Introduction to Integration

Core topics in Mathematics

Lecture 15

Learning Outcomes

- Understand what integration is, in terms of (i) the inverse of differentiation, and (ii) finding the area under a curve.
- Learn the standard rules for integration, using the tables in the formula booklet.
- Understand the difference between **definite** and **indefinite** integration.

Introduction

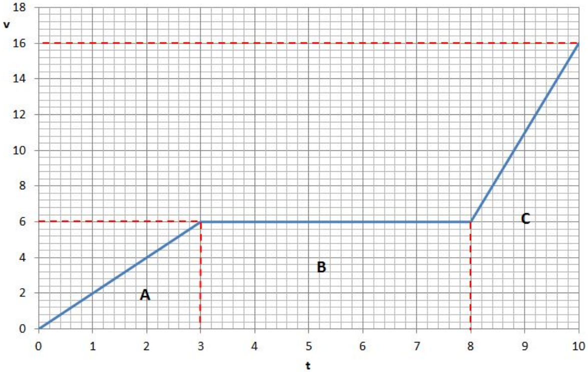
Integration is used to solve many problems, such as determining how much of a quantity has accumulated over time:

- Integrate power over time to determine total energy.
- Integrate force over distance to determine energy spent (valuable in potential energy problems).
- Integrate flow rate over time to determine accumulation of the flowing quantity e.g. mass, volume, charge etc.

Integration concerns calculating the **area underneath a curve.**

Example

Consider again this piece-wise graph which illustrates the velocity v (m/s) of an object over time t (s).



Example

Considering region B we can determine that the object is travelling at 6 m/s between $t = 3$ s and $t = 8$ s.

So in this region the object has travelled 6 m after 1 second, 12 m after 2 seconds and 30 m after 5 seconds.

But we could instead calculate the total distance travelled in region B by calculating the **area underneath the curve:**

$$\begin{aligned} \text{Area} &= \text{Width} \times \text{Height} \\ &= 5\ (\text{s}) \times 6\ (\text{ms}^{-1}) \\ &= 30\ \text{m} \end{aligned}$$

Integration: Opposite of Differentiation

We could similarly calculate the displacement in regions A and C, using the formulae for the area of a trapezium.

However, calculating the area underneath a more general **curve** requires approximations by trapezia, and is a non-trivial process.

So, just as we calculated gradients by referring to our table of derivatives, we will now look at integrating functions by referring to a table of integrals.

Integration is the **inverse of differentiation**, just as division is the inverse of multiplication and subtraction is the inverse of addition. This means that many of the rules will be familiar, but reversed!

Integration Table - **find this in the formula booklet!**

y	$\int y\,dx$
a (any constant)	$ax + C$
ax^n	$\frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$
ae^{nx}	$\frac{ae^{nx}}{n} + C$
$\frac{a}{x}$	$a \ln x + C$
$\frac{a}{nx+b}$	$\frac{a \ln(nx+b)}{n} + C$
$a \sin nx$	$\frac{-a \cos nx}{n} + C$
$a \cos nx$	$\frac{a \sin nx}{n} + C$

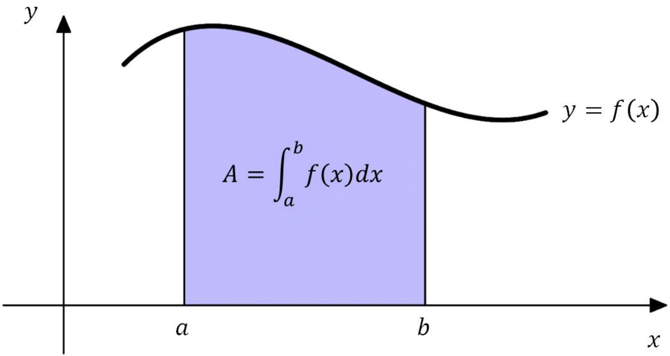
Definite Integration

Integration can be either **definite** or **indefinite**.

Definite integration allows us to calculate the exact area enclosed between:

- a curve,
- two vertical lines and
- the x-axis.

Notation for Definite Integration



Area A is given by the definite integral of function $f(x)$ between the limits $x = a$ and $x = b$.

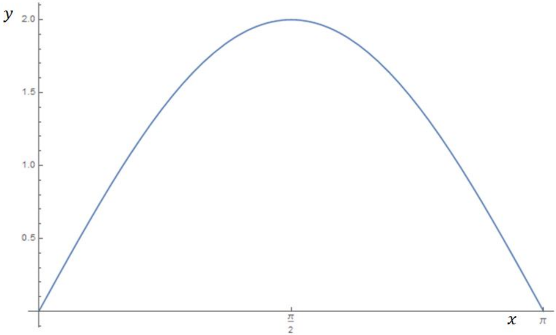
Definite Integration Example 1

Using formal integration, let's confirm the answer from the earlier example. In region B, between $x = 3$ and $x = 8$, the function was a constant: $y = 6$

$$\begin{aligned} \text{disp} &= \int_3^8 6 \, \mathrm{d}x \\ &= \left[6x\right]_3^8 \quad \text{Now sub. in upper and lower limits:} \\ &= (6 \times 8) - (6 \times 3) \quad \text{(upper) - (lower)} \\ &= 48 - 18 \\ &= 30\text{m} \quad \text{as expected.} \end{aligned}$$

Definite Integration Example 2

Determine the area under the curve $y = 2\sin x$ between the limits of $x = 0$ and $x = \pi$. Note that we will always use radians for trigonometric functions in this context unless stated otherwise.



Definite Integration Example 2 - Solution

$$\begin{aligned}
\text{Area} &= \int_0^{\pi} 2\sin x \, \mathrm{d}x \\
&= \Big[-2\cos x \Big]_0^{\pi} \\
&= \big(-2\cos(\pi)\big) - \big(-2\cos(0)\big) \quad \text{(use radians!)} \\
&= 2 + 2 \\
&= 4 \text{ units squared}
\end{aligned}$$

Physical Example 3

Determine the displacement, S m, of an object between the times $t = 2$ s and $t = 5$ s given that the expression for its velocity is:

$$\mathrm{v} = 3t^2 - 6t + 7$$

Physical Example 3 - Solution

$$
\begin{aligned}
S &= \int_2^5 \mathrm{v}\ \mathrm{d}t \\
&= \int_2^5 3t^2 - 6t + 7\ \mathrm{d}t \\
&= \left[\frac{3t^3}{3} - \frac{6t^2}{2} + 7t\right]_2^5 \\
&= \left[t^3 - 3t^2 + 7t\right]_2^5 \\
&= (5^3 - 3 \times 5^2 + 7 \times 5) - (2^3 - 3 \times 2^2 + 7 \times 2) \\
&= 85 - 10 \quad = \quad 75\ \text{m}
\end{aligned}
$$

Indefinite Integration

The other kind is **indefinite integration**, where we do not have specific upper and lower limits. The solutions will always contain the constant of integration, C, to account for this uncertainty.

Example 4: Integrate the function $5x^2 + 7x - 2$ with respect to x:

$$\int 5x^2 + 7x - 2 \, \mathrm{d}x = \frac{5x^3}{3} + \frac{7x^2}{2} - 2x + C$$

Note:

- only one $+C$ is required at the end of the answer, as if there were multiple unknown constants they could be merged.
- $+C$ could appear in definite integration, but it would disappear when subtracting the lower limit from the upper.

Indefinite Integration Example 5

Integrate:

$$y = \int 4x^3 - 7e^{3x} \, \mathrm{d}x$$

Indefinite Integration Example 5 - Solution

$$
\begin{aligned}
y &= \int 4x^3 - 7e^{3x} \, \mathrm{d}x \\
&= \frac{4x^4}{4} - \frac{7e^{3x}}{3} + C \\
&= x^4 - \frac{7}{3}e^{3x} + C
\end{aligned}
$$

We can only determine the constant of integration if we are given additional information about the curve or the physical problem, such as a coordinate, or an initial condition.

Indefinite Integration Example 5

For example, if we were also told that $y = 2$ when $x = 0$, then substitute this in and solve for C:

$$y = x^4 - \frac{7e^{3x}}{3} + C$$

$$\text{At } (0,2): \qquad 2 = 0^4 - \frac{7e^{3\times 0}}{3} + C$$

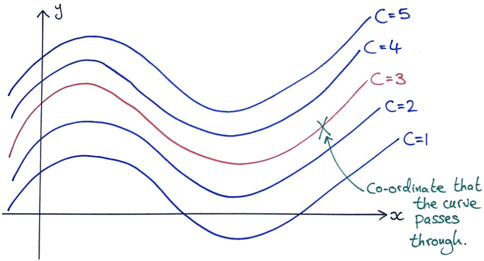
$$2 = -\frac{7}{3} + C$$

$$\therefore \qquad C = 2 + \frac{7}{3} = \frac{13}{3}$$

Therefore, the particular solution is: $y = x^4 - \frac{7e^{3x}}{3} + \frac{13}{3}$

Indefinite integration

How does a condition help us find C? Remember that integration is the inverse of differentiation, so indefinite integrate shows us the *set* of curves whose gradient all obey the same function. Thus, they are all parallel and vertically-shifted depending on the value of C.



A co-ordinate assigns a value to C, selecting a particular curve.

Exercises

Integrate the following:

1) $\int 6\sin(3x) + 5x^{-2}\ \mathrm{d}x$

2) $\int \frac{7x^2}{3} + e^{-2x} + \frac{5}{x}\ \mathrm{d}x$

3) $\int_1^3 \frac{2}{x^3} + 2x + 8\ \mathrm{d}x$

4) Determine the particular solution of $y = \int 3 - 7x + 12x^2\ \mathrm{d}x$, given that when $x = -2$, $y = 0.5$.

Exercises - Solutions (I/IV)

1) Indefinite integration:

$$\int 6\sin(3x) + 5x^{-2}\ \mathrm{d}x = -\frac{6}{3}\cos(3x) + \frac{5}{-1}x^{-1} + C$$

$$= -2\cos(3x) - \frac{5}{x} + C$$

2) Indefinite integration:

$$\int \frac{7x^2}{3} + e^{-2x} + \frac{5}{x}\ \mathrm{d}x = \frac{7}{9}x^3 + \frac{1}{-2}e^{-2x} + 5\ln(x) + C$$

$$= \frac{7}{9}x^3 - \frac{1}{2}e^{-2x} + 5\ln(x) + C$$

Exercises - Solutions (II/IV)

3) Definite integration:

$$\int_1^3 \frac{2}{x^3} + 2x + 8 \ \mathrm{d}x = \int_1^3 2x^{-3} + 2x + 8 \ \mathrm{d}x$$

$$= \left[\frac{2}{-2}x^{-2} + \frac{2}{2}x^2 + 8x\right]_1^3$$

$$= \left[-x^{-2} + x^2 + 8x\right]_1^3$$

$$= \left(\frac{-1}{9} + 9 + 24\right) - \left(\frac{-1}{1} + 1 + 8\right)$$

$$= 32\,\frac{8}{9} - 8 = 24\,\frac{8}{9}$$

Exercises - Solutions (III/IV)

4) First, the indefinite integral:

$$\begin{aligned} y &= \int 3 - 7x + 2x^2 \, \mathrm{d}x \\ &= 3x - \frac{7}{2}x^2 + \frac{2}{3}x^3 + C \end{aligned}$$

Then substitute in the condition $x = -2$ and $y = 0.5$:

$$0.5 = 3(-2) - \frac{7}{2}(-2)^2 + \frac{2}{3}(-2)^3 + C$$

Exercises - Solutions (IV/IV)

Simplifying, and solving for C:

$$0.5 = -6 - \frac{7}{2} \times 4 + \frac{2}{3} \times -8 + C$$

$$0.5 = -6 - 14 - \frac{16}{3} + C$$

$$C = \frac{1}{2} + 20 + \frac{16}{3} = \frac{155}{6}$$

So the particular solution is:

$$y = 3x - \frac{7}{2}x^2 + \frac{2}{3}x^3 + \frac{155}{6}$$