Exponentials and Logarithms

Core topics in Mathematics

Lecture 7

Learning Outcomes

- Recognise logarithmic and exponential functions.
- Sketch logarithmic and exponential functions.
- Apply the laws of logarithms.
- Solve exponential equations.

Introduction - Exponential Functions

General exponential functions (with base b) are of the form:

$$y = Ab^{kx},$$

where A, b, k are constants:

- A is a coefficient and is the value of y when $x = 0$. This is because if $x = 0$, $y = Ab^{k \times 0} = Ab^0 = A \times 1 = A$.
- b is the base.
- k determines how fast the function grows (growth rate).

Certain values for the base b are more common than others, esp.

$$y = 10^x \quad \text{and} \quad y = e^x = \exp(x)$$

THE Exponential Function

One particular base is very important: $e = 2.718281828\ldots$, called Euler's number.

The general form of this type of equation is:

Exponential function:

$$y = Ae^{Bx} + C$$

where A, B and C are constants.

Note, when $x = 0$, $y = A + C$ (this is the y-intercept).

Exercises

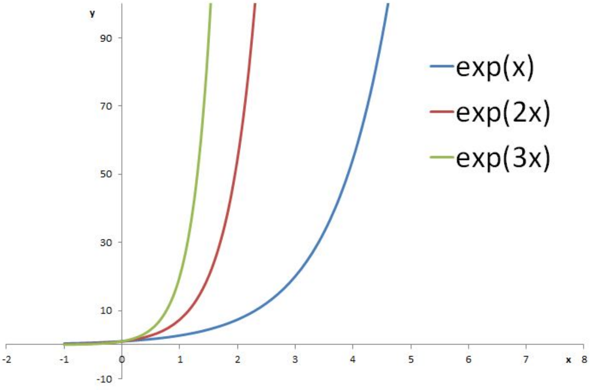
Use your calculator to determine the following:

1) e^4

2) $4e^{7.2}$

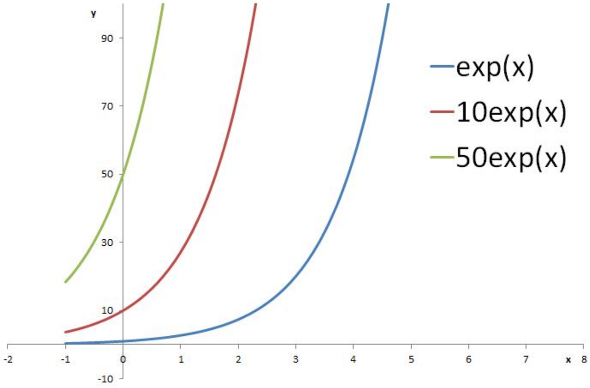
3) $2.9e^{29.7} + 2.3$

Visualising exponential functions



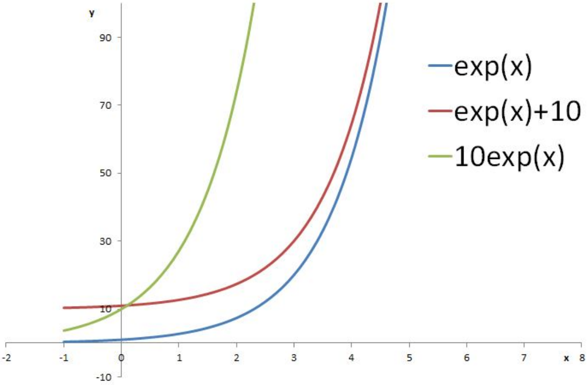
Changing the *magnitude* of B affects the gradient.

Visualising exponential functions



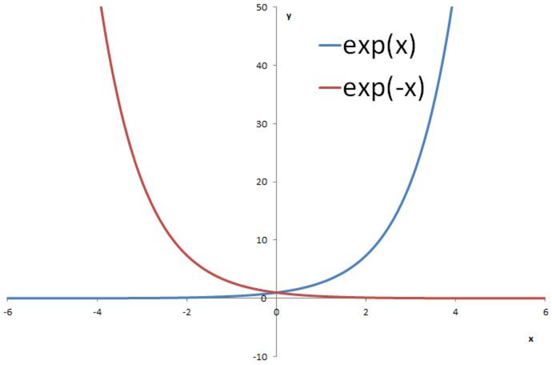
Changing A affects the y-intercept.

Visualising exponential functions



Changing both A and C affect the y-intercept.

Visualising exponential functions



Changing the *sign* of B reflects the e^x curve in the y-axis: positive B gives exponential growth, negative B gives decay.

Applications of the Exponential Function

The exponential function is used frequently across engineering.

It is used in growth and decay models such as:

- Tension in belts: $T_1 = T_0 e^{\mu\theta}$
- Newton's law of cooling: $\theta = \theta_0 e^{-kt}$
- Atmospheric pressure at altitude h: $p = p_0 e^{-h/c}$
- Discharge of a capacitor: $q = Qe^{-t/CR}$

Introduction - The Logarithm Function

The logarithm function is written as follows:

Logarithm function

$$y = \log_a(x),$$

where a and x are positive and $a \neq 1$ is constant.

This can be interpreted as:

"y is the power to which one must raise a (the base), to get x (the argument)."

That is:

$$a^y = x$$

Introduction

Example:

What power of 2 is exactly equal to 8?

Answer: $2 \times 2 \times 2 = 2^3 = 8$.

So, we need exactly 3 "2's" to get 8. This means that the logarithm of 8, to base 2, is 3. We write this as:

$$\log_2(8) = 3$$

Examples

Calculate x:

- $3^x = 81 \qquad \therefore \quad \log_3(81) = 4$
- $6^x = 1 \qquad \therefore \quad \log_6(1) = 0$
- $2^x = 0.125 \qquad \therefore \quad \log_2(0.125) = -3$

So one application of logs is to solve equations where the desired variable is in the index.

Logarithms

The most commonly used bases are 10 and e.

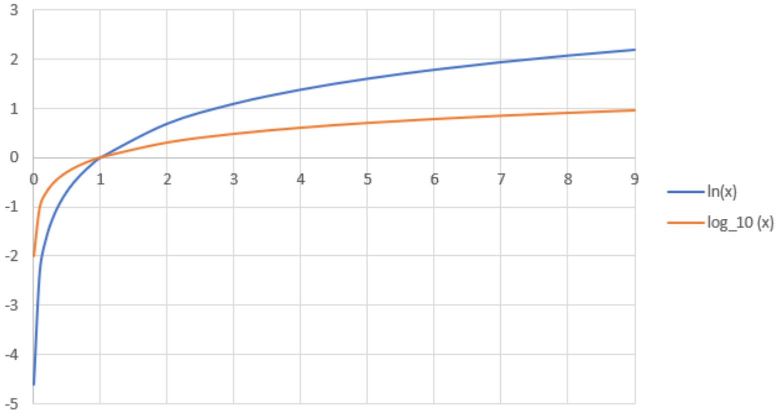
The log button on your calculator is $\log_{10}$. This is the common logarithm.

The ln button on your calculator is $\log_e$. This is the natural logarithm and is usually written ln, so:

$$\log_e 6 = \ln 6$$

Engineers mainly deal with the natural logarithm.

Visualising Logarithms



Remember, the input to a log function must be positive.

Visualising Logarithms

More generally natural logarithm functions are in the form:

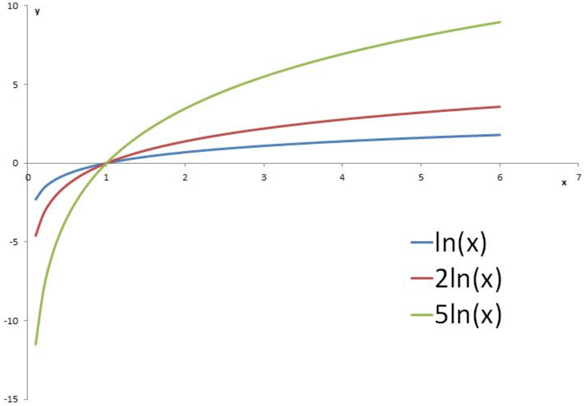
General natural log functions:

$$y = A\ln(x) + B$$

where A and B are constants.

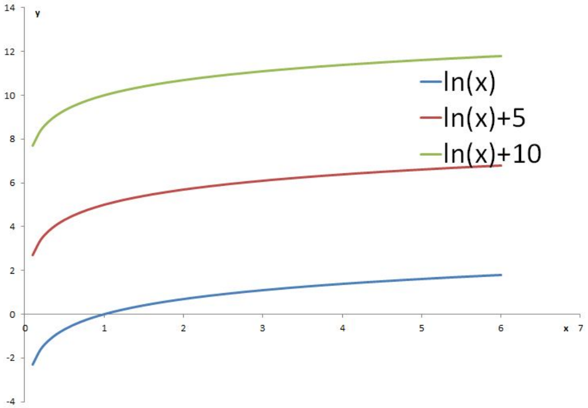
Let's look at what A and B influence. . .

Visualising Logarithms



Changing A stretches the curve vertically.

Visualising Logarithms



Changing B shifts the curve vertically.

Laws of Logarithms

The following are useful for manipulating equations (they are true for *any* base, as long as all the logs share the same base).

Laws of logarithms:

$$\log\left(A^n\right) = n\log\left(A\right)$$

$$\log\left(AB\right) = \log\left(A\right) + \log\left(B\right)$$

$$\log\left(\frac{A}{B}\right) = \log\left(A\right) - \log\left(B\right)$$

$$\log\left(1\right) = 0$$

Examples

Write each of the following as a single log:

- $\log_{10} 6 + \log_{10} 3 = \log_{10}\ (6 \times 3) = \log_{10} 18$
- $\ln 6 - \ln 3 = \ln\ \left(\frac{6}{3}\right) = \ln 2$
- $2\log x = \log\left(x^2\right)$

Logarithms and Exponentials

The logarithm and exponential functions are the inverses of each other, i.e. one undoes the impact of the other:

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln(x)} = x$$

Example:

$$\ln(e^7) = 7 \quad \text{and} \quad e^{\ln(15)} = 15$$

Example 1

Solve the equation $25e^x = 521$ for x:

$$\frac{25e^x}{25} = \frac{521}{25} \quad \text{Isolate } e^x$$

$$e^x = \frac{251}{25}$$

$$\ln(e^x) = \ln\left(\frac{521}{25}\right) \quad \text{Use ln to undo the exponential}$$

$$x = 3.04 \quad \text{to 2 d.p.}$$

Example 2 - Part I/II

Solve the equation $4e^{-3x} + 5 = 12$ for x:

$$4e^{-3x} + 5 - 5 = 12 - 5$$

$$4e^{-3x} = 7$$

$$\frac{4e^{-3x}}{4} = \frac{7}{4}$$

$$e^{-3x} = \frac{7}{4}$$

Example 2 - Part II/II

$$e^{-3x} = \frac{7}{4}$$

$$\ln(e^{-3x}) = \ln\left(\frac{7}{4}\right)$$

$$-3x = \ln\left(\frac{7}{4}\right) \quad \text{now divide by } -3$$

$$x = -0.187 \quad \text{to 3 d.p.}$$

Example 3

A capacitor of capacitance C is allowed to discharge through a resistor of resistance R such that the voltage across the terminals of the capacitor ν at time t after the discharge started is given by:

$$\nu = \nu_0 e^{-\frac{1}{RC}t},$$

where ν_0 is the voltage across the terminals of the capacitor at the start of the discharge.

If $C = 500$ nF, $R = 200$ kΩ and $\nu_0 = 12$ V, determine the time it takes for ν to drop to 6 V.

Example 3 - Solution

Sub. in the values:

$$6 = 12e^{-\frac{t}{200\times 10^3\times 500\times 10^{-9}}}$$

Simplifying:

$$\frac{6}{12} = e^{\frac{-t}{10^{-1}}} \quad \Longrightarrow \quad \frac{1}{2} = e^{-10t}$$

Using ln to invert the exponential:

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{-10t}\right) \quad \Longrightarrow \quad t = -\frac{1}{10}\ln\left(\frac{1}{2}\right) = 0.069\ldots$$