Transposition and Common Problems in Algebra

Core topics in Mathematics

Lecture 4

Learning Outcomes

- Addressing some of the most common difficulties that arise in algebraic manipulation.
- Practising transposition of more challenging equations.

Algebraic manipulation

There are some smaller aspects of algebraic manipulation that we have seen in these examples and which can be tricky. You will need to become comfortable with:

- Manipulating fractions and writing them in different ways.
- Factorisation.
- Simplifying subfractions (fractions within fractions).

We will learn more about these and then make use of them in some more challenging examples of transposition.

Fractions

We can write algebraic fractions in a variety of different ways (combining or separating their parts by multiplication), as long as all parts maintain their correct position on either the numerator or the denominator.

For example:

$$\frac{3y}{x}$$

can be written correctly as any of the following without changing the meaning:

$$3 \times \frac{1}{x} \times y \qquad \frac{3}{x}y \qquad 3\frac{y}{x} \qquad (3y) \div x$$

Factorisation

We already know how to expand brackets:

$$3x(x + y) = 3x^2 + 3xy$$

Factorisation is the reverse of this process. We look at two or more terms, and ask what "factors" are shared by all terms?

Factor: the whole numbers or symbols that a term can be perfectly divided by. For example, 3, 9, x, x^2 and any combinations such as $3x$, $9x$ or $3x^2$ are all factors of $9x^2$.

Factorisation - Example 1

Factorise as much as possible:

$$3x + 2xy$$

The simplest factors of the first term are 3 and x, and those of the second are 2, x and y.

As x is the only common (shared) factor, we can only remove it - leaving behind 3 and $2y$ respectively:

$$3x + 2xy = x(3 + 2y)$$

Factorisation - Example 2

Factorise as much as possible:

$$12x^2 - 8xy^2$$

The simplest factors of the first term are 2 and x, and those of the second are 2, x and y. However, to factorise fully we want to choose the largest shared factors.

All factors of the first term: 2, 3, 4, 6, 12 and x and x^2
All factors of the second term: 2, 4, 8 and x and y and y^2.

Therefore the largest common factor is $4x$:

$$12x^2 - 8xy^2 = 4x(3x - 2y^2)$$

Factorisation - Example 3

Factorise and simplify as much as possible:

$$\frac{28}{\pi x} + 16x$$

It is good practice when factorising to try to ensure that any fractions are also *outside* of the brackets, if this is not overly complicated to achieve.

In this case, in addition to the common factor of 4, we could factor out the πx on the denominator, which will require multiplying the second term by these in order to maintain balance.

$$\frac{28}{\pi x} + 16x = 4\left(\frac{7}{\pi x} + 4x\right) = \frac{4}{\pi x}(7 + 4\pi x^2)$$

Dealing with subfractions

When we have a fraction where either the numerator or the denominator (or both) themselves consist of a fraction, it is **always** possible to simplify them and express as a simple fraction.

This can be achieved with explicit fraction division.

For example:

$$\frac{\frac{15}{4}}{2} = \frac{15}{4} \div 2 = \frac{15}{4} \div \frac{2}{1} = \frac{15}{4} \times \frac{1}{2} = \frac{15}{8}$$

Dealing with subfractions - Example

Simplify:

$$\frac{\frac{15}{y^2}}{\frac{x}{y}}$$

Solution:

$$\frac{\frac{15}{y^2}}{\frac{x}{y}} = \frac{15}{y^2} \div \frac{x}{y} = \frac{15}{y^2} \times \frac{y}{x} = \frac{15y}{xy^2} = \frac{15}{xy}$$

Dealing with subfractions

As a shortcut, we may instead simply multiply the numerator and denominator of the main fraction by the denominator of the subfraction(s):

$$\frac{\frac{15}{y^2}}{\frac{x}{y}} = \frac{\frac{15}{y^2} \times y^2}{\frac{x}{y} \times y^2} = \frac{15}{\frac{xy^2}{y}} = \frac{15}{xy}$$

Transposing Equations

Recall the general principles of transposition:

- Get rid of fractions by multiplying.
- Get rid of brackets by expanding.
- Gather all terms with the unknown to one side by addition/subtraction.
- Remove everything else to the other side by addition/subtraction.
- Use division to leave the unknown by itself.

Let's consider some more challenging examples.

Example 1

Solve:

$$\frac{1}{x} - \frac{2y+1}{3} = 5y$$

for x

Example 1 - Solution (I/IV)

$$\frac{1}{x} - \frac{2y+1}{3} = 5y$$

What do we need to consider in this example?

- Remember that the minus sign applies to **all** of $(2y+1)/3$, not just the $2y$.
- Start by multiplying away all of the fractions.
- Only gather like terms after that.

Example 1 - Solution (II/IV)

$$\frac{1}{x} - \frac{2y+1}{3} = 5y$$

Multiply all terms by x to get rid of the first fraction:

$$x\left(\frac{1}{x}\right) - x\left(\frac{2y+1}{3}\right) = x(5y)$$

$$\therefore \frac{1}{1} - \frac{x(2y+1)}{3} = 5xy$$

$$\therefore 1 - \frac{x(2y+1)}{3} = 5xy$$

Example 1 - Solution (III/IV)

$$1 - \frac{x(2y+1)}{3} = 5xy$$

Now multiply all terms by 3 to get rid of the remaining fraction:

$$3(1) - 3\left(\frac{x(2y+1)}{3}\right) = 3(5xy)$$

$$\therefore 3 - x(2y+1) = 15xy$$

$$\therefore 3 - 2xy - x = 15xy$$

Example 1 - Solution (IV/IV)

$$3 - 2xy - x = 15xy$$

Finally, gather all terms containing x together and simplify:

$$15xy + 2xy + x = 3$$

$$\therefore 17xy + x = 3$$

$$\therefore x(17y + 1) = 3$$

$$\therefore x = \frac{3}{17y + 1}$$

Re-writing $17xy + x$ as $x(17y + 1)$ is an example of **factorisation**.

Example 2

The following formula arises in the study of relativistic motion.

$$T = \frac{T_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

In this case, c denotes the speed of light (3×10^8 m/s). How is it related to the other variables?

Example 2 - Solution (I/IV)

$$T = \frac{T_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

Begin by multiplying both sides by the denominator of the fraction:

$$T\left(1 - \frac{v^2}{c^2}\right)^{1/2} = T_0$$

Undo the power of 1/2 by squaring both sides of the equation:

$$\left(T\left(1 - \frac{v^2}{c^2}\right)^{1/2}\right)^2 = \left(T_0\right)^2$$

$$\therefore\ T^2\left(1 - \frac{v^2}{c^2}\right) = T_0^2$$

Example 2 - Solution (II/IV)

Divide both sides by T^2:

$$1 - \frac{v^2}{c^2} = \frac{T_0^2}{T^2}$$

Isolate the term containing c:

$$-\frac{v^2}{c^2} = \frac{T_0^2}{T^2} - 1$$

Now multiply both sides by c^2 to extract it from the denominator:

$$-v^2 = c^2\left(\frac{T_0^2}{T^2} - 1\right)$$

Example 2 - Solution (III/IV)

To get c^2 alone, divide both sides by the contents of the brackets:

$$c^2 = \frac{-v^2}{\frac{T_0^2}{T^2} - 1}$$

This can be simplified slightly by changing the sign of all terms within the fraction:

$$c^2 = \frac{v^2}{1 - \frac{T_0^2}{T^2}}$$

Example 2 - Solution (IV/IV)

To simplify further, address the subfraction T_0^2/T^2 by multiplying the numerator and denominator of the main fraction by T^2:

$$c^2 = \frac{v^2 T^2}{T^2 - T_0^2}$$

Finally, take the square root of both sides to obtain an expression for c:

$$c = \sqrt{\frac{v^2 T^2}{T^2 - T_0^2}}$$

Example 3

The following formula describes the relativistic Doppler shift concerning the changes in frequency of light due to relative longitudinal motion of a source and observer:

$$\nu' = \nu \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}$$

Obtain a formula for β.

Example 3 - Solution(I/III)

Divide both sides by ν:

$$\frac{\nu'}{\nu} = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}$$

Now, if we square both sides we can eliminate both square roots:

$$\begin{aligned} \left(\frac{\nu'}{\nu}\right)^2 &= \left(\frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\right)^2 \\ &= \frac{(\sqrt{1-\beta})^2}{(\sqrt{1+\beta})^2} \quad \text{(Rules of indices!)} \\ &= \frac{1-\beta}{1+\beta} \end{aligned}$$

Example 3 - Solution(II/III)

Now multiply both sides by denominator $1 + \beta$ to simplify the fraction:

$$\left(\frac{\nu'}{\nu}\right)^2 (1 + \beta) = 1 - \beta$$

Expand the brackets and gather like terms (with β):

$$\left(\frac{\nu'}{\nu}\right)^2 + \left(\frac{\nu'}{\nu}\right)^2 \beta = 1 - \beta$$

$$\therefore \left(\frac{\nu'}{\nu}\right)^2 \beta + \beta = 1 - \left(\frac{\nu'}{\nu}\right)^2$$

Example 3 - Solution(III/III)

Factorise β from the LHS:

$$\beta\left(\left(\frac{\nu'}{\nu}\right)^2 + 1\right) = 1 - \left(\frac{\nu'}{\nu}\right)^2$$

Divide both sides by the contents of the brackets to isolate β and finally simplify the subfractions:

$$\beta = \frac{1 - \left(\frac{\nu'}{\nu}\right)^2}{1 + \left(\frac{\nu'}{\nu}\right)^2} = \frac{1 - \frac{\nu'^2}{\nu^2}}{1 + \frac{\nu'^2}{\nu^2}} = \frac{\nu^2 - \nu'^2}{\nu^2 + \nu'^2}$$