Differentiation: the Product and Quotient rules

Core topics in Mathematics

Lecture 13

Learning Outcomes

- Apply the product and quotient rules to differentiate more complicated functions.

The Product Rule: Example 1

The function

$$y = 9x^2 e^{7x}$$

is comprised of one function $9x^2$ **multiplied** by another function e^{7x}. This is more complicated than any of our standard functions, but it also isn't a "function of a function" (there are no obvious inner and outer parts), so the chain rule cannot help either.

In order to differentiate this, we need to use the **product rule**.

The Product Rule

The product rule tells us how to differentiate a function that is the product (multiple) of two functions.

Product Rule:

If $y = u \cdot v$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

This formula is made up of two functions, u and v. Note that these are two elements of the overall function y that we want to differentiate. To determine $\mathrm{d}v/\mathrm{d}x$ we must differentiate v w.r.t. x and similarly to determine $\mathrm{d}u/\mathrm{d}x$ we must differentiate u w.r.t. x.

Return to Example 1: $y = 9x^2e^{7x}$

In this example:

$$u = 9x^2 \qquad \therefore \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 18x$$

$$v = e^{7x} \qquad \therefore \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 7e^{7x}$$

Substituting these values into the product rule gives:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= (9x^2) \times (7e^{7x}) + (e^{7x}) \times (18x) \\ &= 63x^2e^{7x} + 18xe^{7x}\end{aligned}$$

Example 2

Differentiate:

$$y = -5x^4 \sin(3x)$$

Let $u = -5x^4 \quad \therefore \quad \frac{\mathrm{d}u}{\mathrm{d}x} = -20x^3$

and $v = \sin(3x) \quad \therefore \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 3\cos(3x)$

Substituting these values into the product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} = (-5x^4) \times (3\cos(3x)) + (\sin(3x)) \times (-20x^3)$$

Example 2

This should be simplified as much as possible:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (-5x^4) \times (3\cos(3x)) + (\sin(3x)) \times (-20x^3) \\ &= -15x^4\cos(3x) - 20x^3\sin(3x)\end{aligned}$$

This could be further simplified by factorisation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -5x^3(3x\cos(3x) + 4\sin(3x))$$

The Quotient Rule - Example 3

The equation

$$y = \frac{9\cos(3x)}{5x^4}$$

is comprised of one function $9\cos(3x)$ **divided** by another function $5x^4$. Again, none of our existing rules are able to handle this[1] so in order to differentiate this function we need to use the **quotient rule**.

[1] Can you think of a way to re-write this function so that we could use the product rule?

The Quotient Rule

The quotient rule tells us how to differentiate a function that is a fraction (quotient) of two functions.

Quotient Rule:

If $y = \frac{u}{v}$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

This is very similar to the product rule method, but we substitute the four terms into a different equation. Note that it is essential that u is the numerator, and v the denominator.

Return to Example 3: $y = \frac{9\cos(3x)}{5x^4}$

In this example:

Numerator: $u = 9\cos(3x)$ $\therefore$ $\frac{\mathrm{d}u}{\mathrm{d}x} = -27\sin(3x)$

Denominator: $v = 5x^4$ $\therefore$ $\frac{\mathrm{d}v}{\mathrm{d}x} = 20x^3$

Substituting these values into the quotient rule gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

The Quotient Rule

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(5x^4)\times(-27\sin(3x)) - (9\cos(3x))\times(20x^3)}{(5x^4)^2} \\
&= \frac{-135x^4\sin(3x) - 180x^3\cos(3x)}{25x^8} \\
&= \frac{-45x^3}{25x^8}\big(3x\sin(3x) + 4\cos(3x)\big) \\
&= \frac{-9}{5x^5}\big(3x\sin(3x) + 4\cos(3x)\big)
\end{aligned}
$$

Example 4

Differentiate:

$$y = \frac{9x^3}{2\sin(5x)}$$

Let the numerator be $u = 9x^3 \quad \therefore \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 27x^2$

and the denominator be $v = 2\sin(5x) \quad \therefore \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 10\cos(5x)$

Example 4

Substituting these values into the quotient rule gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

$$= \frac{(2\sin(5x)) \times (27x^2) - (9x^3) \times (10\cos(5x))}{(2\sin(5x))^2}$$

$$= \frac{54x^2\sin(5x) - 90x^3\cos(5x)}{4\sin^2(5x)}$$