Differentiation: the Chain Rule

Core topics in Mathematics

Lecture 12

Learning Outcomes

- Recognise "functions of functions".
- Apply the chain rule to differentiate these.

Introduction

It is not possible to differentiate every function using the standard rules. For example, if we wished to differentiate:

$$y = 7x^3 \sin(5x)$$

there is no formula in the table for the precise form $ax^n \sin(mx)$.

Similarly we can't (yet) differentiate:

$$y = \frac{8e^{-6x} + 3x}{\cos(2x)} \quad \text{or} \quad y = 9(2x-4)^3,$$

We will be learning additional rules to cover cases like these.

The Chain Rule

To differentiate a function of a function $y = f(g(x))$ (i.e. one function inside another function), we must use the **chain rule**.

The Chain Rule:

If $y = f(u)$ and $u = g(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

f is the "outer" function, and g is the "inner" function, which we designate as a new variable u.

Functions of functions

We would use the chain rule for functions that look like:

$y = 5(3x - 8)^4$ where the inner function $3x - 8$ lies within the outer function $5(\mathrm{X})^4$

$y = -2\cos(4x + 7)$ where the function $4x + 7$ lies within the outer function $-2\cos(\mathrm{X})$

$y = 7e^{5x^2}$ where the function $5x^2$ lies within the function $7e^{\mathrm{X}}$

Example 1 (I/III)

To determine the derivative of

$$y = 3(5x - 7)^4$$

we must first recognise that we have one function $5x - 7$ inside another function $3X^4$.

We make a substitution u, usually for the "thing" inside the brackets. Thus, if we let $u = 5x - 7$, we can re-write the original equation (the outer function) as:

$$y = 3u^4$$

By introducing u, we have separated the original "'function of a function" into two "simple" functions: $y = 3u^4$ and $u = 5x - 7$

Example 1 (II/III)

The chain rule formula requires y to be differentiated w.r.t. u and u to be differentiated w.r.t. x:

$$u = 5x - 7 \quad \Longrightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 5$$

$$y = 3u^4 \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}u} = 12u^3$$

Substituting these into the rule:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= (12u^3) \times (5) \\ &= 60u^3 \end{aligned}$$

Example 1 (III/III)

However, this is not the final answer, as we must now substitute $u = 5x - 7$ back into the answer:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 60u^3 \\ &= 60(5x-7)^3 \end{aligned}$$

Always state the final answer in terms of the original variables (in this case x) and not u, which we introduced during the process of solving the problem.

Example 2 (I/II)

Determine the derivative of

$$y = -5\cos(2x+3)$$

First, substitute the inner function: $u = 2x + 3$.

Second, re-write the original equation: $y = -5\cos(u)$.

Now calculate the derivatives of $u = g(x)$ and $y = f(u)$:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}u} = 5\sin(u)$$

Example 2 (II/II)

Now, substitute both results into the chain rule formula:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= (5\sin u) \times (2) \\ &= 10\sin(u)\end{aligned}$$

Finally substitute $u = 2x + 3$ back into the answer:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 10\sin(2x + 3)$$

Example 3 (I/II)

Determine the derivative of

$$y = 3e^{(5x^2-3x+1)}$$

First, substitute the inner function $u = 5x^2 - 3x + 1$

Then re-write the original equation (the outer function): $y = 3e^u$

Now calculate the derivatives of u $(= g(x))$ and y $(= f(u))$:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 10x - 3 \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}u} = 3e^u$$

Example 3 (II/II)

Substitute both results into the chain rule formula:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= (3e^u) \times (10x - 3) \\ &= 3(10x - 3)e^u\end{aligned}$$

Finally substitute $u = 5x^2 - 3x + 1$ back in to obtain the answer in terms of x only:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3(10x - 3)e^{(5x^2 - 3x + 1)}$$