Applications of Integration

Core topics in Mathematics

Lecture 18

Learning Outcomes

- Apply integration to problems in an engineering context.
- Determine the area between two different curves plotted in the same plane.

Integration in an Engineering Context

Both processes of calculus can be applied to consider how certain quantities are changing with respect to another (usually time).

Differentiation gives us the rate of change of a quantity.

Integration instead can be used to find the **total amount of a quantity that accumulates over time**, given that we know something about it's rate of change or flow.

Example 1

The volume of water V accumulated in a tank is given by:

$$\mathrm{V} = \int Q \, \mathrm{d}t$$

where $Q = (1-t)^2 + 16$ is the (volumetric) flow rate of water into the tank and t is time.

If $\mathrm{V} = 4$ when $t = 0$, determine the relationship between V and t.

Note that all quantities are in SI units.

Example 1 - Solution (I/II)

Substitute in the formula:

$$\begin{aligned} \mathrm{V} &= \int (1-t)^2 + 16 \, \mathrm{d}t \\ &= \int 1 + t^2 - 2t + 16 \, \mathrm{d}t \\ &= \int t^2 - 2t + 17 \, \mathrm{d}t \\ &= \frac{1}{3}t^3 - t^2 + 17t + C \end{aligned}$$

Example 1 - Solution (II/II)

Finally, we substitute in the condition ($\mathrm{V} = 4$ when $t = 0$) to determine the value of C:

$$\mathrm{V}(t = 0) = 4 \quad \Longrightarrow \quad 4 = \frac{1}{3}(0)^3 - (0)^2 + 17(0) + C$$

Simplifying,

$$4 = 0 + C \quad \Longrightarrow \quad C = 4$$

So the particular solution for the volume is:

$$\therefore\ \mathrm{V} = \frac{1}{3}t^3 - t^2 + 17t + 4$$

Example 2

The amount of charge q passing a point in a circuit during a period of time τ is governed by:

$$q = \int_0^\tau i(t)\ \mathrm{d}t$$

where $i(t)$ is the current flow (in μA) at time $0 \leq t \leq \tau$

If $i(t) = 40e^{-0.1t}$, find a relationship between the variables q and t.

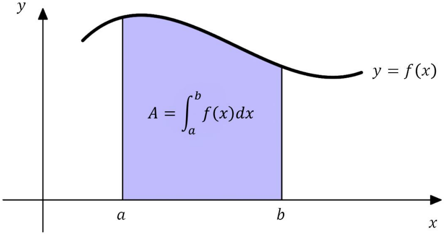
Example 2 - Solution

Substitute in the formula and evaluate the integral:

$$
\begin{aligned}
q(\tau) &= \int_0^{\tau} 40e^{-0.1t}\, \mathrm{d}t \\
&= \left[\frac{40}{-0.1}e^{-0.1t}\right]_0^{\tau} \\
&= \big[-400e^{-0.1t}\big]_0^{\tau} \\
&= \big(-400e^{-0.1\times\tau}\big) - \big(-400e^{-0.1\times 0}\big) \\
&= 400\big(1 - e^{-0.1\tau}\big)
\end{aligned}
$$

Areas above curves

Consider a curve $y = f(x)$ which is **above** the x-axis in the region $a < x < b$. Suppose A is the area bounded by the curve $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$:



The definite integral of this function over the range $a < x < b$ yields a positive value, which is exactly this area A.

Example 3

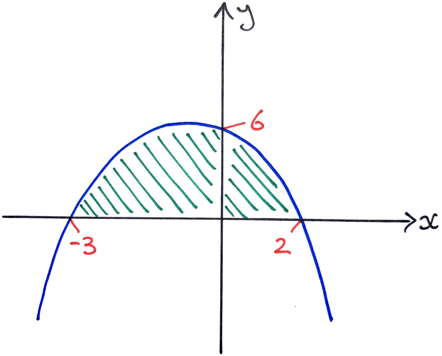
Find the area enclosed between the curve $y = -x^2 - x + 6$ and the x-axis.

No limits are given, so we shall have to determine appropriate limits by considering the shape of this curve.

Example 3 - Solution (I/II)

As this is a ∩-shaped parabola, if there are two real roots then they will define the limits of the region "contained" between the curve and the x-axis. Thus, we find them by solving for $y = 0$:

$$-x^2 - x + 6 = 0$$
$$\therefore \; x^2 + x - 6 = 0$$
$$\therefore \; (x+3)(x-2) = 0$$
$$\therefore \; x = -3 \quad \text{and} \quad x = 2$$



Or we could have used the quadratic formula to obtain these roots.

Example 3 - Solution (II/II)

Now conduct the definite integral between these limits:

$$\int_{-3}^{2} -x^2 - x + 6 \; \mathrm{d}x = \Big[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x \Big]_{-3}^{2}$$

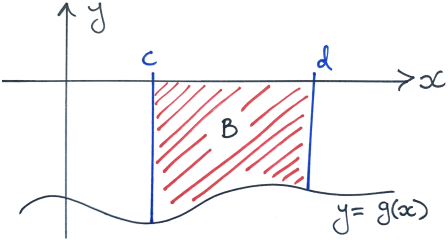
$$= \Big(-\frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 + 6(2) \Big) - \Big(-\frac{1}{3}(-3)^3 - \frac{1}{2}(-3)^2 + 6(-3) \Big)$$

$$= \Big(-\frac{8}{3} - \frac{4}{2} + 12 \Big) - \Big(\frac{27}{3} - \frac{9}{2} - 18 \Big)$$

$$= \frac{125}{6} \quad = \quad 20.8 \;\text{ square units (1 d.p.)}$$

Areas beneath curves

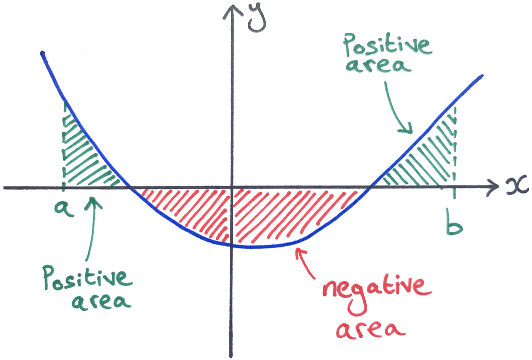
However, integrating the function of a curve over a range of x where it is *below* the x-axis gives a negative value, which is precisely $-1\times$ the area between the curve and the x-axis.



Thus: $$B = -\int_c^d g(x)\,\mathrm{d}x = \left|\int_c^d g(x)\,\mathrm{d}x\right|$$

Areas both above and beneath curves

So what if we wish to calculate the area enclosed by a curve that is both above and below the x-axis in different regions?



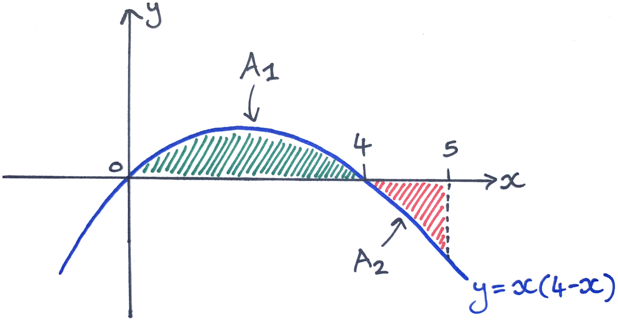
Areas both above and beneath curves

In this case, when asked to calculate the total *area* we must determine this positive value by separately calculating the integrals for regions where the curve is above and below the x-axis, and then summing their **magnitudes** (i.e. absolute values - always a positive value!).

Therefore, we must begin by solving an equation to find where the curve crosses the x-axis, as this may occur within the region that we are interested in. That is, where do the roots occur?

Example 4:

Find the area between the curve $y = 4x - x^2$ and the x-axis from $x = 0$ to $x = 5$.



Example 4 - Solution (I/II)

Either by using the quadratic formula, or factorising to $y = x(4 - x)$, we first deduce that the roots of this quadratic are at $x = 0$ and $x = 4$.

Since a root occurs in the range, the total area is split in two parts: above the x-axis in $0 < x < 4$, and below the x-axis in $4 < x < 5$. We must formulate these two integrals separately, then add their magnitudes:

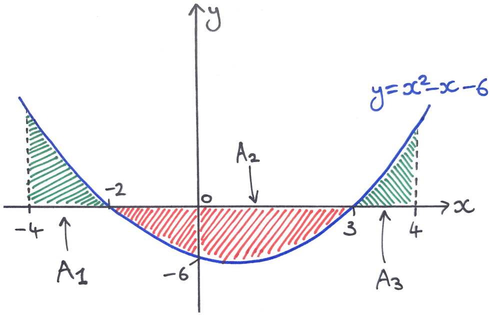
Example 4 - Solution (II/II)

$$
\begin{aligned}
A &= A_1 + A_2 \\
&= \left| \int_0^4 4x - x^2 \, \mathrm{d}x \right| + \left| \int_4^5 4x - x^2 \, \mathrm{d}x \right| \\
&= \left| \left[2x^2 - \frac{1}{3}x^3 \right]_0^4 \right| + \left| \left[2x^2 - \frac{1}{3}x^3 \right]_4^5 \right| \\
&= \left| \frac{32}{3} \right| + \left| \frac{-7}{3} \right| \\
&= \frac{32}{3} + \frac{7}{3} \quad = \quad 13 \;\text{ square units}
\end{aligned}
$$

Example 5:

Find the area between the curve $y = x^2 - x - 6$ and the x-axis between $x = -4$ and $x = 4$, and compare this with the integral:

$$\int_{-4}^{4} x^2 - x - 6 \, \mathrm{d}x$$



Example 5 - Solution (I/III)

Either by using the quadratic equation, or factorising to $y = (x - 3)(x + 2)$, deduce roots at $x = -2$ and $x = 3$.

This time two roots occur in the range, and so the graph shows three regions. The integrals over $-4 < x < -2$ and $3 < x < 4$ will give positive results, while the integral over $-2 < x < 3$ will be negative.

To find the total area we add the magnitudes of the three areas:

Example 5 - Solution (II/III)

$$
\begin{aligned}
A &= A_1 + A_2 + A_3 \\
&= \left| \int_{-4}^{-2} x^2 - x - 6 \,\mathrm{d}x \right| + \left| \int_{-2}^{3} x^2 - x - 6 \,\mathrm{d}x \right| + \left| \int_{3}^{4} x^2 - x - 6 \,\mathrm{d}x \right| \\
&= \left| \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x \right]_{-4}^{-2} \right| + \left| \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x \right]_{-2}^{3} \right| + \left| \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x \right]_{3}^{4} \right| \\
&= \left| \frac{38}{3} \right| + \left| \frac{-125}{6} \right| + \left| \frac{17}{6} \right| \\
&= \frac{38}{3} + \frac{125}{6} + \frac{17}{6} = \frac{109}{3} \text{ square units}
\end{aligned}
$$

Example 5 - Solution (III/III)

By comparison, the single integral (where the middle region contributes a negative area) gives a different result:

$$\int_{-4}^{4} x^2 - x - 6 \, \mathrm{d}x \quad = \quad \frac{38}{3} - \frac{125}{6} + \frac{17}{6} \quad = \quad -\frac{16}{3}$$

This is the *net* area under the curve. It is negative because the middle area A_2 beneath the curve is larger in size than the combined "positive" areas of A_1 and A_3 that lie above the curve.

Areas Between Curves

We can also define the areas enclosed *between two curves* by taking this concept further and integrating the *difference between the functions*. . .

Example 6

Find the area enclosed **between** the curves:

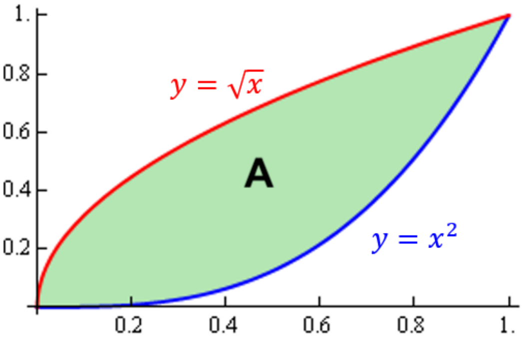
$$y = \sqrt{x}$$

and

$$y = x^2$$

Example 6 - Solution (I/III)

The trick here is to subtract the area under the lower curve from the area under the upper curve. The limits are found by solving the two equations simultaneously, to find their points of intersection.



Example 6 - Solution (II/III)

Finding the values of x where the curves intersect:

$$\sqrt{x} = x^2 \quad \text{square both sides} \ldots$$
$$x = x^4$$
$$x^4 - x = 0 \quad \text{factorising} \ldots$$
$$x(x^3 - 1) = 0$$
$$x = 0 \quad \text{or} \quad x^3 - 1 = 0$$

If $x^3 - 1 = 0$, then $x^3 = 1$ and so:

$$x = \sqrt[3]{1} = 1$$

So the limits are $x = 0$ and $x = 1$

Example 6 - Solution (III/III)

Evaluate the integral of the upper curve minus the lower curve with these limits:

$$\begin{aligned} A &= \int_0^1 \sqrt{x} - x^2 \,\mathrm{d}x = \int_0^1 x^{\frac{1}{2}} - x^2 \,\mathrm{d}x \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1 \\ &= \left(\frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{3}(1)^3\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}} - \frac{1}{3}(0)^3\right) \\ &= \left(\frac{2}{3}\cdot 1 - \frac{1}{3}\cdot 1\right) - (0-0) = \frac{1}{3} \end{aligned}$$