Modelling with Functions

Core topics in Mathematics

Lecture 8

Contents

In this session, we will not introduce new ideas but will practice synthesising what we have learned about functions in the context of mathematical modelling of real-world behaviour and systems.

We will also practice some problems that may require us to combine different theories that we have learned separately.

Example 1

A consultant charges an upfront cost of £60, and an additional hourly rate of £15/hour *pro rata* for services.

If your business has a £250 consulting budget, how long can you hire them for?

Example 1 - Solution (I/III)

To formulate this problem mathematically, assign variables t and C to the quantities of interest - the amount of time that the consultant works for, and how much this will cost. Formally, let t be the number of hours that the consultant works for, and $C(t)$ the total cost in pounds of engaging them for time t.

Because the cost C increases at a fixed rate with the amount of time t worked, this is fundamentally a **linear** relationship:

$$C = at + b$$

And we need to determine the values of the parameters (constants) a and b.

Example 1 - Solution (II/III)

Because of the initial charge of £60, the initial value of $C(t = 0) = 60$, hence:

$$60 = a(0) + b \quad \Longrightarrow \quad b = 60$$

And because the value of C increases by 15 with every increase in t by 1 hour, this means that the gradient $a = 15$ because this represents the rate of change. Hence:

$$C(t) = 15t + 60$$

Example 1 - Solution (III/III)

$$C(t) = 15t + 60$$

Finally, we want to solve for t such that $C(t) = 250$:

$$250 = 15t + 60$$
$$\therefore 15t = 190$$
$$\therefore t = 190/15 = 12.666\ldots$$

So we can hire the consultant to work for 12 hours and 40 minutes.

Example 2

In population dynamics, the logistic function is often used in order to predict the growth of a population P at time t:

$$P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right) \mathrm{e}^{-rt}}$$

where K, P_0 and r are positive constants.

i) How does the population size behave as time increases forever?
ii) A lab technician sets up an experiment with 100 bacteria, and sufficient nutrients to sustain a maximum population of 50,000. One week later they record a population of 32,000. If this function is a reasonable model of growth, how much longer will it be for the population to exceed 40,000? Confirm the solution using EXCEL.

Example 2 - Solution (I/VIII)

i) In terms of the variables, this means $t \to \infty$. Since we know $r > 0$, then e^{-rt} exhibits exponential decay with t, and thus:

$$e^{-rt} \to 0 \quad \text{as} \quad t \to \infty$$

Hence:

$$\begin{aligned}
\lim_{t\to\infty} P &= \lim_{t\to\infty} \frac{K}{1 + \left(\frac{K-P_0}{P_0}\right) e^{-rt}} \\
&= \frac{K}{1 + \left(\frac{K-P_0}{P_0}\right) \cdot 0} \\
&= \frac{K}{1} \\
&= K, \ \text{which is the maximum sustainable population.}
\end{aligned}$$

Example 2 - Solution (II/VIII)

ii) Before the model can be applied, we need to fit the three parameters. From part (i), we have $K = 50,000$. The initial population is given by P when $t = 0$:

$$\begin{aligned} P(0) &= \frac{K}{1 + \left(\frac{K-P_0}{P_0}\right) e^{-r\times 0}} = \frac{K}{1 + \left(\frac{K-P_0}{P_0}\right) e^{0}} \\ &= \frac{K}{1 + \left(\frac{K-P_0}{P_0}\right) \cdot 1} = \frac{K}{1 + \frac{K-P_0}{P_0}} \\ &= \frac{KP_0}{P_0 + K - P_0} = \frac{KP_0}{K} \\ &= P_0, \quad \text{and so we know that } P_0 = 100. \end{aligned}$$

Example 2 - Solution (III/VIII)

Use the final information provided to determine parameter r. Let t be the time in days from the start, then we have $P(7) = 32000$. Before substituting this in, transpose to obtain a general formula for r:

$$P = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right) e^{-rt}}$$

Removing the fraction:

$$P\left(1 + \left(\frac{K - P_0}{P_0}\right) e^{-rt}\right) = K$$

$$\therefore \left(\frac{K - P_0}{P_0}\right) e^{-rt} = \frac{K}{P} - 1$$

Example 2 - Solution (IV/VIII)

Multiply both sides by P_0 and divide by $K - P_0$ to isolate the exponential:

$$e^{-rt} = \left(\frac{P_0}{K - P_0}\right)\left(\frac{K}{P} - 1\right)$$

Taking logs:

$$-rt = \ln\left(\frac{P_0}{K - P_0}\left(\frac{K}{P} - 1\right)\right) \tag{1}$$

And so:

$$r = -\frac{1}{t}\ln\left(\frac{P_0}{K - P_0}\left(\frac{K}{P} - 1\right)\right)$$

Example 2 - Solution (V/VIII)

Then substituting in $K = 50000$, $P_0 = 100$, $P = 32000$ and $t = 7$:

$$\begin{aligned} r &= -\frac{1}{7}\ln\left(\frac{100}{50000-100}\left(\frac{50000}{32000}-1\right)\right) \\ &= -\frac{1}{7}\ln\left(\frac{1}{499}\left(\frac{25}{16}-1\right)\right) \\ &= -\frac{1}{7}\ln\left(\frac{9}{16\times 499}\right) = -\frac{1}{7}\ln\left(\frac{9}{7984}\right) \\ &= 0.9697100\ldots \end{aligned}$$

As expected, this is a positive value. We will need to keep r to a high precision.

Example 2 - Solution (VI/VIII)

From equation (1) above, we can then easily obtain a formula for t:

$$t = -\frac{1}{r} \ln \left(\frac{P_0}{K - P_0} \left(\frac{K}{P} - 1 \right) \right)$$

To find when the population exceeds 40,000, evaluate this formula with $P = 40000$, $K = 50000$, $P_0 = 100$, and $r = -0.9697100$:

$$\begin{aligned} t &= -\frac{1}{0.96971} \ln \left(\frac{100}{49900} \left(\frac{50000}{40000} - 1 \right) \right) \\ &= -\frac{1}{0.96971} \ln \left(\frac{1}{499} \left(\frac{5}{4} - 1 \right) \right) \\ &= -\frac{1}{0.96971} \ln \left(\frac{1}{4 \times 499} \right) \end{aligned}$$

Example 2 - Solution (VII/VIII)

Hence:

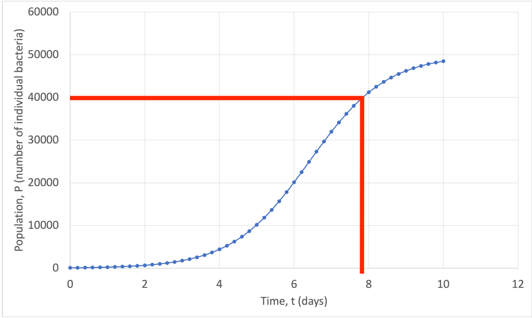
$$\begin{aligned} t &= -\frac{1}{0.96971} \ln\left(\frac{1}{1996}\right) \\ &= 7.83626\ldots \end{aligned}$$

Thus 7.84 days from the starting point. So in fact, due to the behaviour of exponential growth, the bacterial population will exceed 40,000 before the end of the next (eighth) day.

Example 2 - Solution (VIII/VIII)

Plotting the logistic function in EXCEL, we can confirm our result directly from the graph or from the calculated values:

7.6		38041.08654
7.8	t	39715.73095
8		41210.05557
8.2	P0	42528.10119



Notice the exponential-like initial growth, before the population saturates at the carrying capacity K.

Example 3

Indicial equations such as the following are used in the solutions to certain kinds of differential equation problems:

$$2^{x+1} + 2^{3-x} = 17$$

By making a substitution $y = 2^x$, determine all the values of x that satisfy this equation.

Example 3 - Solution (I/III)

Before implementing the substitution, we need to see how 2^x appears in the equation using rules of indices:

$$2^{x+1} + 2^{3-x} = 2^x \cdot 2^1 + 2^3 \cdot 2^{-x}$$

Hence:

$$2\left(2^x\right) + \frac{8}{2^x} = 17$$

Substituting in $y = 2^x$:

$$2y + \frac{8}{y} = 17$$

Example 3 - Solution (II/III)

This gives a simpler equation which we solve for y. Multiply all terms by y to remove the fraction:

$$2y^2 + 8 = 17y$$

$$\therefore 2y^2 - 17y + 8 = 0$$

This is then a quadratic equation in y, which can be solved by the quadratic formula or factorised to:

$$(2y - 1)(y - 8) = 0$$

Hence the two solutions for y are $y = \frac{1}{2}$ and $y = 8$.

Example 3 - Solution (III/III)

In terms of the original variable x, this means we have $2^x = 8$ and $2^x = \frac{1}{2}$. Hence, for one solution:

$$\ln\left(2^x\right) = \ln\left(8\right)$$

Which using the rules of logarithms, indicates:

$$x\ln\left(2\right) = \ln\left(8\right)$$

and so

$$x = \frac{\ln\left(8\right)}{\ln\left(2\right)} = \frac{\ln\left(2^3\right)}{\ln\left(2\right)} = \frac{3\ln\left(2\right)}{\ln\left(2\right)} = 3$$

Similarly, for the other solution:

$$\ln\left(2^x\right) = \ln\left(\frac{1}{2}\right) = \ln\left(2^{-1}\right) \quad \Longrightarrow \quad x = -1$$