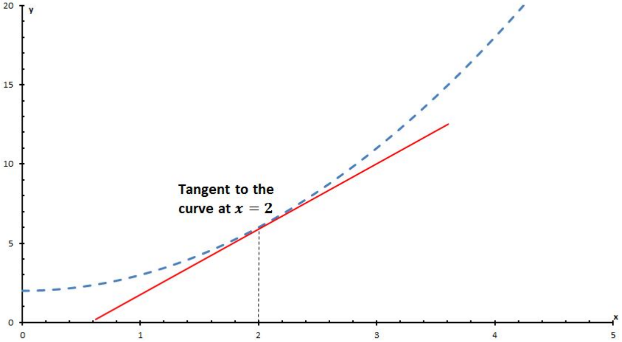
Differentiation

First, we would need a rigorous way of defining the gradient of a curve at a particular point:

The gradient of a curve at a point is equal to the gradient of the tangent line at that point.

A tangent line is a straight line that only just touches the curve at exactly that particular point. So we could draw such a line at the point we were interested in. . .

Differentiation



Then we would set up a triangle (as in linear cases) to calculate the gradient of the tangent. But . . . how could we consistently draw perfect tangents to the curve at all infinitely-many points?

Differentiation from first principles

The gradient can be calculated by taking the *limit* that an estimate of the gradient tends to, as you zoom in on a smaller area around the point. For a general function $y = f(x)$, the gradient at a point x_0 is then given by:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \frac{\text{approximate rise near } x_0}{\text{approximate run near } x_0}$$

This is called "differentiation from first principles". For standard functions, formulae for the derivatives have been proven using this process. We can just use these rules, so we will **not** need to draw tangents of use the above formula ourselves in this module.

Differentiation: Standard rules. **See Formulae booklet!**

For constant *a*, *n*:

y	$\frac{dy}{dx}$
a (any constant)	0
ax	a
ax^n	$n \times ax^{n-1}$
ae^{nx}	$n \times ae^{nx}$
$a \ln nx$	$\frac{a}{x}$
$a \sin nx$	$n \times a \cos nx$
$a \cos nx$	$-n \times a \sin nx$

Example 1

Calculate an expression for the gradient of $y = 7x^3$.

Looking in the left-hand-side of the table, we can see that this is in the form ax^n, where $a = 7$ and $n = 3$. The corresponding right-hand column instructs us on how to differentiate it:

If $y = ax^n$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = n \times ax^{n-1}$

Therefore, in the case of $y = 7x^3$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3 \times 7x^{3-1} \\ &= 21x^2\end{aligned}$$

Example 2

Calculate an expression for the gradient of $y = 4x^2$.

Again, this is in the form ax^n, where $a = 4$ and $n = 2$. Therefore

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2 \times 4x^{2-1} \\ &= 8x\end{aligned}$$

Note that this is an expression for the gradient, which is dependent upon x. If we wanted to calculate the gradient at a *specific point*, say $x = 5$, then we simply substitute this value into the gradient expression:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=5} = 8 \times 5 = 40$$

Exercise

Determine expressions for the gradients of the following curves:

1) $y = 3x^4$

2) $y = -7x^9$

3) $x = 9t^{-2}$

4) $y = \frac{7}{2}x^3$, and calculate the gradient at the point $x = 4$.

5) $y = \frac{2}{\phi^5}$, and calculate the gradient at the point $\phi = -2.4$.

6) $y = 4\sin(5x)$. This has the form $a\sin(nx)$. What are a and n?

Exercise: Solutions (I/II)

1) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(3x^4\right) = 3 \times 4x^{4-1} = 12x^3$

2) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(-7x^9\right) = -7 \times 9x^{9-1} = -63x^8$

3) $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(9t^{-2}\right) = 9 \times (-2)t^{-2-1} = -18t^{-3}$

4) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{7}{2}x^3\right) = \frac{7}{2} \times 3x^{3-1} = \frac{21}{2}x^2$

Hence, at $x = 4$, the gradient is: $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=4} = \frac{21}{2}(4)^2 = 168$

Exercise: Solutions (II/II)

5) First, the function must be rewritten in the form: $y = 2\phi^{-5}$

Then $\dfrac{\mathrm{d}y}{\mathrm{d}\phi} = \dfrac{\mathrm{d}}{\mathrm{d}\phi}\left(2\phi^{-5}\right) = 2 \times (-5)\phi^{-5-1} = -10\phi^{-6}$

At $\phi = -2.4$, we have $\left.\dfrac{\mathrm{d}y}{\mathrm{d}\phi}\right|_{\phi=-2.4} = -10(-2.4)^{-6} = -0.05$

6) $a = 4$ and $n = 5$, then the derivative is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(4\sin(5x)\right) = 4 \times 5\cos(5x) = 20\cos(5x)$$

Example 3

Calculate an expression for the gradient of:

$$y = 3x^2 + 7x - 3 + 2e^{5x}$$

When we have a sum of multiple terms, in order to differentiate this we simply differentiate each term and sum their gradients in the same way (this property of differentiation is called *linearity*).

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(7x\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(3\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(2e^{5x}\right) \\ &= \left(2 \times 3x^{2-1}\right) + \left(7\right) - \left(0\right) + \left(5 \times 2e^{5x}\right) \\ &= 6x + 7 + 10e^{5x} \end{aligned}$$