

Further Matrices and their Applications

Core topics in Mathematics

Lecture 22

Learning Outcomes

- Perform additional matrix operations such as determining the inverse, transpose and determinant (if they exist).
- Solve systems of linear simultaneous equations using matrices.

Transpose of a Matrix

Taking the transpose of a matrix causes the 1st row to become the first column, the 2nd row to become the second column, etc.

For example, if

$$\underline{M} = \begin{pmatrix} 1 & -1 \\ 3 & 0 \\ 2 & -8 \end{pmatrix}$$

Then

$$\underline{M}^T = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & -8 \end{pmatrix}$$

Determinants

Square matrices (with dimensions $n \times n$) have a property called the **determinant**.

The determinant of matrix \underline{A} can be denoted by $\det(\underline{A})$ or $|\underline{A}|$.

For a 2×2 matrix \underline{A} , the determinant is very simple to calculate by multiplying the diagonal entries:

Determinant of a 2×2 matrix:

$$\det(\underline{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: determinant of a 2×2 matrix

Given the square matrix

$$\underline{B} = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$$

The determinant is given by:

$$\begin{aligned} \det(\underline{B}) &= 3 \times 2 - (-1) \times 4 \\ &= 6 + 4 \\ &= 10 \end{aligned}$$

Identity Matrices

Identity matrices are square matrices in which all elements are zero except for the elements on the leading diagonal; these are all 1, e.g.

$$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \underline{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Pre/post-multiplying by \underline{I} has no impact, i.e.

$$\underline{A} \underline{I} = \underline{A} \quad \text{and} \quad \underline{I} \underline{A} = \underline{A}$$

Inverse matrix

For a **square** matrix \underline{A} , there may exist an **inverse matrix** \underline{A}^{-1}

Inverse Matrix

$$\underline{A}\underline{A}^{-1} = \underline{I} \quad \text{and} \quad \underline{A}^{-1}\underline{A} = \underline{I}$$

So an inverse matrix is analogous to the reciprocal of a number - it's what you multiply by to get back to 1 (or the identity):

$$5 \times \frac{1}{5} = 1$$

$$\underline{A} \times \underline{A}^{-1} = \underline{I}$$

Calculating the inverse of a 2×2 matrix

For a general 2×2 square matrix $\underline{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

Inverse of a 2×2 matrix

$$\underline{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{or} \quad \underline{A}^{-1} = \frac{1}{|\underline{A}|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If the determinant of a square matrix is equal to **zero**, then that matrix has **no inverse!**

Example: Inverse of a 2×2 matrix

To find (if it exists) the inverse of 2×2 square matrix \underline{A} :

$$\underline{A} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

First obtain the determinant:

$$\det(\underline{A}) = (1)(2) - (-1)(0) = 2$$

Then as the determinant is non-zero, the inverse exists and is:

$$\underline{A}^{-1} = \frac{1}{\det(\underline{A})} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}$$

Exercise: Inverse of a 2×2 matrix

For the following square matrices, find the determinant and the inverse matrix if it exists:

$$\underline{B} = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

$$\underline{C} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Solution: Inverse of a 2×2 matrix

$$\underline{B} = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

$$\underline{B}^{-1} = \frac{1}{(1)(2) - (0)(-3)} \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3/2 & 1/2 \end{pmatrix}$$

$$\underline{C} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\det(\underline{C}) = (1)(-1) - (1)(-1) = 0$$

Hence \underline{C} has zero determinant \implies its inverse does not exist.

Motivation

Many engineering problems can be modelled as a system of simultaneous equations.

For example, let's say that there are two materials A and B, whose densities are unknown. You have two samples of different composites of these: one is 15% A and 85% B and has a density of 1kgm^{-3} , while the other is 40% A and 60% B but twice as dense. This could be written as:

$$0.15\underline{A} + 0.85\underline{B} = 1$$

$$0.4\underline{A} + 0.6\underline{B} = 2$$

We wish to determine the densities of the constituents A and B.

Introduction

This is an example of a pair of simultaneous linear equations.
Another example:

$$3x + 2y = 16$$

$$-x + 4y = 7$$

We will learn to solve them (i.e. find the unique values of x and y for which both equations are true) using a matrix method.

Method (1)

Given a pair of simultaneous equations, ensure they are in this form first:

$$ax + by = p$$

$$cx + dy = q$$

- ① Then write the pair of equations as a matrix equation:

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\underline{AX} = \underline{B}$$

Method (2)

① So the square matrix of coefficients is $\underline{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the vector \underline{X} contains x and y which we want to find.

② Calculate the inverse matrix \underline{A}^{-1}

③ Pre-multiply both sides by the inverse matrix to obtain \underline{X} :

$$\underline{A}\underline{X} = \underline{B} \implies \underline{A}^{-1}\underline{A}\underline{X} = \underline{A}^{-1}\underline{B} \implies \underline{X} = \underline{A}^{-1}\underline{B}$$

④ From the entries in vector \underline{X} , read off the values of x and y .

⑤ Substitute the values of x and y back into the original equations to verify solutions.

Example 1 (I/II)

Solve for x and y :

$$5x + 2y = 10$$

$$4x - 3y = 14$$

Re-writing this as a matrix equation,

$$\begin{pmatrix} 5 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

so we have $\underline{A}\underline{X} = \underline{B}$, where

$$\underline{A} = \begin{pmatrix} 5 & 2 \\ 4 & -3 \end{pmatrix}, \quad \underline{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

Example 1 (II/II)

Then,

$$\underline{A}^{-1} = \frac{1}{(5)(-3) - (2)(4)} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix} = \frac{-1}{23} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix}$$

and so

$$\underline{X} = \underline{A}^{-1}\underline{B} = \frac{-1}{23} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 10 \\ 14 \end{pmatrix} = \begin{pmatrix} 58/23 \\ -30/23 \end{pmatrix}$$

Thus we find $x = 58/23$ and $y = -30/23$.

Example 2

Solve for x and y :

$$3x = 7 + 5y$$

$$4y + 2x = 20$$

Example 2 - Solution (I/II)

First, re-write both of these in a consistent format:

$$3x - 5y = 7$$

$$2x + 4y = 20$$

Re-writing this as a matrix equation,

$$\begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 20 \end{pmatrix}$$

so we have $\underline{A}\underline{X} = \underline{B}$, where

$$\underline{A} = \begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix}, \quad \underline{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} 7 \\ 20 \end{pmatrix}$$

Example 2 - Solution (II/II)

Then,

$$\underline{A}^{-1} = \frac{1}{(3)(4) - (-5)(2)} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix}$$

and so

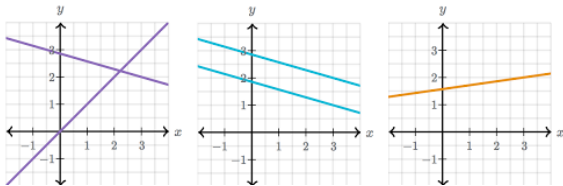
$$\underline{X} = \underline{A}^{-1}\underline{B} = \frac{1}{22} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ 20 \end{pmatrix} = \begin{pmatrix} 64/11 \\ 23/11 \end{pmatrix}$$

Thus we find $x = 64/11$ and $y = 23/11$.

Special cases

A linear equation $ax + by = d$ can be re-written in the form $y = mx + c$. In other words, we have been trying to find the co-ordinates of the point where two straight lines intersect.

What if the pair of lines are **parallel** or actually **the same**?



In these cases (**zero solutions** or **infinitely many solutions**), the matrix of coefficients will be **uninvertible** (its determinant = 0).

Special cases

If the matrix of coefficients has determinant $= 0$, examine the two equations and determine if they are the same equation (infinitely-many solutions), or if they are contradictory (zero solutions).

$$x - 3y = 10$$

$$2x - 6y = 20$$

$$-2x + y = 3$$

$$4x - 2y = 17$$

The first pair are the **same**, and the second pair are **contradictory**.

Exercises

Use the matrix method to solve the following systems of simultaneous equations:

$$\begin{aligned} \text{(a)} \quad & 7x + 2y = 4 \\ & 3x - 5y = 6 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & 5x = 10 + 2y \\ & 3x + 4y = 6 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad & 2 + \frac{8y}{x} = \frac{12}{x} \\ & -4y = x + 5 \end{aligned}$$