Polar form of Complex Numbers

Core topics in Mathematics

Lecture 20

Learning Outcomes

- Represent complex numbers in an Argand diagram.
- Express complex numbers in rectangular/Cartesian and polar form, and convert between these.

Note: Make sure your calculator is set to **radians**, not degrees!

Argand Diagrams

Complex numbers written in the form

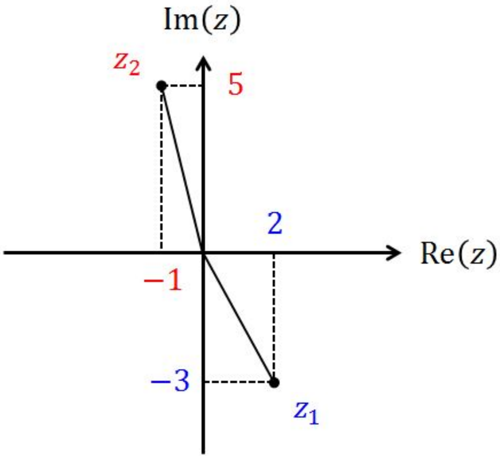
$$z = x + jy$$

are said to be in rectangular form (also called Cartesian form).

In this form we can represent a complex number graphically using the co-ordinate (x, y) in an Argand diagram, where the x-axis is the Real part and the y-axis represents the Imaginary part.

Argand Diagrams

Plotting $z_1 = 2 - j3$ and $z_2 = -1 + j5$ in an Argand diagram:



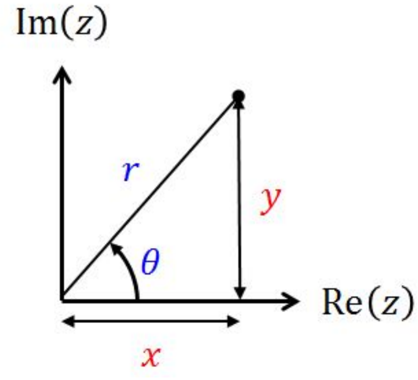
Modulus and Argument

The Argand diagram suggests an alternative way of representing complex numbers.

Instead of using the co-ordinates (x, y) to fix the position of the end of a line in the Argand diagram, we could define the line's position using the modulus r (length of the line) and the argument θ (angle relative to the positive real axis).

Polar Form

Given a complex number, $z = x + jy$, where both $x, y > 0$:



We can calculate the modulus using Pythagoras:

$$r = \sqrt{x^2 + y^2}$$

and the argument can be calculated using trigonometry:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Polar Form

So, if

$$z = x + jy$$

is a complex number written in Cartesian form, then

Polar form:

$$z = r\cos\theta + jr\sin\theta$$

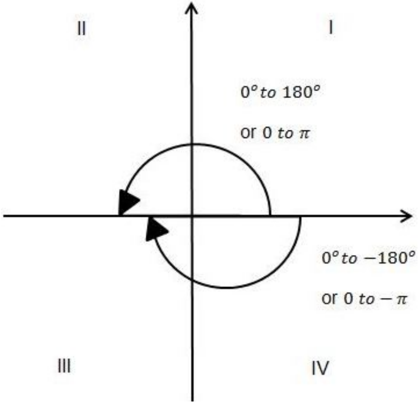
where r is the modulus and θ is the argument

is the same complex number, but written in polar form. The shorthand form for this is:

$$z = r\angle\theta$$

Measuring the Argument

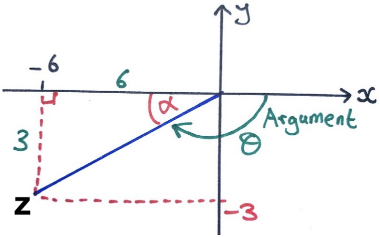
If z is not in the first quadrant, we need to do more to find the argument, as it is measured **anti-clockwise** from the **positive real axis**. By convention, it should be in the range $-\pi < \theta \leq +\pi$:



Measuring the Argument - an example:

Consider:

$$z = -6 - j3$$



This complex number is in the third quadrant.

Using right angle trig. we initially determine the angle α by:

$$\begin{aligned} \alpha &= \tan^{-1}\left(\frac{|y|}{|x|}\right) = \tan^{-1}\left(\frac{3}{6}\right) \\ &= 0.464 \end{aligned}$$

But this is not the argument, rather:

$$\theta = \pi - \alpha = 2.678$$

and finally as it is rotating the "wrong" way, the argument is:

$$Arg(z) = -\theta = -2.678$$

Measuring the Argument

An alternative approach is to *always* calculate θ according to:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

and then:

- If the complex number lies in quadrant II on the Argand diagram, then we add 180^o or π to the result.
- If the complex number lies in quadrant III on the Argand diagram, then subtract 180^o or π from the result.

You may use this rule if preferred. Applying to the previous example:

$$Arg(z) = \tan^{-1}\left(\frac{-3}{-6}\right) - \pi = -2.678$$

Examples 1

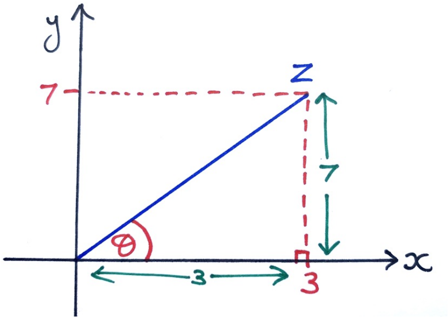
Express the following complex numbers in polar form:

1) $z = 3 + j7$

2) $z = -4 + j3$

Examples 1 - Solutions

1)



Modulus:

$$\begin{aligned} r &= \sqrt{3^2 + 7^2} \\ &= \sqrt{58} \\ &= 7.616 \end{aligned}$$

Argument:

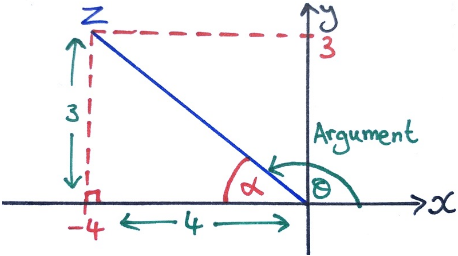
$$\theta = \tan^{-1}\left(\frac{7}{3}\right) = 1.166$$

Hence,

$$z = 7.616\angle 1.166$$

Examples 1 - Solutions

2)



Modulus:

$$\begin{aligned} r &= \sqrt{3^2 + 4^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Argument:

$$\alpha = \tan^{-1}\left(\frac{3}{4}\right) = 0.644$$

$$\theta = \pi - 0.644 = 2.498$$

Hence, $z = 5\angle 2.498$

Converting to Rectangular Form

Given a complex number written in polar form:

$$z = r\cos\theta + jr\sin\theta$$

We are easily able to convert to rectangular form (by using trigonometry) by using the formulae:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Note that in this case in **does not matter which quadrant** the complex number lies in.

Examples 2

Express the following complex numbers in rectangular form:

1) $z = 8\angle 2.1$

2) $z = 5.3\angle -3$

Examples 2 - Solutions

1)

$$x = 8\cos(2.1) = -4.039 \quad \text{and} \quad y = 8\sin(2.1) = 6.906$$

$$\therefore\ z = -4.039 + j\,6.906$$

2)

$$x = 5.3\cos(-3) = -5.247 \quad \text{and} \quad y = 5.3\sin(-3) = -0.748$$

$$\therefore\ z = -5.247 - j\,0.748$$

Polar Form Arithmetic

Polar form can be useful since multiplications and division in polar form are much easier; as shown by the formulae:

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2)$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle(\theta_1 - \theta_2)$$

where

$$z_1 = r_1 \angle \theta_1 \quad \text{and} \quad z_2 = r_2 \angle \theta_2$$

Examples 3

Given that $z_1 = 5.3\angle 2.1$ and $z_2 = 2.7\angle - 0.3$, determine:

1) $z_1 z_2$

2) $\frac{z_1}{z_2}$

Examples 3 - Solutions

1)

$$z_1 z_2 = (5.3 \times 2.7)\angle\big(2.1 + (-0.3)\big)$$
$$= 14.31\angle 1.8$$

2)

$$\frac{z_1}{z_2} = \left(\frac{5.3}{2.7}\right)\angle\big(2.1 - (-0.3)\big)$$
$$= 1.963\angle 2.4$$