Quadratic functions

Core topics in Mathematics

Lecture 6

Learning Outcomes

- Recognise quadratic functions.
- Recognise typical shapes of quadratic graphs.
- Solve quadratic equations.

Introduction

We previously introduced polynomial functions. Apart from constant functions, the simplest was a linear equation with highest power x^1 (describing a straight line). The next simplest are second-order polynomials:

Quadratic equation

$$y = ax^2 + bx + c,$$

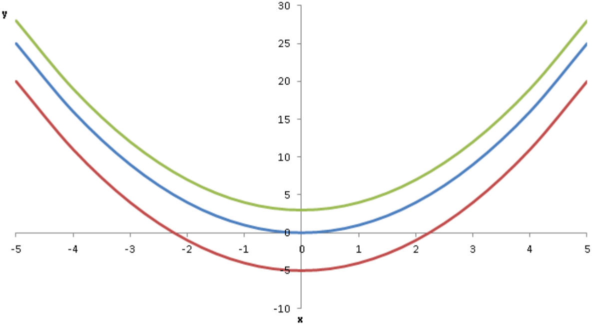
where a, b and c are constants and $a \neq 0$

This represents a curve with a single turning point, called a parabola. All quadratics can be represented in this general form.

There are two types, depending on the value of a.

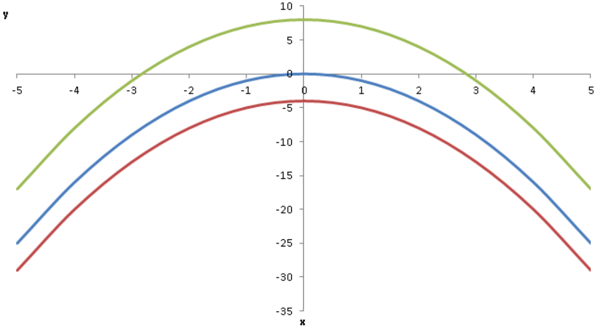
Introduction

When $a > 0$ the curve is $\cup$-shaped



Introduction

When $a < 0$ the curve is $\cap$-shaped



Introduction

When trying to visualise a quadratic function, consider:

- What is the orientation?
  - $a > 0$: upturned.
  - $a < 0$: downturned.
- Is it broad or narrow compared to $y = x^2$?
  - $|a| > 1$: narrower.
  - $|a| < 1$: broader.
- Where is the turning point?
  - Positive c will push it up.
  - Negative c will push it down.
  - Positive or negative b will push it down if $a > 0$ (up if $a < 0$).
  - $b = 0$: on the y-axis.
  - $b > 0$: left of the y-axis if $a > 0$ (right if $a < 0$).
  - $b < 0$: right of the y-axis if $a > 0$ (left if $a < 0$).

Introduction

The constant c is the y-intercept, as in the linear case (if $x = 0$ then the equation becomes $y = a \times 0^2 + b \times 0 + c = c$).

The curve can cross the x-axis (at $y = 0$) twice, once (just touching it) or never. If it does cross the x-axis, we can calculate the values of x where this occurs by solving:

$$ax^2 + bx + c = 0$$

The **solutions** of $0 = ax^2 + bx + c$ are the same as the x-intercepts of $y = ax^2 + bx + c$ and are also known as the **roots** of $ax^2 + bx + c$.

Factorisation

There are two ways to solve a quadratic equation. Sometimes we can "factorise", which is the reverse of expanding brackets. If $a = 1$, then we seek two numbers that multiply to c and add to b.

Example:

$$x^2 + 4x + 3 = 0 \qquad \text{What pair multiplies to 3 and adds to 4?}$$
$$x^2 + 3x + 1x + 3 = 0 \qquad \text{3 and 1 of course!}$$
$$(x + 3)(x + 1) = 0$$

This means that either $x + 3 = 0$, so $x = -3$, or that $x + 1 = 0$ so $x = -1$. This approach isn't always possible, so the most reliable method to use (which *always* works!) is. . .

The Quadratic Formula

The quadratic formula:

If $ax^2 + bx + c = 0$ and $a \neq 0$ then:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

The **discriminant** is the "bit under the square root." It indicates how many roots exist and of what type:

- $b^2 - 4ac > 0$ indicates two real and distinct roots (x_1 and x_2)
- $b^2 - 4ac = 0$ indicates real and repeated roots ($x_1 = x_2$)
- $b^2 - 4ac < 0$ indicates complex roots ($x = \alpha + j\beta$)

Example

Let's use this method to solve the same equation as before:

$$x^2 + 4x + 3 = 0$$

In this case, the constants are $a = 1$, $b = 4$ and $c = 3$, so we substitute these into the formula:

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 3}}{2 \times 1} = \frac{-4 \pm \sqrt{16 - 12}}{2} \\ &= \frac{-4 \pm \sqrt{4}}{2} = \frac{-4 \pm 2}{2} = -2 \pm 1 \end{aligned}$$

Thus, the solutions are $x = -3$ and $x = -1$, as we had found by factorisation.

Roots and Discriminants

Furthermore, if there are two distinct, real roots x_1 and x_2 to the quadratic $y = ax^2 + bx + c$, then it is possible to re-write the quadratic in the form:

$$y = a(x - x_1)(x - x_2)$$

However, if there are real and repeated roots, $x_1 = x_2$, then it is possible to re-write the quadratic in the form:

$$y = a(x - x_1)^2$$

Exercises

Determine the roots of the following quadratics:

1) $y = 3x^2 + 13x - 10$

2) $y = x^2 - 14x + 49$

3) $y = x^2 + 6x + 34$

Factorise the following quadratics:

4) $y = x^2 + 7x + 12$

5) $y = x^2 - 10x + 25$

Exercises - Solutions: $y = 3x^2 + 13x - 10$

1) Here, the coefficients are $a = 3$, $b = 13$ and $c = -10$.
Using the formula:

$$
\begin{aligned}
x &= \frac{-13 \pm \sqrt{13^2 - 4 \times 3 \times -10}}{2 \times 3} \\
&= \frac{-13 \pm \sqrt{289}}{6} \quad \text{Positive discriminant.} \\
&= \frac{-13 \pm 17}{6} \\
&= \frac{4}{6} \quad \text{or} \quad -\frac{30}{6} \\
&= \frac{2}{3} \quad \text{or} \quad -5 \quad \text{So we have two distinct roots.}
\end{aligned}
$$

Exercises - Solutions: $y = x^2 - 14x + 49$

2) Here, the coefficients are $a = 1$, $b = -14$ and $c = 49$.
Using the formula:

$$
\begin{aligned}
x &= \frac{-(-14) \pm \sqrt{(-14)^2 - 4 \times 1 \times 49}}{2 \times 1} \\
&= \frac{14 \pm \sqrt{0}}{2} \quad \text{Discriminant is zero.} \\
&= \frac{14 \pm 0}{2} \\
&= \frac{14}{2} \\
&= 7 \quad \text{This time we have one repeated solution.}
\end{aligned}
$$

Exercises - Solutions: $y = x^2 + 6x + 34$

3) Here, the coefficients are $a = 1$, $b = 6$ and $c = 34$.
Using the formula:

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 34}}{2 \times 1} \\ &= \frac{-6 \pm \sqrt{36 - 136}}{2} \\ &= \frac{-6 \pm \sqrt{-100}}{2} \quad \text{Discriminant is negative.} \end{aligned}$$

There are no real solutions - we can't proceed any further. This corresponds to a parabola that sits above the x-axis and never touches it.

Exercises - Solutions: $y = x^2 + 7x + 12$

4) Solving by the quadratic formula:

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4 \times 1 \times 12}}{2 \times 1} \\ &= \frac{-7 \pm \sqrt{1}}{2} \\ &= \frac{-7 \pm 1}{2} \\ &= -4 \quad \text{or} \quad -3 \end{aligned}$$

Thus we can factorise as:

$$\begin{aligned} y &= 1(x - (-3))(x - (-4)) \\ &= (x+3)(x+4) \end{aligned}$$

Exercises - Solutions: $y = x^2 - 10x + 25$

5) Solving by the quadratic formula:

$$
\begin{aligned}
x &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4 \times 1 \times 25}}{2 \times 1} \\
&= \frac{10 \pm \sqrt{0}}{2} \\
&= \frac{10 \pm 0}{2} \\
&= 5 \quad \text{(repeated)}
\end{aligned}
$$

Thus we can write the factorised quadratic function as:

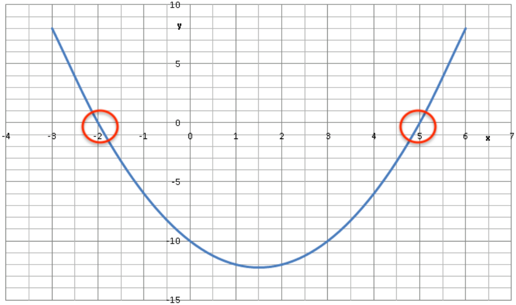
$$y = 1(x-5)^2 = (x-5)^2$$

Observe that -5 and -5 multiply to 25 and add to -10 as required.

Determining the Quadratic Equation from a Graph

There are two ways to do this:

Method 1



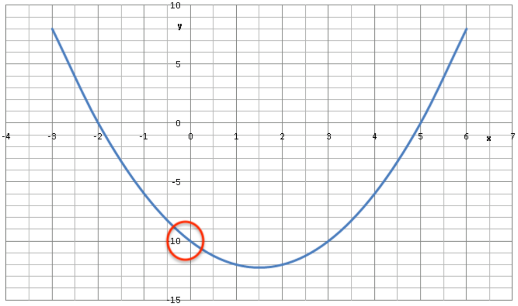
We can see the roots are $x_1 = -2$ and $x_2 = 5$, which means in factorised form the quadratic equation is:

$$\begin{aligned} y &= a(x+2)(x-5) \\ &= a(x^2 - 3x - 10) \end{aligned}$$

We would need to substitute in another point (e.g. the y-intercept) to confirm that the value of a is 1 in this case.

Determining the Quadratic Equation from a Graph

Method 2



We can see that the y-intercept is -10.

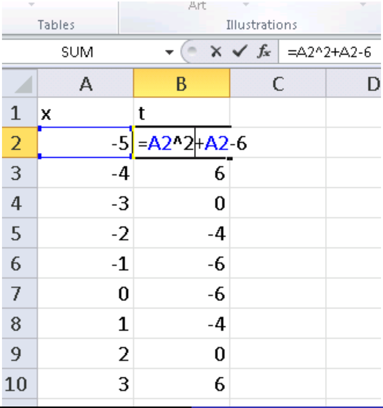
$$\therefore y = ax^2 + bx - 10$$

Now, if we choose two coordinates, e.g. $(-1, -6)$ and $(6, 8)$, we can solve $y = ax^2 + bx - 10$ simultaneously. We will learn more on this later.

Using Excel to Plot Polynomials

To plot higher order functions we make use of the $\wedge$ symbol, which means to the power of.

Example: To plot $y = x^2 + x - 6$:



	A	B	C	D
1	x	t		
2	-5	=A2^2+A2-6		
3	-4	6		
4	-3	0		
5	-2	-4		
6	-1	-6		
7	0	-6		
8	1	-4		
9	2	0		
10	3	6		