Core Topics in Mathematics

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Contents

1 BIDMAS and fractions

1.1 Learning Outcomes

- Communicate calculations to technology.
- Understand the rules of BIDMAS.
- Perform calculations (addition, subtraction, multiplication, division) with fractions.

1.2 Communicating with Technology

In order to effectively communicate with technology (calculators, Excel, computer algebra systems (CAS), etc.) you need to know how it interprets the information you provide.

For example, how would you tell a calculator or Excel to compute the following?

$$\frac{9-3}{5} \times 2^4$$

1.3 BIDMAS

BIDMAS is an acronym to help you remember the correct **order of operations**. This is the unambiguous way in which software evaluates a complicated calculation. It stands for:

- Brackets
- Index (powers and roots)
- Division and Multiplication,
- Addition and Subtraction.

Highest priority is given to brackets while lowest priority is given to addition and subtraction. Division and multiplication have the same priority level, as do addition and subtraction.

1.3.1 Example of BIDMAS

If we wish to evaluate:

$$10 + 5 \times 4$$

We must first perform the multiplication, then the addition:

$$\begin{aligned} & 10 + 5 \times 4 \\ = & \ 10 + 20 \\ = & \ 30 \end{aligned}$$

A common error is to perform the sum from left to right, i.e. $15 \times 4 = 60$. But say four people visited a theme park. It cost £5 each to enter and there's a fixed car park fee of £10. What would you expect to pay in total for entry? Not £60!

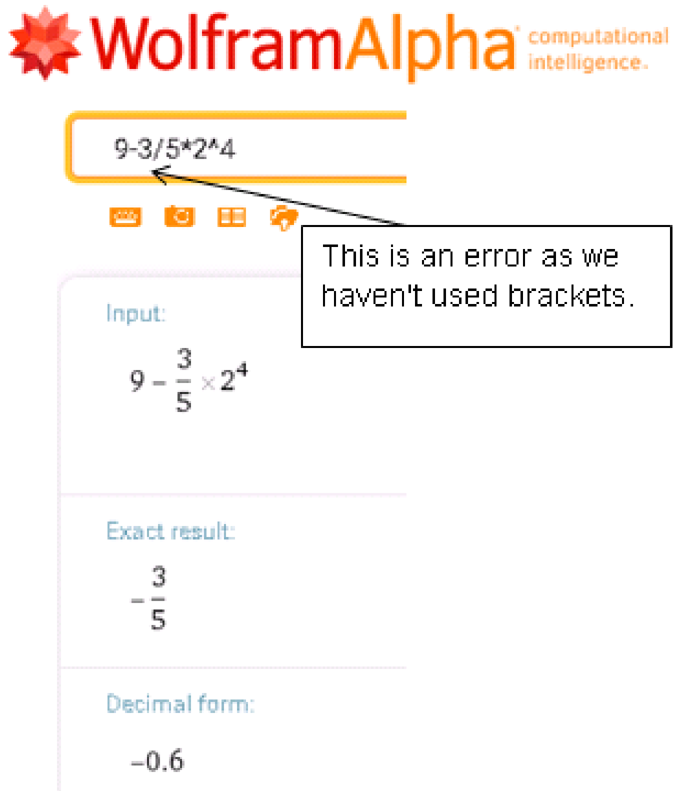
BIDMAS means that we can use brackets to tell our technology to do a particular calculation first, e.g.

$$\frac{9-3}{5} \times 2^4 \tag{1}$$

should be written in a calculator or Excel, etc. as:

$$(9-3)/5 \times 2^4 \ = \ \frac{6}{5} \times 2^4 \ = \ \frac{6}{5} \times 16 \ = \ 19.2$$

Note that modern calculators have a fraction button which means that the sum can be input as in equation 1.



The web-based CAS Wolfram Alpha gives an interpretation of your entry (under "input") in addition to the answer to your calculation.

1.4 Fractions

1.4.1 Working with Fractions: Addition and Subtraction

Fractions may be added/subtracted if every denominator is the same, e.g.

$$\frac{4}{7}+\frac{2}{7} = \frac{6}{7}$$

and

$$\frac{1}{3}-\frac{2}{3}+\frac{5}{3} = \frac{4}{3}$$

If the denominators are **not** the same, then we must first make them so, by scaling both numerator and denominator of one or both of the fractions by an appropriate amount.

Example 1:

$$\begin{aligned}\frac{3}{5}-\frac{1}{2} &= \frac{3\times 2}{5\times 2}-\frac{1\times 5}{2\times 5}\\ &= \frac{6}{10}-\frac{5}{10}\\ &= \frac{1}{10}\end{aligned}$$

Example 2:

$$\begin{aligned}\frac{2}{5}+\frac{7}{15} &= \frac{2\times 3}{5\times 3}+\frac{7}{15}\\ &= \frac{6}{15}+\frac{7}{15}\\ &= \frac{13}{15}\end{aligned}$$

1.4.2 Working with Fractions: Mixed Fractions

We may also have to deal with **mixed fractions** (a combination of a whole number and a fraction less than 1). In this case, we can convert it to an "improper" or "top-heavy" fraction:

$$
\begin{aligned}
4\frac{5}{6} &= 4+\frac{5}{6} \\
&= \frac{4}{1}+\frac{5}{6} \\
&= \frac{4\times 6}{1\times 6}+\frac{5}{6} \quad = \quad \frac{24}{6}+\frac{5}{6} \\
&= \frac{29}{6}
\end{aligned}
$$

1.4.3 Working with Fractions: Multiplication

The following rule can be used to deal with all multiplications:

> **Multiplying fractions:**
>
> $$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

For example:

$$\frac{4}{9}\times\frac{3}{5} \quad = \quad \frac{4\times 3}{9\times 5} \quad = \quad \frac{12}{45} \quad = \quad \frac{4}{15}$$

1.4.4 Working with Fractions: Division

The following rule can be used to deal with all divisions:

Fraction Division:

$$\frac{a}{b} \div \frac{c}{d} \quad \text{or} \quad \frac{\frac{a}{b}}{\frac{c}{d}} \quad = \quad \frac{a}{b} \times \frac{d}{c} \quad = \quad \frac{ad}{bc}$$

For example:

$$\frac{\frac{3}{7}}{\frac{2}{5}} \quad = \quad \frac{3}{7} \times \frac{5}{2} \quad = \quad \frac{3 \times 5}{7 \times 2} \quad = \quad \frac{21}{10}$$

2 Indices

2.1 Learning Outcomes

- Use rules of indices.

2.2 Indices

Indices are also known as exponents, powers and orders. The index of a number is simply the power to which you are raising it, e.g. the index of 3^4 is 4 (the "base" is 3).

If no index is given then the index must be 1.

Example:

$$5 = 5^1 \quad \text{and} \quad x = x^1$$

You will need to know how to interpret and simplify various expressions involving indices.

2.3 Rules of Indices

When multiplying identical bases, we can use the rule:

Multiplying indices:

$$a^m \times a^n = a^{m+n}$$

Example:

$$4^3 \times 4^5 = 4 \times 4 \times 4 \;\; \times \;\; 4 \times 4 \times 4 \times 4 \times 4 = 4^8$$

Or more simply using this rule:

$$4^3 \times 4^5 = 4^{3+5} = 4^8$$

When dividing identical bases, we can use the rule:

Dividing indices:

$$a^m \div a^n \quad \text{or} \quad \frac{a^m}{a^n} \quad = \quad a^{m-n}$$

Example:

$$\frac{6^5}{6^2} = \frac{6 \times 6 \times 6 \times 6 \times 6}{6 \times 6} = 6 \times 6 \times 6 = 6^3$$

Or more simply using this rule:

$$\frac{6^5}{6^2} = 6^{5-2} = 6^3$$

Now consider the expression $\left(a^6\right)^2$

$$\begin{aligned}
\left(a^6\right)^2 &= (a \times a \times a \times a \times a \times a)^2 \\
&= (a \times a \times a \times a \times a \times a) \times (a \times a \times a \times a \times a \times a) \\
&= a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a \\
&= a^{12}
\end{aligned}$$

It turns out that another rule is:

$$(a^6)^2 = a^{12} = a^{6 \times 2}$$

So we obtain a general rule for powers of powers:

Powers of indices:

$$\left(a^m\right)^n = a^{m \times n} = \left(a^n\right)^m$$

Example:

$$\left(5.03^{0.75}\right)^{1.8} = 5.03^{0.75\times1.8} = 5.03^{1.35} = 8.85 \quad \text{(2 d.p.)}$$

Confirm this on your calculator: check both $\left(5.03^{0.75}\right)^{1.8}$ and $5.03^{1.35}$
Now consider the expression $\frac{a^2}{a^6}$. We already know that:

$$\frac{a^2}{a^6} = a^{2-6} = a^{-4}$$

However:

$$\frac{a^2}{a^6} = \frac{aa}{aaaaaa} = \frac{1}{aaaa} = \frac{1}{a^4}$$

Therefore, we may conclude that another rule is:

$$a^{-4} = \frac{1}{a^4}$$

Negative indices:

$$a^{-m} = \frac{1}{a^m}$$

Negative indices denote reciprocals.

Example:

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

Now consider the expression $\frac{a^2}{a^2}$. We know that:

$$\frac{a^2}{a^2} = a^{2-2} = a^0$$

But it is also true that:

$$\frac{a^2}{a^2} = \frac{aa}{aa} = 1$$

Therefore, we can conclude that:

$$a^0 = 1$$

This is again a general rule, that *any* number raised to the power of zero is exactly one:

Zero index:

$$a^0 = 1$$

Example:

$$17^0 = 1 \qquad \pi^0 = 1 \qquad \big(47.01\pi + 13\big)^0 = 1 \qquad \big(zy - d\big)^0 = 1$$

Finally, consider the expression $a^{\frac{1}{2}}\,a^{\frac{1}{2}}$. We know that:

$$a^{\frac{1}{2}}\,a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$$

But this is also the definition of square roots:

$$\sqrt[2]{a}\,\sqrt[2]{a} = a$$

Therefore, we can conclude that:

$$a^{\frac{1}{2}} = \sqrt{a}$$

This is true not just for square roots, but more generally for n^{th}-roots:

Fractional indices:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Example:

$$64^{\frac{1}{3}} = \sqrt[3]{64} = 4$$

We also know that:

$$(\sqrt[n]{a})^m = (a^{\frac{1}{n}})^m = a^{\frac{1}{n}\times m} = a^{\frac{m}{n}}$$

Therefore, we can conclude that:

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

So fractional indices indicate roots and powers.

Example:

$$25^{\frac{3}{2}} = (\sqrt{25})^3 = 5^3 = 125$$

2.4 Summary

For any numbers a, m, n:

Rules of indices:

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{m\times n}$$

$$a^{-n} = \frac{1}{a^n}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

$$a^0 = 1 \quad \text{and} \quad a^1 = a$$

We will use these rules constantly throughout the module. You must come to know them instinctively.

3 Solving and transposing equations

3.1 Learning Outcomes

- Solve equations.
- Transpose equations.
- Addressing some of the most common difficulties that arise in algebraic manipulation.

3.2 Transposing and Solving Equations

Transposition is the process of rearranging an equation into a different form using logically valid steps.

All steps in transposition originate from a single logical principle:

If two initially equal things are changed in an identical manner, then they must still be equal *after* the change.

This means we can make changes to, say, the left-hand side (LHS) of an equation, provided we make precisely the **same change** to the right-hand side (RHS) also.

General principles are to:

- Get rid of fractions by multiplying.
- Get rid of brackets by expanding.
- Gather all terms with the unknown to one side by addition/subtraction.
- Remove everything else to the other side by addition/subtraction.
- Use division to leave the unknown by itself.

But this is a skill mainly acquired by practising many examples.

3.3 Examples

Example 1:

Solve $2x - 4 = 10$ for x.

First, we add +4 to both sides to remove the -4 term on the LHS:

$$2x - 4 + 4 = 10 + 4$$

$$\therefore\ 2x = 14 \qquad \text{Now } 2x \text{ is alone.}$$

$$\therefore\ \frac{2x}{2} = \frac{14}{2} \qquad \text{Remove the factor of 2 by division.}$$

$$\therefore\ x = 7$$

Example 2:

Solve $3x + 4 = 31$.

$$3x + 4 - 4 = 31 - 4 \qquad \text{Subtract away the +4 on the LHS.}$$

$$\therefore\ 3x = 27 \qquad \text{Now the only } x\text{-term is alone.}$$

$$\therefore\ \frac{3x}{3} = \frac{27}{3}$$

$$\therefore\ x = 9$$

Example 3:

Solve $5x - 6 = 3x - 8$.

$$5x - 6 - 3x = 3x - 8 - 3x \quad \text{Gather the } x\text{-terms on LHS.}$$

$$\therefore\ 2x - 6 = -8$$

$$\therefore\ 2x - 6 + 6 = -8 + 6 \quad \text{Remove the other term.}$$

$$\therefore\ 2x = -2$$

$$\therefore\ \frac{2x}{2} = \frac{-2}{2} \quad \text{Divide away the factor of 2.}$$

$$\therefore\ x = -1$$

Example 4:

Solve $2(3x - 7) + 4(3x + 2) = 6(5x + 9) + 3$.

First, always expand all the brackets. Then we will gather the x-terms and the constants together:

$$6x - 14 + 12x + 8 = 30x + 54 + 3$$

$$\therefore\ 18x - 6 = 30x + 57$$

$$\therefore\ 18x - 6 - 30x = 30x + 57 - 30x$$

$$\therefore\ -12x - 6 = 57$$

Now proceeding as in previous examples:

$$-12x - 6 = 57$$

$$\therefore\ -12x - 6 + 6 = 57 + 6$$

$$\therefore\ -12x = 63$$

$$\therefore\ \frac{-12x}{-12} = \frac{63}{-12}$$

$$\therefore x = -\frac{63}{12} \quad \text{or} \quad -5.25$$

Example 5:

Solve $\mu = u + at$ for a.

$$\mu - u = u + at - u$$

$$\therefore\ \mu - u = at$$

$$\therefore\ \frac{\mu - u}{t} = \frac{at}{t}$$

$$\therefore\ \frac{\mu - u}{t} = a$$

Or rather:

$$a = \frac{\mu - u}{t}$$

Example 6:

Solve $\dfrac{2+t}{3} = 2(t-k)$ for t.

$$3\times\frac{(2+t)}{3} = 3\times 2(t-k)$$

$$\therefore\ 2+t = 6(t-k)$$

$$\therefore\ 2+t = 6t-6k$$

$$\therefore\ 2+t-6t = 6t-6k-6t$$

$$\therefore\ 2-5t = -6k$$

$$\therefore\ 2-5t-2 = -6k-2$$

$$\therefore\ -5t = -6k-2$$

$$\therefore\ \frac{-5t}{-5} = \frac{-6k-2}{-5}$$

$$t = \frac{-6k-2}{-5}$$

The solution could also be written as:

$$t = \frac{6k+2}{5}$$

Or:

$$t = \frac{2(3k+1)}{5}$$

Or:

$$t = \frac{2}{5}(3k+1)$$

There are often multiple ways to present the solution.

Example 7:

Solve $\frac{1}{f} = \frac{1}{u} + \frac{1}{\mu}$ for μ.

First, let's get the term containing μ on its own:

$$\frac{1}{f} - \frac{1}{u} = \frac{1}{u} + \frac{1}{\mu} - \frac{1}{u}$$

$$\therefore \quad \frac{1}{f} - \frac{1}{u} = \frac{1}{\mu}$$

Now multiply both sides by μ to get it out of the denominator:

$$\mu\left(\frac{1}{f} - \frac{1}{u}\right) = \mu\frac{1}{\mu}$$

$$\therefore \quad \mu\left(\frac{1}{f} - \frac{1}{u}\right) = 1$$

$$\therefore \quad \frac{\mu\left(\frac{1}{f} - \frac{1}{u}\right)}{\frac{1}{f} - \frac{1}{u}} = \frac{1}{\frac{1}{f} - \frac{1}{u}}$$

$$\therefore \quad \mu = \frac{1}{\frac{1}{f} - \frac{1}{u}}$$

Often there are many paths that can be taken to the solution. Let's now see an alternative approach to solving $\frac{1}{f} = \frac{1}{u} + \frac{1}{\mu}$ for μ.

Begin by multiplying all terms by μ to remove it from the denominator, then gather together all resulting terms that contain μ:

$$\mu\times\frac{1}{f}=\mu\times\frac{1}{u}+\mu\times\frac{1}{\mu}$$

$$\therefore \quad \frac{\mu}{f}=\frac{\mu}{u}+1$$

$$\therefore \quad \frac{\mu}{f}-\frac{\mu}{u}=\frac{\mu}{u}+1-\frac{\mu}{u}$$

$$\therefore \quad \frac{\mu}{f}-\frac{\mu}{u}=1$$

$$\therefore \quad \mu\left(\frac{1}{f}-\frac{1}{u}\right)=1$$

$$\therefore \quad \frac{\mu\left(\frac{1}{f}-\frac{1}{u}\right)}{\frac{1}{f}-\frac{1}{u}}=\frac{1}{\frac{1}{f}-\frac{1}{u}}$$

$$\therefore \quad \mu=\frac{1}{\frac{1}{f}-\frac{1}{u}}=\frac{1}{\frac{u-f}{fu}}=\frac{fu}{u-f}$$

In ther final step we have simplified the “fractions inside a fraction” by multiplying both the numerator and denominator of the main fraction by the common denominator of the internal fractions. We never want to have nested fractions in our final solution - they can always be simplified in this way.

Example 8:

Solve $A = 2\pi r^2 + 2\pi rh$ for h.

Start by isolating the term containing h:

$$A - 2\pi r^2 = 2\pi r^2 + 2\pi rh - 2\pi r^2$$

$$\therefore\ A - 2\pi r^2 = 2\pi rh$$

$$\therefore\ \frac{A - 2\pi r^2}{2\pi r} = \frac{2\pi rh}{2\pi r}$$

$$\therefore\ \frac{A - 2\pi r^2}{2\pi r} = h$$

Thus,

$$h = \frac{A - 2\pi r^2}{2\pi r}$$

Example 9:

Solve $T = 2\pi\sqrt{\dfrac{L}{g}}$ for L.

In this case, the desired variable L is contained inside a root. We first need to isolate the root, by dividing both sides by 2π:

$$\frac{T}{2\pi} = \frac{2\pi}{2\pi}\sqrt{\frac{L}{g}}$$

$$\therefore\ \frac{T}{2\pi} = \sqrt{\frac{L}{g}}$$

Now that the entire right-hand side consists of a square root, we can extract L by taking the square of both sides:

$$\left(\frac{T}{2\pi}\right)^2 = \left(\sqrt{\frac{L}{g}}\right)^2$$

$$\therefore\ \left(\frac{T}{2\pi}\right)^2 = \frac{L}{g}$$

$$\therefore\ \left(\frac{T}{2\pi}\right)^2 = \frac{L}{g}$$

$$\therefore\ \left(\frac{T}{2\pi}\right)^2 \times g = \frac{L}{g}\times g$$

$$\therefore\ g\left(\frac{T}{2\pi}\right)^2 = L$$

Hence,

$$L = g\left(\frac{T}{2\pi}\right)^2$$

3.4 More challenging examples

Example 1

Solve:

$$\frac{1}{x} - \frac{2y+1}{3} = 5y$$

for x

Solution:

$$\frac{1}{x} - \frac{2y+1}{3} = 5y$$

What do we need to consider in this example?

- Remember that the minus sign applies to **all** of $(2y + 1)/3$, not just the $2y$.
- Start by multiplying away all of the fractions.
- Only gather like terms after that.

$$\frac{1}{x} - \frac{2y+1}{3} = 5y$$

Multiply all terms by x to get rid of the first fraction:

$$x\left(\frac{1}{x}\right) - x\left(\frac{2y+1}{3}\right) = x\big(5y\big)$$

$$\therefore \frac{1}{1} - \frac{x(2y+1)}{3} = 5xy$$

$$\therefore 1 - \frac{x(2y+1)}{3} = 5xy$$

Now multiply all terms by 3 to get rid of the remaining fraction:

$$3(1) - 3\left(\frac{x(2y+1)}{3}\right) = 3(5xy)$$

$$\therefore 3 - x(2y+1) = 15xy$$

$$\therefore 3 - 2xy - x = 15xy$$

Finally, gather all terms containing x together and simplify:

$$15xy + 2xy + x = 3$$

$$\therefore 17xy + x = 3$$

$$\therefore x(17y+1) = 3$$

$$\therefore x = \frac{3}{17y+1}$$

Re-writing $17xy + x$ as $x(17y+1)$ is called **factorisation**, which we shall practice more in this section.

Example 2

The following formula arises in the study of relativistic motion.

$$T = \frac{T_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

In this case, c denotes the speed of light (3×10^8 m/s). How is it related to the other variables?

Solution:

$$T = \frac{T_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

Begin by multiplying both sides by the denominator of the fraction:

$$T\left(1 - \frac{v^2}{c^2}\right)^{1/2} = T_0$$

Undo the power of $1/2$ by squaring both sides of the equation:

$$\left(T\left(1 - \frac{v^2}{c^2}\right)^{1/2}\right)^2 = \left(T_0\right)^2$$

$$\therefore T^2\left(1 - \frac{v^2}{c^2}\right) = T_0^2$$

Divide both sides by T^2:

$$1 - \frac{v^2}{c^2} = \frac{T_0^2}{T^2}$$

Isolate the term containing c:

$$-\frac{v^2}{c^2} = \frac{T_0^2}{T^2} - 1$$

Now multiply both sides by c^2 to extract it from the denominator:

$$-v^2 = c^2\left(\frac{T_0^2}{T^2} - 1\right)$$

To get c^2 alone, divide both sides by the contents of the brackets:

$$c^2 = \frac{-v^2}{\dfrac{T_0^2}{T^2} - 1}$$

This can be simplified slightly by changing the sign of all terms within the fraction:

$$c^2 = \frac{v^2}{1 - \dfrac{T_0^2}{T^2}}$$

To simplify further, address the subfraction T_0^2/T^2 by multiplying the numerator and denominator of the main fraction by T^2:

$$c^2 = \frac{v^2 T^2}{T^2 - T_0^2}$$

(We will practice this technique more later.) Finally, take the square root of both sides to obtain an expression for c:

$$c = \sqrt{\frac{v^2 T^2}{T^2 - T_0^2}}$$

Example 3

The following formula describes the relativistic Doppler shift concerning the changes in frequency of light due to relative longitudinal motion of a source and observer:

$$\nu' = \nu \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}$$

Obtain a formula for β.

Solution:

Divide both sides by ν:

$$\frac{\nu'}{\nu} = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}$$

Now, if we square both sides we can eliminate both square roots:

$$\begin{aligned} \left(\frac{\nu'}{\nu}\right)^2 &= \left(\frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\right)^2 \\ &= \frac{(\sqrt{1-\beta})^2}{(\sqrt{1+\beta})^2} \quad \text{(Rules of indices!)} \\ &= \frac{1-\beta}{1+\beta} \end{aligned}$$

Now multiply both sides by denominator $1+\beta$ to simplify the fraction:

$$\left(\frac{\nu'}{\nu}\right)^2 (1+\beta) = 1-\beta$$

Expand the brackets and gather like terms (with β):

$$\left(\frac{\nu'}{\nu}\right)^2 + \left(\frac{\nu'}{\nu}\right)^2 \beta = 1-\beta$$

$$\therefore \left(\frac{\nu'}{\nu}\right)^2 \beta + \beta = 1 - \left(\frac{\nu'}{\nu}\right)^2$$

Factorise β from the LHS:

$$\beta\left(\left(\frac{\nu'}{\nu}\right)^2 + 1\right) = 1 - \left(\frac{\nu'}{\nu}\right)^2$$

Divide both sides by the contents of the brackets to isolate β and finally simplify the sub-fractions:

$$\beta = \frac{1 - \left(\frac{\nu'}{\nu}\right)^2}{1 + \left(\frac{\nu'}{\nu}\right)^2} \quad = \quad \frac{1 - \frac{\nu'^2}{\nu^2}}{1 + \frac{\nu'^2}{\nu^2}} \quad = \quad \frac{\nu^2 - \nu'^2}{\nu^2 + \nu'^2}$$

3.5 Common difficulties in algebraic manipulation

There are some smaller aspects of algebraic manipulation that we have seen in these examples and which can be tricky. You will need to become comfortable with:

- Manipulating fractions and writing them in different ways.
- Factorisation.
- Simplifying subfractions (fractions within fractions).

3.5.1 Fractions

We can write algebraic fractions in a variety of different ways (combining or separating their parts by multiplication), as long as all parts maintain their correct position on either the numerator or the denominator.

For example:

$$\frac{3y}{x}$$

can be written correctly as any of the following without changing the meaning:

$$3 \times \frac{1}{x} \times y \qquad \frac{3}{x}y \qquad 3\frac{y}{x} \qquad (3y) \div x$$

3.5.2 Factorisation

We already know how to expand brackets:

$$3x(x+y) = 3x^2 + 3xy$$

Factorisation is the reverse of this process. We look at two or more terms, and ask what "factors" are shared by all terms?

Factor: the whole numbers or symbols that a term can be perfectly divided by. For example, 3, 9, x, x^2 and any combinations such as $3x$, $9x$ or $3x^2$ are all factors of $9x^2$.

Example:

Factorise as much as possible:

$$3x + 2xy$$

The simplest factors of the first term are 3 and x, and those of the second are 2, x and y.

As x is the only common (shared) factor, we can only remove it - leaving behind 3 and $2y$ respectively:

$$3x + 2xy = x(3 + 2y)$$

Example: Factorise as much as possible:

$$12x^2 - 8xy^2$$

The simplest factors of the first term are 2 and x, and those of the second are 2, x and y. However, to factorise fully we want to choose the largest shared factors.

All factors of the first term: 2, 3, 4, 6, 12 and x and x^2
All factors of the second term: 2, 4, 8 and x and y and y^2.

Therefore the largest common factor is $4x$:

$$12x^2 - 8xy^2 = 4x(3x - 2y^2)$$

Factorise and simplify as much as possible:

$$\frac{28}{\pi x} + 16x$$

It is good practice when factorising to try to ensure that any fractions are also *outside* of the brackets, if this is not overly complicated to achieve.

In this case, in addition to the common factor of 4, we could factor out the πx on the denominator, which will require multiplying the second term by these in order to maintain balance.

$$\frac{28}{\pi x} + 16x = 4\left(\frac{7}{\pi x} + 4x\right) = \frac{4}{\pi x}(7 + 4\pi x^2)$$

3.5.3 Dealing with subfractions

When we have a fraction where either the numerator or the denominator (or both) themselves consist of a fraction, it is **always** possible to simplify them and express as a simple fraction.

This can be achieved with explicit fraction division.

For example:

$$\frac{\frac{15}{4}}{2} \quad = \quad \frac{15}{4} \div 2 \quad = \quad \frac{15}{4} \div \frac{2}{1} \quad = \quad \frac{15}{4} \times \frac{1}{2} \quad = \quad \frac{15}{8}$$

Example:

Simplify:

$$\frac{\frac{15}{y^2}}{\frac{x}{y}}$$

Solution:

$$\frac{\frac{15}{y^2}}{\frac{x}{y}} \quad = \quad \frac{15}{y^2} \div \frac{x}{y} \quad = \quad \frac{15}{y^2} \times \frac{y}{x} \quad = \quad \frac{15y}{xy^2} \quad = \quad \frac{15}{xy}$$

As a shortcut, we may instead simply multiply the numerator and denominator of the main fraction by the denominator of the subfraction(s):

$$\frac{\frac{15}{y^2}}{\frac{x}{y}} \quad = \quad \frac{\frac{15}{y^2} \times y^2}{\frac{x}{y} \times y^2} \quad = \quad \frac{15}{\frac{xy^2}{y}} \quad = \quad \frac{15}{xy}$$

4 Linear functions

4.1 Learning Outcomes

- Recognise polynomials
- Interpret linear equations
- Recognise typical shapes of polynomial graphs (constant and linear)
- Plot linear graphs using Excel

4.2 Polynomial functions

A **polynomial function** is one that *only* involves non-negative integer powers of x, for example:

- $y = 7x + 4$ (polynomial of order/degree 1, linear)
- $y = 3x^2 - 5x - 1$ (polynomial of order/degree 2, quadratic)
- $y = -x^3 + 5x^2 - 7x + 12.01$ (poly. of order/degree 3, cubic)

Functions containing negative or non-integer powers, or other function such as trigonometric) are *not* polynomials, e.g.

- $y = x^2 + 4\sqrt{x} - 5$
- $y = x^2 + \sin(x)$
- $y = \dfrac{5}{x^2} - 7x^3 + 6x - 4$

4.3 Function notation

To express a function we may write, for example, $y = 3x^2 + 8x - 7$. Here the **independent variable** is x and the **dependent variable** is y; we say that y is dependent upon x. That is, the value of y depends on the value of x that we put in.

We could also express the function as:

$$y(x) = 3x^2 + 8x - 7$$

or

$$f(x) = 3x^2 + 8x - 7$$

or

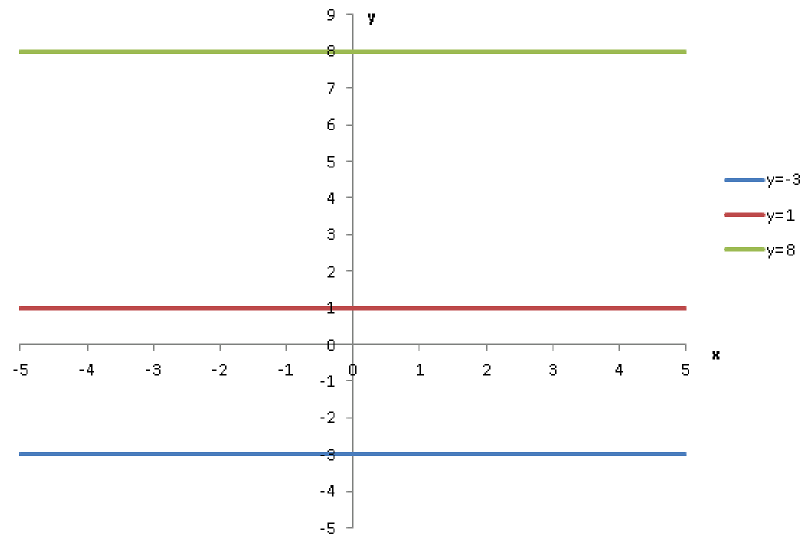
$$g(x) = 3x^2 + 8x - 7$$

etc.

Here the independent variable is explicitly x, and the function is named y, f, or g

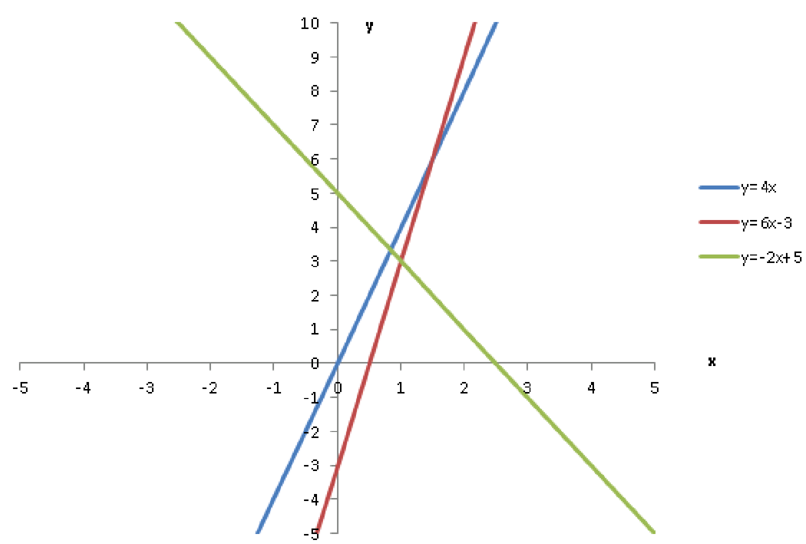
4.4 Graphs of Polynomials

Constant functions:



Graphs of **constant functions** (no dependency on x) are always straight, horizontal lines.

Linear functions:



Graphs of **linear functions** are always straight lines.

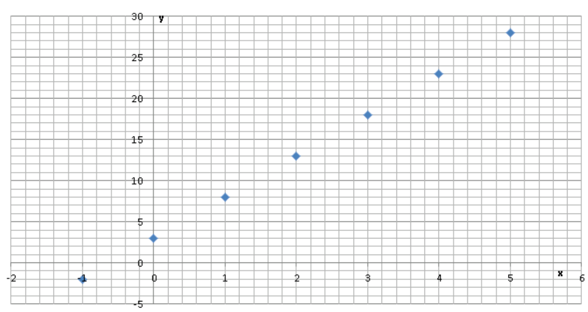
4.5 Linear Graphs

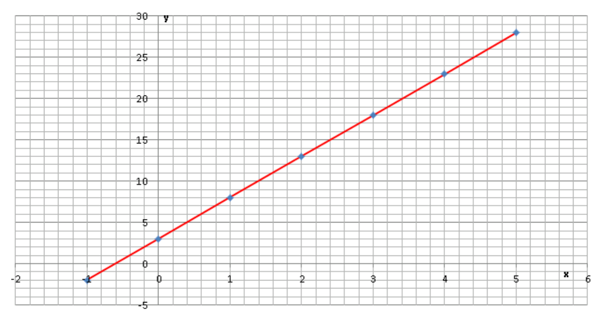
To plot graphs manually we first have to define the x range, if not already specified. Then we need to calculate the value of the function, y, for the specific values of x.

Plot the function $y = 5x + 3$ in the range $-1 \leq x \leq 5$.

x	y
-1	5(-1)+3 = -2
0	3
1	8
2	13
3	18
4	23
5	28

We can then plot the (x, y) coordinates on a graph:





4.6 Determining the Linear Equation

Equation of a straight line:

$$y = mx + c \tag{2}$$

where m and c are constants, represents a straight line.

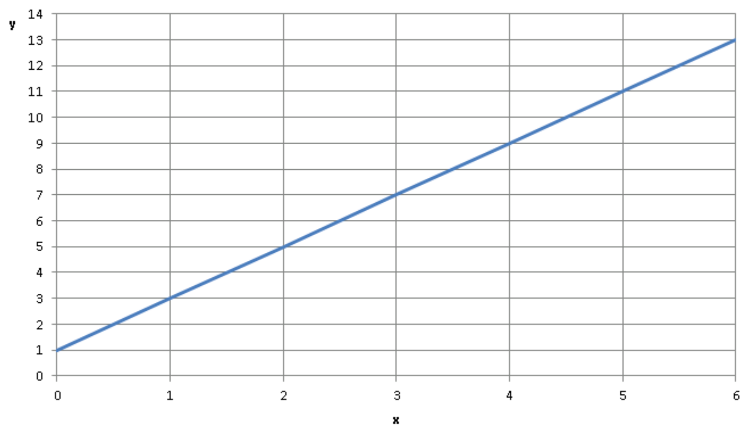
m is the **gradient** (slope) of the line and can be calculated as

$$m = \frac{\text{vertical change (rise)}}{\text{horizontal change (run)}} = \frac{\Delta y}{\Delta x}$$

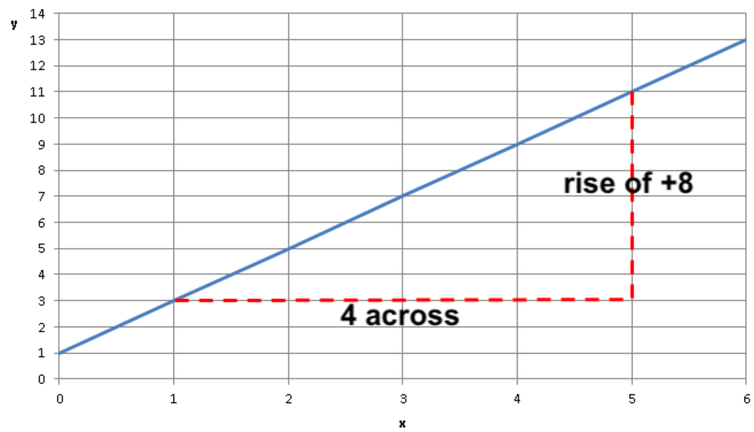
c is the value of y when the line crosses the y-axis (at $x = 0$), known as the y**-intercept**.

Note: to find where the line crosses the x-axis, simply let $y = 0$.

Example: Find the equation of this line:



First, we can see that $c = 1$ as this is the height where the y-axis is crossed.



and

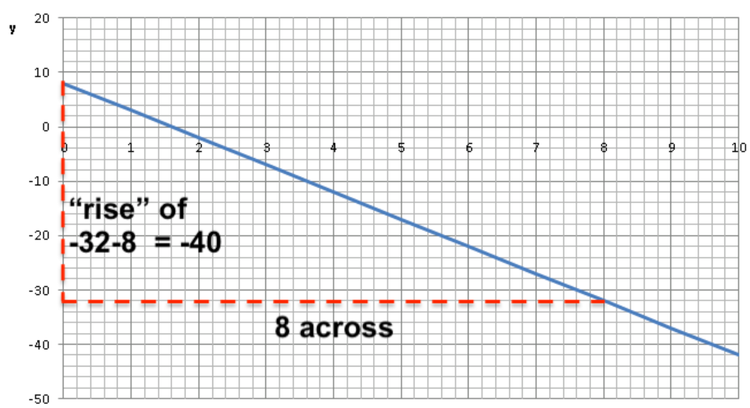
$$m = \frac{rise}{run} = \frac{8}{4} = 2$$

Thus,

$$y = 2x + 1$$

Note: if the straight line graph is *decreasing* then we expect a **negative gradient**.

This is because the “rise” will actually be a fall - a decrease in y.



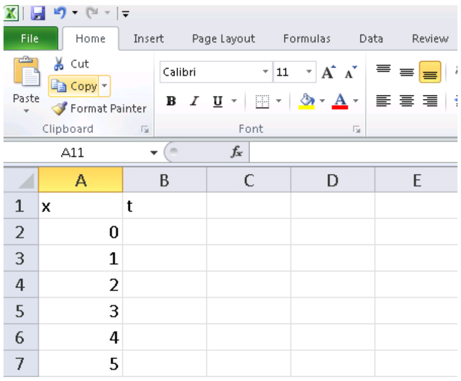
This time the gradient is negative as there is a decrease from left to right:

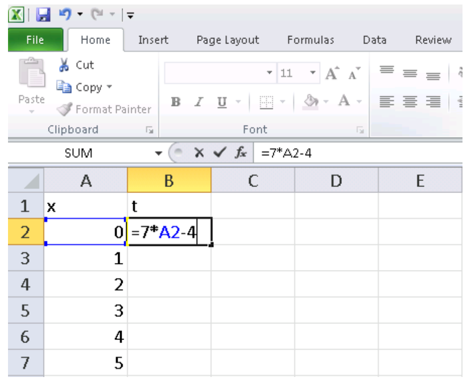
$$m = \frac{rise}{run} = \frac{-40}{8} = -5$$

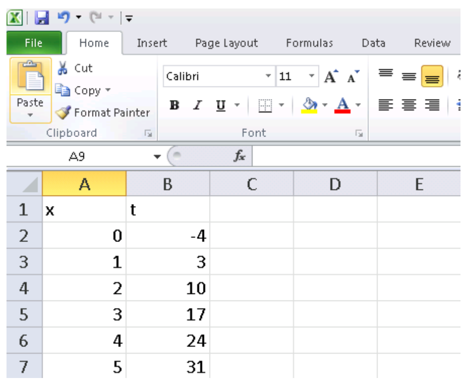
4.7 Using Excel to Plot Polynomials

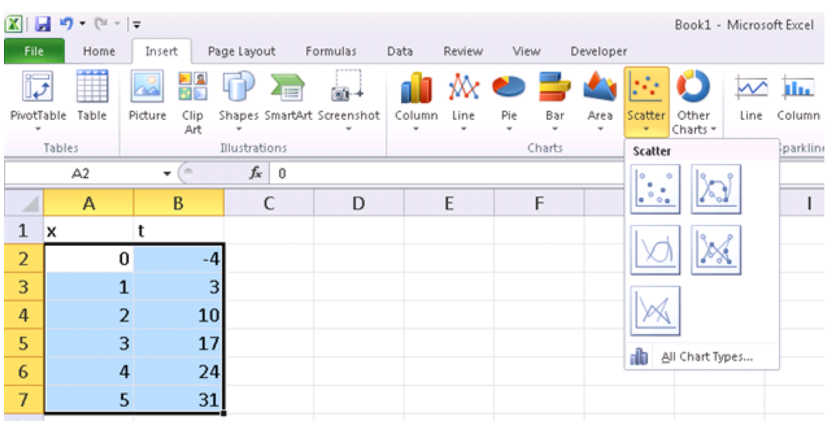
Using Excel to plot a function allows us to automate the process.

To plot the linear function $y = 7x - 4$ in the range $0 \le x \le 5$:









5 Quadratic functions

5.1 Learning Outcomes

- Recognise quadratic equations.
- Recognise typical shapes of quadratic graphs.
- Solve quadratic equations.

5.2 What are quadratics?

In the last lecture, we discussed polynomials. The simplest kind was a linear equation with highest power x^1 (describing a straight line). The next simplest are second-order polynomials:

Quadratic equation:

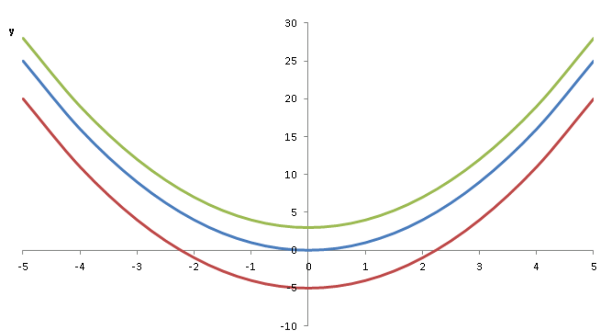
$$y = ax^2 + bx + c, \tag{3}$$

where a, b and c are constants and $a \neq 0$

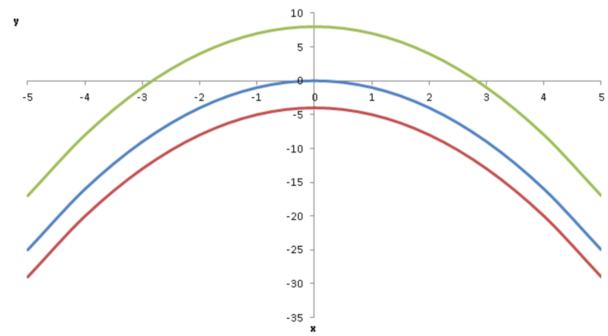
This represents a curve with a single turning point, called a parabola. All quadratics take the general form of equation (3).

There are two types, depending on the value of a.

When $a > 0$ the curve is $\cup$-shaped:



When $a < 0$ the curve is $\cap$-shaped:



When trying to visualise a quadratic function, consider:

- What is the orientation?
  - $a > 0$: upturned.
  - $a < 0$: downturned.
- Is it broad or narrow compared to $y = x^2$?
  - $|a| > 1$: narrower.
  - $|a| < 1$: broader.
- Where is the turning point?
  - Positive c will push it up.
  - Negative c will push it down.
  - Positive or negative b will push it down if $a > 0$ (up if $a < 0$).
  - $b = 0$: on the y-axis.
  - $b > 0$: left of the y-axis if $a > 0$ (right if $a < 0$).
  - $b < 0$: right of the y-axis if $a > 0$ (left if $a < 0$).

The constant c is the y-intercept, as in the linear case (if $x = 0$ then the equation becomes $y = a \times 0^2 + b \times 0 + c = c$).

The curve can cross the x-axis (at $y = 0$) twice, once (just touching it) or never. If it does cross the x-axis, we can calculate the values of x where this occurs by solving:

$$ax^2 + bx + c = 0$$

The **solutions** of $0 = ax^2 + bx + c$ are the same as the x-intercepts of $y = ax^2 + bx + c$ and are also known as the **roots** of $ax^2 + bx + c$.

5.3 Solving quadratic equations

Factorisation:

There are various ways to solve a quadratic equation. Sometimes we can “factorise”, which is the reverse of expanding brackets. If $a = 1$, then we seek two numbers that multiply to c and add to b.

Example:

$x^2 + 4x + 3 = 0$ What pair multiplies to 3 and adds to 4?

$x^2 + 3x + 1x + 3 = 0$ 3 and 1 of course!

$(x + 3)(x + 1) = 0$

This means that either $x + 3 = 0$, so $x = -3$, or that $x + 1 = 0$ so $x = -1$. This approach isn't always possible, so the most reliable method to use (which *always* works!) is...

The quadratic formula:

The quadratic formula:

If $ax^2 + bx + c = 0$ and $a \neq 0$ then:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{4}$$

The **discriminant** is the “bit under the square root.” It indicates how many roots exist and of what type:

- $b^2 - 4ac > 0$ indicates two real and distinct roots (x_1 and x_2)
- $b^2 - 4ac = 0$ indicates real and repeated roots ($x_1 = x_2$)
- $b^2 - 4ac < 0$ indicates complex roots ($x = \alpha + j\beta$)

Roots and Discriminants:
Furthermore, if there are two distinct, real roots x_1 and x_2 to the quadratic $y = ax^2+bx+c$, then it is possible to re-write the quadratic in the form:

$$y = a(x - x_1)(x - x_2)$$

However, if there are real and repeated roots, $x_1 = x_2$, then it is possible to re-write the quadratic in the form:

$$y = a(x - x_1)^2$$

5.4 Examples:

Determine the roots of the following quadratics:

1) $y = 3x^2 + 13x - 10$

2) $y = x^2 - 14x + 49$

3) $y = x^2 + 6x + 34$

Factorise the following quadratics:

4) $y = x^2 + 7x + 12$

5) $y = x^2 - 10x + 25$

Solutions:

1) Here, the coefficients are $a = 3$, $b = 13$ and $c = -10$.

Using the formula:

$$
\begin{aligned}
x &= \frac{-13 \pm \sqrt{13^2 - 4 \times 3 \times -10}}{2 \times 3} \\
&= \frac{-13 \pm \sqrt{289}}{6} \quad \text{Positive discriminant.} \\
&= \frac{-13 \pm 17}{6} \\
&= \frac{4}{6} \quad \text{or} \quad -\frac{30}{6} \\
&= \frac{2}{3} \quad \text{or} \quad -5 \quad \text{So we have two distinct roots.}
\end{aligned}
$$

2) Here, the coefficients are $a = 1$, $b = -14$ and $c = 49$.
Using the formula:

$$
\begin{aligned}
x &= \frac{-(-14) \pm \sqrt{(-14)^2 - 4 \times 1 \times 49}}{2 \times 1} \\
&= \frac{14 \pm \sqrt{0}}{2} \quad \text{Discriminant is zero.} \\
&= \frac{14 \pm 0}{2} \\
&= \frac{14}{2} \\
&= 7 \quad \text{This time we have one repeated solution.}
\end{aligned}
$$

3) Here, the coefficients are $a = 1$, $b = 6$ and $c = 34$.

Using the formula:

$$\begin{aligned}
x &= \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 34}}{2 \times 1} \\
&= \frac{-6 \pm \sqrt{36 - 136}}{2} \\
&= \frac{-6 \pm \sqrt{-100}}{2} \quad \text{Discriminant is zero.}
\end{aligned}$$

There are no real solutions - we can't proceed any further. This corresponds to a parabola that sits above the x-axis and never touches it.

4) Solving by the quadratic formula:

$$\begin{aligned}
x &= \frac{-7 \pm \sqrt{7^2 - 4 \times 1 \times 12}}{2 \times 1} \\
&= \frac{-7 \pm \sqrt{1}}{2} \\
&= \frac{-7 \pm 1}{2} \\
&= -4 \quad \text{or} \quad -3
\end{aligned}$$

Thus we can factorise as:

$$\begin{aligned}
y &= 1(x - (3))(x - (-4)) \\
&= (x + 3)(x + 4)
\end{aligned}$$

5) Solving by the quadratic formula:

$$\begin{aligned}
x &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4 \times 1 \times 25}}{2 \times 1} \\
&= \frac{10 \pm \sqrt{0}}{2} \\
&= \frac{10 \pm 0}{2} \\
&= 5 \quad \text{(repeated)}
\end{aligned}$$

Thus we can write the factorised quadratic function as:

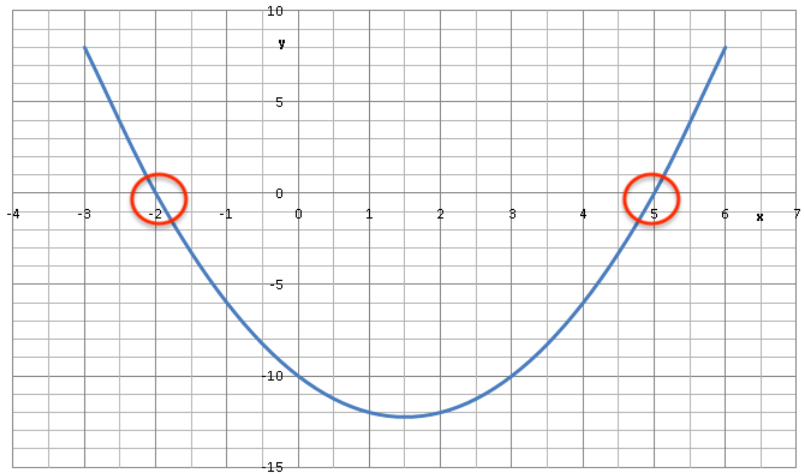
$$y = 1(x - 5)^2 = (x - 5)^2$$

Observe that -5 and -5 multiply to 25 and add to -10 as required.

5.5 Determining the Quadratic Equation from a Graph

There are two ways to do this:

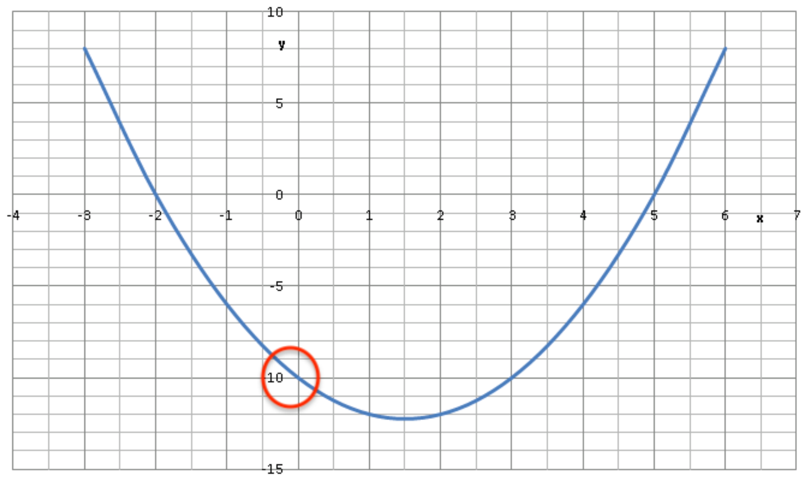
Method 1



We can see the roots are $x_1 = -2$ and $x_2 = 5$, which means in factorised form the quadratic equation is:

$$\begin{aligned} y &= (x+2)(x-5) \\ &= x^2 - 3x - 10 \end{aligned}$$

Method 2



We can see that the y-intercept is -10.

$$\therefore y = ax^2 + bx - 10$$

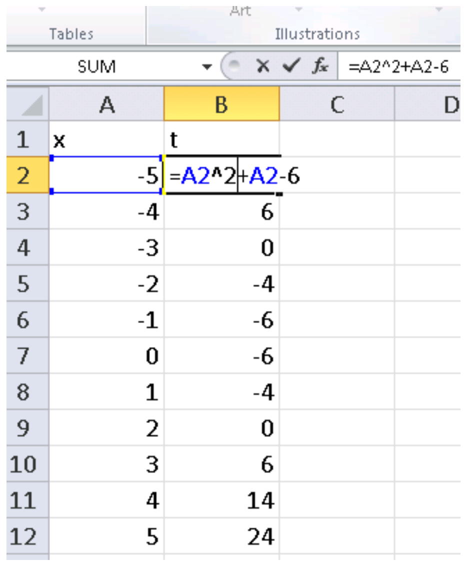
Now, if we choose two coordinates, e.g. $(-1, -6)$ and $(6, 8)$, we can solve $y = ax^2 + bx - 10$ simultaneously. We will learn more on this later.

5.6 Plotting in Excel

To plot higher order functions we make use of the $\wedge$ symbol, which means to the power of.

Example:

To plot $y = x^2 + x - 6$:



	A	B	C	D
1	x	t		
2	-5	=A2^2+A2-6		
3	-4	6		
4	-3	0		
5	-2	-4		
6	-1	-6		
7	0	-6		
8	1	-4		
9	2	0		
10	3	6		
11	4	14		
12	5	24		

6 Exponential and logarithmic functions

6.1 Learning Outcomes

- Recognise logarithmic and exponential functions.
- Sketch logarithmic and exponential functions.
- Apply the laws of logarithms.
- Solve exponential equations.

6.2 Exponential Functions

The general exponential function is of the form:

$$y = Ab^{kx},$$

where A, b, k are constants:

- A is a coefficient and is the value of y when $x = 0$. This is because if $x = 0$, $y = Ab^{k\times 0} = Ab^0 = A \times 1 = A$.
- b is the base.
- k determines how fast the function grows (growth rate).

Certain values for the base b are more common than others, especially:

$$y = 10^x \quad \text{and} \quad y = e^x = \exp(x)$$

One particular base is very important: $\mathrm{e} = 2.718281828\ldots$, called Euler's number.

The general form of this type of equation is:

The Exponential function:

$$y = Ae^{Bx} + C$$

where A, B and C are constants.

Note, when $x = 0$, $y = A + C$ (this is the y-intercept).

6.2.1 Examples

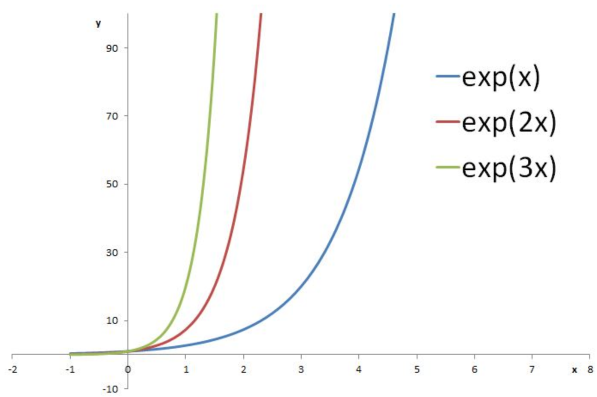
Use your calculator to determine the following:

1) e^4

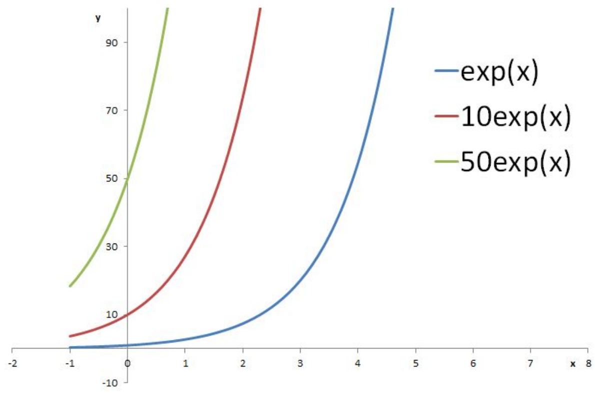
2) $4e^{7.2}$

3) $2.9e^{29.7} + 2.3$

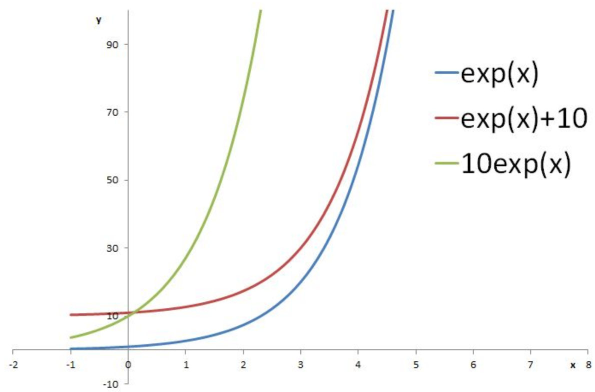
6.2.2 Graphs of the Exponential Function



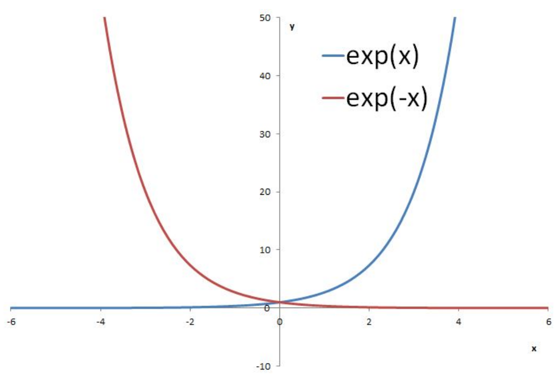
Changing the *magnitude* of B affects the gradient.



Changing A affects the y-intercept.



Changing both A and C affect the y-intercept.



Changing the *sign* of B reflects the e^x curve in the y-axis: positive B gives exponential growth, negative B gives decay.

6.2.3 Applications of the Exponential Function

The exponential function is used frequently across engineering.

It is used in growth and decay models such as:

- Tension in belts: $T_1 = T_0 e^{\mu\theta}$
- Newton's law of cooling: $\theta = \theta_0 e^{-kt}$
- Atmospheric pressure at altitude h: $p = p_0 e^{-h/c}$
- Discharge of a capacitor: $q = Qe^{-t/CR}$

6.3 The Logarithm Function

The logarithm function is written as follows:

Logarithm function:

$$y = \log_a(x),$$

where a and x are positive and $a \neq 1$ is constant.

This can be interpreted as:

"y is the power to which one must raise a (the base), to get x (the argument)."

That is:

$$a^y = x$$

Example:

What power of 2 is exactly equal to 8?

Answer: $2 \times 2 \times 2 = 2^3 = 8$

So, we need exactly 3 "2's" to get 8. This means that the logarithm of 8, to base 2, is 3. We write this as:

$$\log_2(8) = 3$$

Examples:

Calculate x:

- $3^x = 81 \quad \therefore \ \log_3(81) = 4$
- $6^x = 1 \quad \therefore \ \log_6(1) = 0$
- $2^x = 0.125 \quad \therefore \ \log_2(0.125) = -3$

So one application of logs is to solve equations where the desired variable is in the index.

The most commonly used bases are 10 and e.

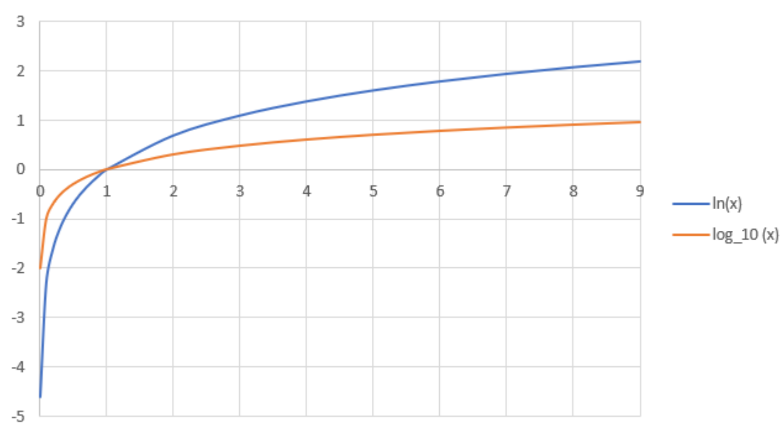
The log button on your calculator is $\log_{10}$. This is the common logarithm.

The ln button on your calculator is $\log_e$. This is the natural logarithm and is usually written ln, so:

$$\log_e 6 = \ln 6$$

Engineers mainly deal with the natural logarithm.

6.3.1 Visualising Logarithms



Remember, the input to a log function must be positive.

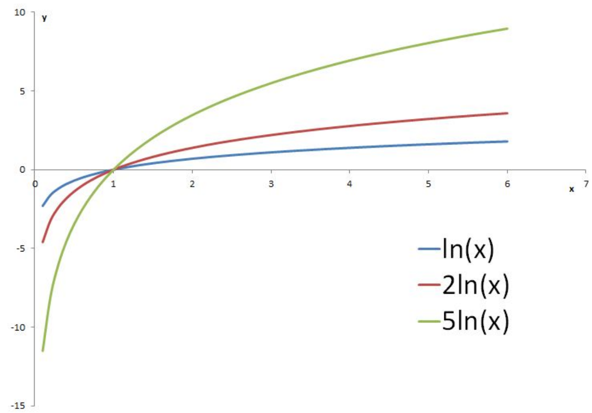
More generally natural logarithm functions are in the form:

General natural log functions:

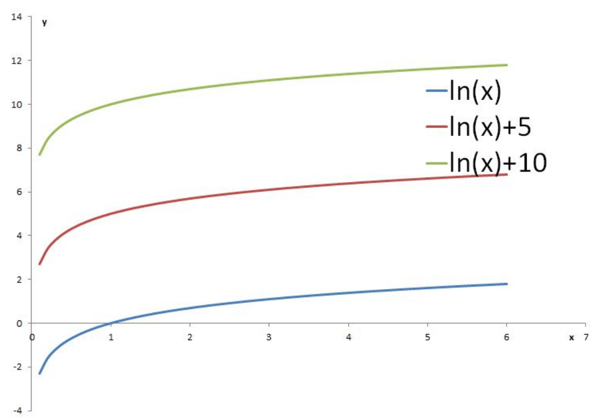
$$y = A\ln(x) + B$$

where A and B are constants.

Let's look at what A and B influence...



Changing A stretches the curve vertically.



Changing B shifts the curve vertically.

6.3.2 Laws of Logarithms

The following are useful for manipulating equations (they are true for *any* base, as long as all the logs share the same base).

Laws of logarithms:

$$\log\left(A^n\right) = n\log\left(A\right)$$

$$\log\left(AB\right) = \log\left(A\right) + \log\left(B\right)$$

$$\log\left(\frac{A}{B}\right) = \log\left(A\right) - \log\left(B\right)$$

$$\log\left(1\right) = 0$$

Examples:

Write each of the following as a single log:

- $\log_{10} 6 + \log_{10} 3 = \log_{10}\ (6 \times 3) = \log_{10} 18$
- $\ln 6 - \ln 3 = \ln\ \left(\frac{6}{3}\right) = \ln 2$
- $2\log x = \log\left(x^2\right)$

6.4 Interaction between Logarithms and Exponentials

The logarithm and exponential functions are the inverses of each other, i.e. one undoes the impact of the other:

$$\ln(e^x) = x \text{ and } e^{\ln(x)} = x$$

For example:

$$\ln(e^7) = 7 \text{ and } e^{\ln(15)} = 15$$

Example 1

Solve the equation $25e^x = 521$ for x:

$$\frac{25e^x}{25} = \frac{521}{25} \quad \text{Isolate } e^x$$

$$e^x = \frac{251}{25}$$

$$\ln(e^x) = \ln\left(\frac{521}{25}\right) \quad \text{Use ln to undo the exponential}$$

$$x = 3.04 \text{ to 2 d.p.}$$

Example 2

Solve the equation $4e^{-3x} + 5 = 12$ for x:

$$4e^{-3x} + 5 - 5 = 12 - 5$$

$$4e^{-3x} = 7$$

$$\frac{4e^{-3x}}{4} = \frac{7}{4}$$

$$e^{-3x} = \frac{7}{4}$$

$$\ln(e^{-3x}) = \ln\left(\frac{7}{4}\right)$$

$$-3x = \ln\left(\frac{7}{4}\right) \quad \text{now divide by } -3$$

$$x = -0.187 \quad \text{to 3 d.p.}$$

Example 3

A capacitor of capacitance C is allowed to discharge through a resistor of resistance R such that the voltage across the terminals of the capacitor ν at time t after the discharge started is given by:

$$\nu = \nu_0 e^{-\frac{1}{RC}t},$$

where ν_0 is the voltage across the terminals of the capacitor at the start of the discharge.

If $C = 500$ nF, $R = 200$ kΩ and $\nu_0 = 12$ V, determine the time it takes for ν to drop to 6 V.

Sub. in the values:

$$6 = 12e^{-\frac{t}{200\times10^3\times500\times10^{-9}}}$$

Simplifying:

$$\frac{6}{12} = e^{\frac{-t}{10^{-1}}} \qquad \Longrightarrow \qquad \frac{1}{2} = e^{-10t}$$

Using ln to invert the exponential:

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{-10t}\right) \qquad \Longrightarrow \qquad t = -\frac{1}{10}\ln\left(\frac{1}{2}\right) = 0.069\ldots$$

7 Modelling with Functions

7.1 Introduction

In this session, we practice synthesising what we have learned about functions in the context of mathematical modelling of real-world behaviour and systems.

We will also practice some problems that may require us to combine different theories that we have learned separately.

7.2 Example 1

A consultant charges an upfront cost of £60, and an additional hourly rate of £15/hour *pro rata* for services.

If your business has a £250 consulting budget, how long can you hire them for?

Solution:

To formulate this problem mathematically, assign variables t and C to the quantities of interest - the amount of time that the consultant works for, and how much this will cost. Formally, let t be the number of hours that the consultant works for, and $C(t)$ the total cost in pounds of engaging them for time t.

Because the cost C increases at a fixed rate with the amount of time t worked, this is fundamentally a **linear** relationship:

$$C = at + b$$

And we need to determine the values of the parameters (constants) a and b.

Because of the initial charge of £60, the initial value of $C(t = 0) = 60$, hence:

$$60 = a(0) + b \qquad \Longrightarrow \qquad b = 60$$

And because the value of C increases by 15 with every increase in t by 1 hour, this means that the gradient $a = 15$ because this represents the rate of change. Hence:

$$C(t) = 15t + 60$$

Finally, we want to solve for t such that $C(t) = 250$:

$$250 = 15t + 60$$

$$\therefore 15t = 190$$

$$\therefore t = 190/15 = 12.666\ldots$$

So we can hire the consultant to work for 12 hours and 40 minutes.

7.3 Example 2

In population dynamics, the logistic function is often used in order to predict the growth of a population P at time t:

$$P(t) = \frac{K}{1 + \left(\frac{K-P_0}{P_0}\right) \mathrm{e}^{-rt}}$$

where K, P_0 and r are positive constants.

i) How does the population size behave as time increases forever?
ii) A lab technician sets up an experiment with 100 bacteria, and sufficient nutrients to sustain a maximum population of 50,000. One week later they record a population of 32,000. If this function is a reasonable model of growth, how much longer will it be for the population to exceed 40,000? Confirm the solution using EXCEL.

Solution:

i) In terms of the variables, this means $t \to \infty$. Since we know $r > 0$, then e^{-rt} exhibits exponential decay with t, and thus:

$$\mathrm{e}^{-rt} \to 0 \quad \text{as} \quad t \to \infty$$

Hence:

$$
\begin{aligned}
\lim_{t\to\infty} P &= \lim_{t\to\infty} \frac{K}{1+\left(\frac{K-P_0}{P_0}\right)\mathrm{e}^{-rt}} \\
&= \frac{K}{1+\left(\frac{K-P_0}{P_0}\right)\cdot 0} \\
&= \frac{K}{1} \\
&= K, \quad \text{which is the maximum sustainable population.}
\end{aligned}
$$

ii) Before the model can be applied, we need to fit the three parameters. From part (i), we have $K = 50,000$. The initial population is given by P when $t = 0$:

$$
\begin{aligned}
P(0) &= \frac{K}{1+\left(\frac{K-P_0}{P_0}\right)\mathrm{e}^{-r\times 0}} = \frac{K}{1+\left(\frac{K-P_0}{P_0}\right)\mathrm{e}^{0}} \\
&= \frac{K}{1+\left(\frac{K-P_0}{P_0}\right)\cdot 1} = \frac{K}{1+\frac{K-P_0}{P_0}} \\
&= \frac{KP_0}{P_0+K-P_0} = \frac{KP_0}{K} \\
&= P_0, \quad \text{and so we know that } P_0 = 100.
\end{aligned}
$$

Use the final information provided to determine parameter r. Let t be the time in days from the start, then we have $P(7) = 32000$. Before substituting this in, transpose to obtain a general formula for r:

$$P = \frac{K}{1+\left(\frac{K-P_0}{P_0}\right)\mathrm{e}^{-rt}}$$

Removing the fraction:

$$P\left(1+\left(\frac{K-P_0}{P_0}\right)\mathrm{e}^{-rt}\right) = K$$

$$\therefore \left(\frac{K-P_0}{P_0}\right)\mathrm{e}^{-rt} = \frac{K}{P} - 1$$

Multiply both sides by P_0 and divide by $K - P_0$ to isolate the exponential:

$$\mathrm{e}^{-rt} = \left(\frac{P_0}{K - P_0}\right)\left(\frac{K}{P} - 1\right)$$

Taking logs:

$$-rt = \ln\left(\frac{P_0}{K - P_0}\left(\frac{K}{P} - 1\right)\right) \tag{5}$$

And so:

$$r = -\frac{1}{t}\ln\left(\frac{P_0}{K - P_0}\left(\frac{K}{P} - 1\right)\right)$$

Then substituting in $K = 50000$, $P_0 = 100$, $P = 32000$ and $t = 7$:

$$\begin{aligned} r &= -\frac{1}{7}\ln\left(\frac{100}{50000 - 100}\left(\frac{50000}{32000} - 1\right)\right) \\ &= -\frac{1}{7}\ln\left(\frac{1}{499}\left(\frac{25}{16} - 1\right)\right) \\ &= -\frac{1}{7}\ln\left(\frac{9}{16 \times 499}\right) \;=\; -\frac{1}{7}\ln\left(\frac{9}{7984}\right) \\ &= 0.9697100\ldots \end{aligned}$$

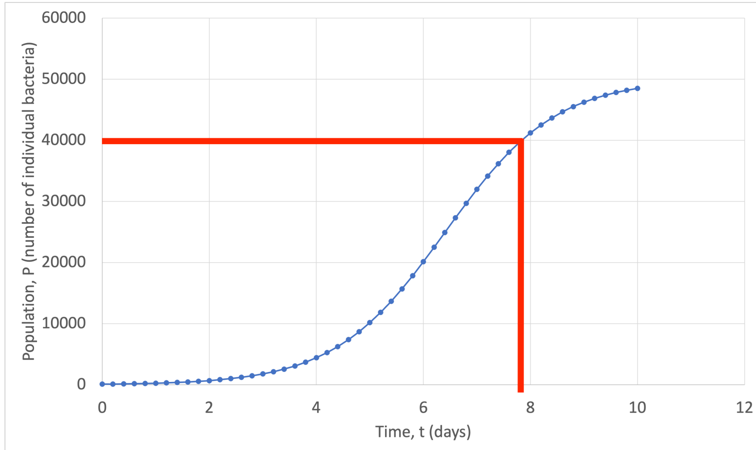
As expected, this is a positive value. We will need to keep r to a high precision.

From equation 5 above, we can then easily obtain a formula for t:

$$t = -\frac{1}{r}\ln\left(\frac{P_0}{K - P_0}\left(\frac{K}{P} - 1\right)\right)$$

To find when the population exceeds 40,000, evaluate this formula with $P = 40000$, $K = 50000$, $P_0 = 100$, and $r = -0.9697100$:

7.6		38041.08654
7.8	t	39715.73095
8		41210.05557
8.2	P0	42528.10119



$$
\begin{aligned}
t &= -\frac{1}{0.96971}\ln\left(\frac{100}{49900}\left(\frac{50000}{40000}-1\right)\right)\\
&= -\frac{1}{0.96971}\ln\left(\frac{1}{499}\left(\frac{5}{4}-1\right)\right)\\
&= -\frac{1}{0.96971}\ln\left(\frac{1}{4\times 499}\right)
\end{aligned}
$$

Hence:

$$
\begin{aligned}
t &= -\frac{1}{0.96971}\ln\left(\frac{1}{1996}\right)\\
&= 7.83626\ldots
\end{aligned}
$$

Thus 7.84 days from the starting point. So in fact, due to the behaviour of exponential growth, the bacterial population will exceed 40,000 before the end of the next (eighth) day.

Plotting the logistic function in EXCEL, we can confirm our result directly from the graph or from the calculated values:
Notice the exponential-like initial growth, before the population saturates at the carrying capacity K.

7.4 Example 3

Indicial equations such as the following are used in the solutions to certain kinds of differential equation problems:

$$2^{x+1} + 2^{3-x} = 17$$

By making a substitution $y = 2^x$, determine all the values of x that satisfy this equation.

Solution:

Before implementing the substitution, we need to see how 2^x appears in the equation using rules of indices:

$$2^{x+1} + 2^{3-x} = 2^x \cdot 2^1 + 2^3 \cdot 2^{-x}$$

Hence:

$$2\left(2^x\right) + \frac{8}{2^x} = 17$$

Substituting in $y = 2^x$:

$$2y + \frac{8}{y} = 17$$

This gives a simpler equation which we solve for y. Multiply all terms by y to remove the fraction:

$$2y^2 + 8 = 17y$$

$$\therefore 2y^2 - 17y + 8 = 0$$

This is then a quadratic equation in y, which can be solved by the quadratic formula or factorised to:

$$(2y - 1)(y - 8) = 0$$

Hence the two solutions for y are $y = \dfrac{1}{2}$ and $y = 8$.

In terms of the original variable x, this means we have $2^x = 8$ and $2^x = \frac{1}{2}$. Hence, for one solution:

$$\ln\left(2^x\right) = \ln\left(8\right)$$

Which using the rules of logarithms, indicates:

$$x \ln\left(2\right) = \ln\left(8\right)$$

and so

$$x = \frac{\ln\left(8\right)}{\ln\left(2\right)} = \frac{\ln\left(2^3\right)}{\ln\left(2\right)} = \frac{3\ln\left(2\right)}{\ln\left(2\right)} = 3$$

Similarly, for the other solution:

$$\ln\left(2^x\right) = \ln\left(\frac{1}{2}\right) = \ln\left(2^{-1}\right) \quad \Longrightarrow \quad x = -1$$

8 Triangle geometry and introduction to Trigonometry

8.1 Learning Outcomes

- Introducing trigonometric functions and radian measure.
- Calculate lengths and angles using right-angled triangle rules.
- Rules for calculating lengths and angles in more general triangles.
- General form of trigonometric functions.

Trigonometric functions are frequently encountered when solving engineering problems, e.g.

- Solutions of differential equations (mechanics, circuit analysis, control theory etc)
- Diffraction
- Surveying
- Navigation
- Optics and acoustics

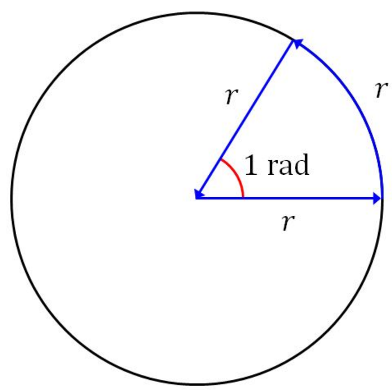
8.2 Radians

You will already be familiar with the degree as a measure of angle, where one full rotation is equivalent to an angle of 360^o.

Radians are an alternative unit for measuring angular rotation.

Radians are often simpler to work with as they lead to helpful approximate formulae, and they are more commonly used throughout mathematics except when discussing angles of triangles or other shapes.

The radian is defined as the angle between two radii that create a circular arc with a length equal to one radius:



Since the circumference of a circle has a length of 2π radii, there must be 2π radians in a full rotation.

Therefore:

Radians - degrees exchange rate:

$$2\pi \text{ rad} = 360^o$$

and also

$$1 \text{ rad} = 57.3^o \text{ to 1 d.p.}$$

Radians	Degrees
0	0
$\frac{\pi}{2}$	90
π	180
$\frac{3\pi}{2}$	270
2π	360
1	57.3

8.3 The basic Trigonometric functions

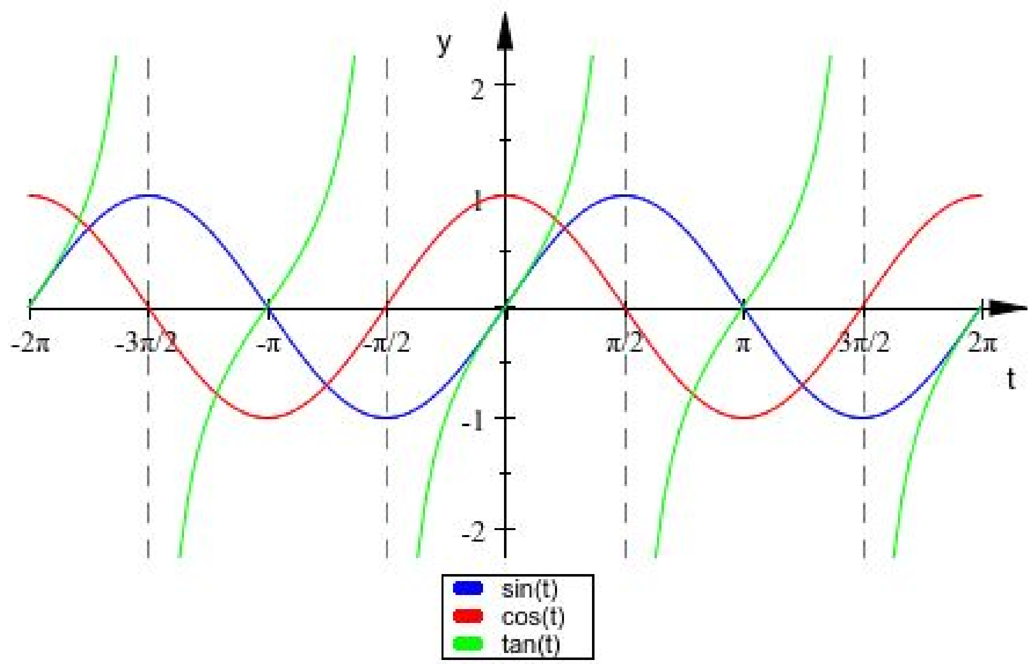
Trigonometric functions take an angle (usually measured in radians) as their input.

They are **periodic**, meaning that they repeat a pattern indefinitely.

There are three main functions of this kind, and we need to be familiar with their appearance:

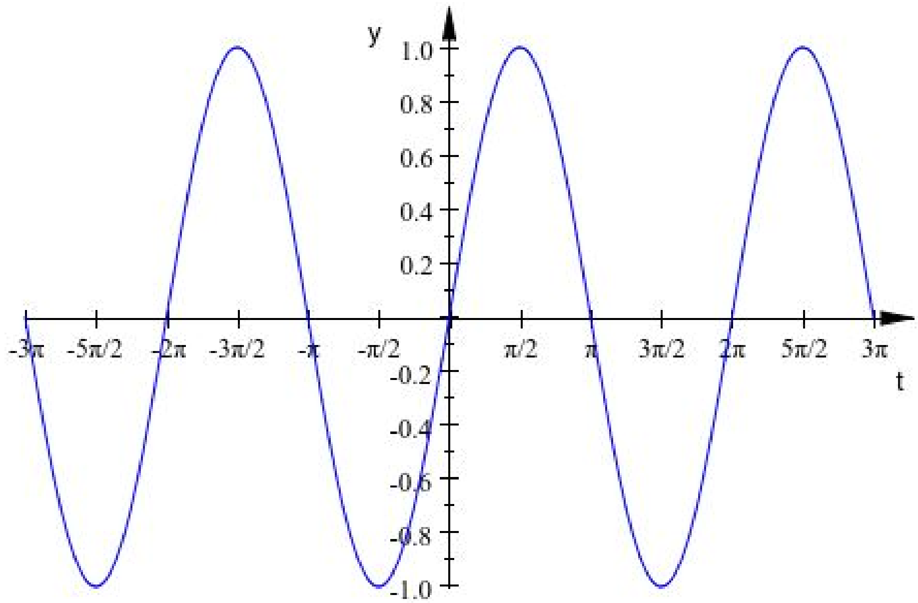
- $y = \sin(t)$
- $y = \cos(t)$
- $y = \tan(t)$

Sketching Sinusoids:



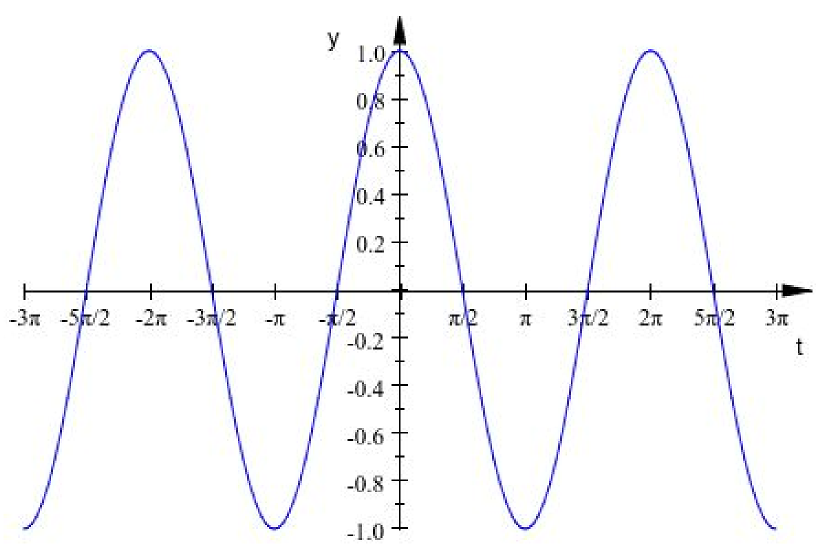
Sketching the sine function:

$\sin(t) = 0$ when $t = 0$. It varies between -1 and 1 and takes 2π rad (or 360^o) to complete one full cycle.



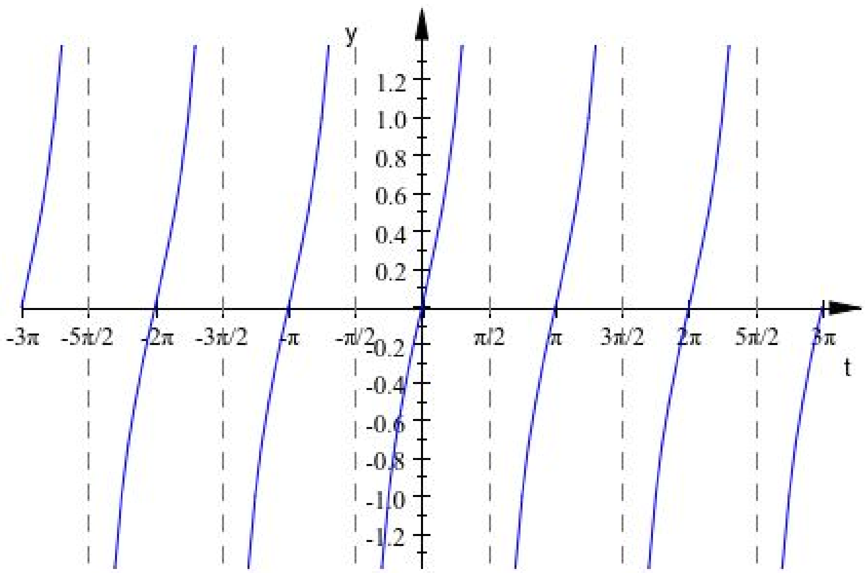
Sketching the cosine function:

$\cos(t) = 1$ when $t = 0$. It varies between -1 and 1 and takes 2π rad (or 360^o) to complete one full cycle. It is identical to sine, but shifted left by $\pi/2$ radians.



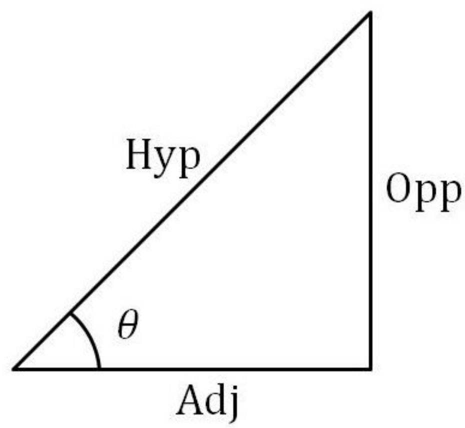
Sketching the tangent function:

$\tan(t) = 0$ when $t = 0$. It possesses asymptotes at $t = \pm\frac{\pi}{2}$ rad (or $\pm 90^o$) and takes π rad (or 180^o) to complete one full cycle.



8.4 Trigonometric Ratios

You may have encountered the trigonometric rules of a right-angled triangle:



Right-angled triangle trigonometric rules:

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

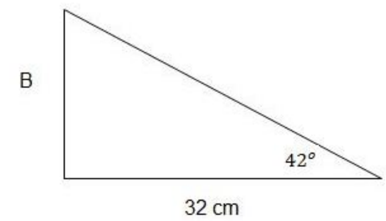
$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$$

We can use these ratios to find unknown angles or side lengths.

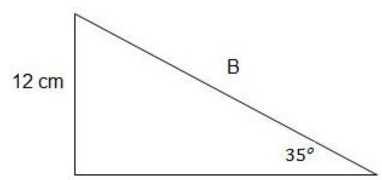
8.4.1 Example 1:

Calculate B in each of the following right-angled triangles:

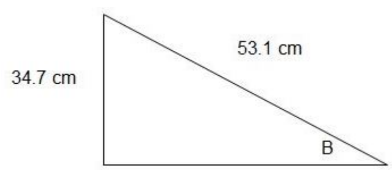
a)



b)

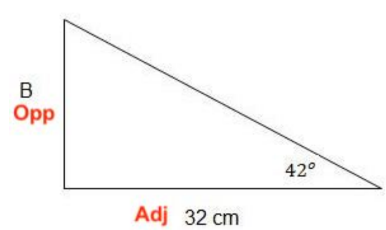


c)



8.4.2 Solutions:

Example 1(a) - Solution



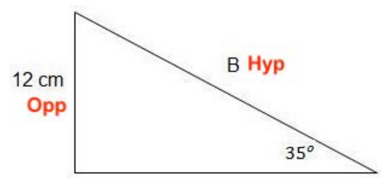
As the two sides of interest are the opposite and the adjacent, we can use the tangent ratio:

$$\tan(42°) = \frac{Opp}{Adj} = \frac{B}{32}$$

Transposing:

$$B = 32\tan(42°) = 28.81\text{cm} \quad (2\text{ d.p.})$$

Example 1(b) - Solution



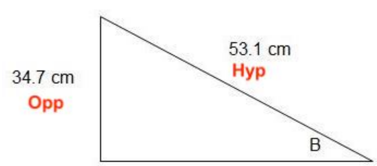
As the two sides of interest are the opposite and the hypotenuse, we can use the sine ratio:

$$\sin(35°) = \frac{Opp}{Hyp} = \frac{12}{B}$$

Transposing:

$$B = \frac{12}{\sin(35°)} = 20.92\text{cm} \quad (2\text{ d.p.})$$

Example 1(c) - Solution



As the two sides of interest are the opposite and the hypotenuse, we can use the sine ratio:

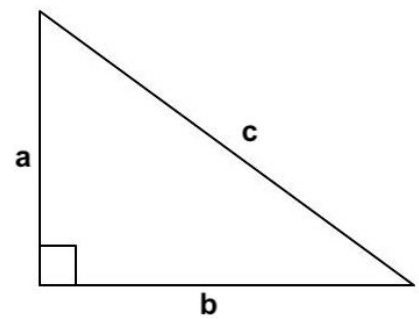
$$\sin(B°) = \frac{Opp}{Hyp} = \frac{34.7}{53.1} = 0.65348\ldots$$

Using the inverse sine (or *arcsin*):

$$B = \sin^{-1}(0.65348) = 40.80° \quad (2\text{ d.p.})$$

8.5 Pythagoras' Theorem

We can also calculate the length of a side, if the other two sides are known, by using the **Pythagorean theorem**:



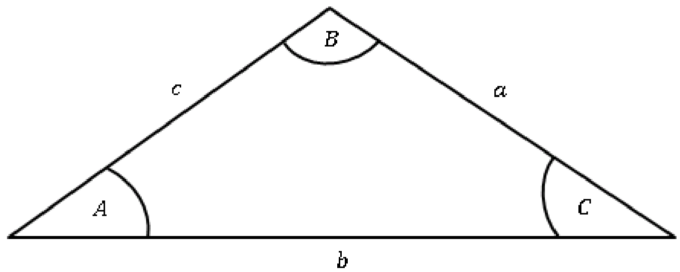
Pythagoras' Theorem:

$$a^2 + b^2 = c^2$$

where c is the length of the hypotenuse.

8.6 Trigonometric Ratios: Non-right-angled Triangles

Often the unknown angle or side of a triangle is from a non-right angled triangle:



There are two rules that can be used depending on what information we have and what requires calculation.

8.6.1 The Sine Rule

Sine rule:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \tag{6}$$

Or, equivalently

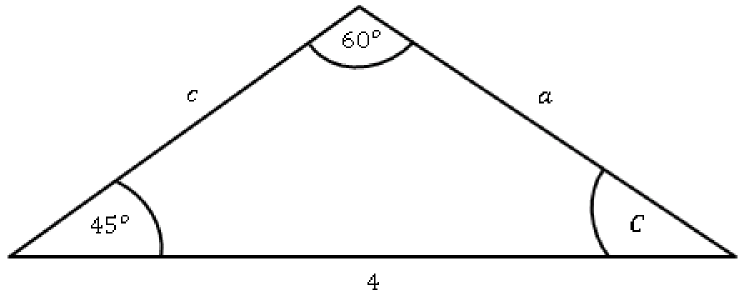
$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \tag{7}$$

These are equivalent, but it is easier to use equation (6) if a side is the unknown and equation (7) if an angle is the unknown.

Note: in order to use the Sine Rule, a complete pair must be known, i.e. a and A (a side and the angle facing it)

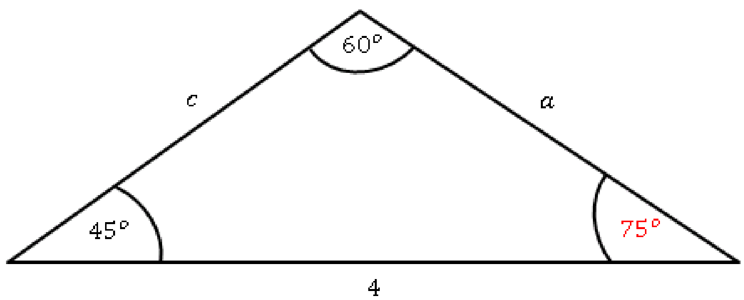
8.6.2 Example 2:

Find a, c and C.



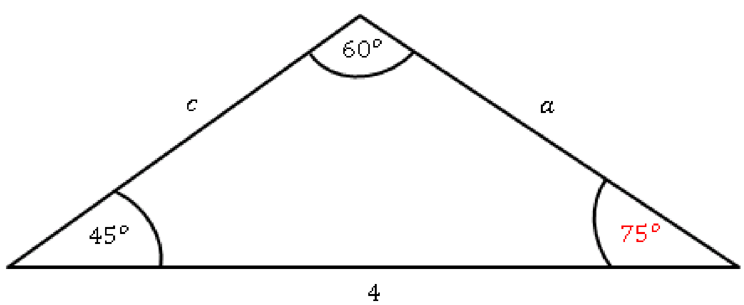
First, calculate C by the fact that angles in a triangle sum to 180^o

$\therefore C = 180 - 60 - 45 = 75^o$



We have a complete pair (4 and 60^o), so we can use the Sine rule.

Let's use it to calculate a, as we also have the angle opposing it.



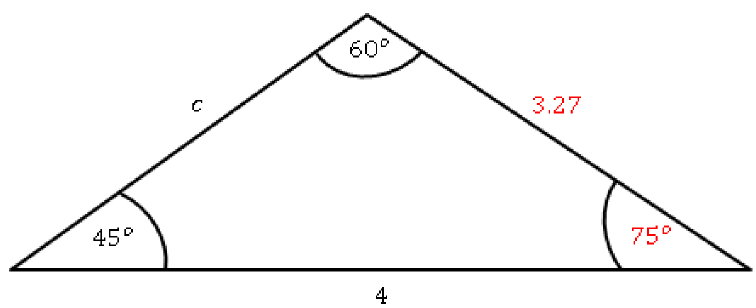
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

$$\frac{a}{\sin(45)} = \frac{4}{\sin(60)}$$

$$a = \sin(45) \times \frac{4}{\sin(60)}$$

$$a = 3.27 \ \text{ to 2 d.p.}$$

Now for c:



$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

$$\frac{4}{\sin(60)} = \frac{c}{\sin(75)}$$

$$c = \sin(75) \times \frac{4}{\sin(60)}$$

$$c = 4.46 \ \text{ to 2 d.p.}$$

8.6.3 The Cosine Rule

Cosine rule:

$$a^2 = b^2 + c^2 - 2bc\,\cos(A) \tag{8}$$

Or

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc} \tag{9}$$

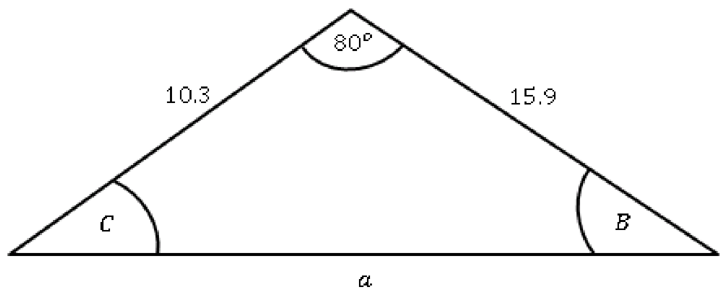
Equation (8) would be used to calculate a side and equation (9) would be used to calculate an angle.

Note: we use the Cosine Rule when we know either:

- **all three sides** a, b, c or
- **two sides** b, c **and the angle** A **inbetween them.**

8.6.4 Example 3

Determine a:



Here, we don't have a complete pair so we cannot use the Sine rule. However, we *do* know two sides and the angle (80°) inbetween, so we can use the Cosine Rule.

$$a^2 = b^2 + c^2 - 2bc\ \cos(A)$$

$$a^2 = 10.3^2 + 15.9^2 - 2 \times 10.3 \times 15.9 \times \cos(80)$$

$$a^2 = 302.0233$$

$$a = 17.38 \text{ to 2 d.p.}$$

8.7 General form

It is common in engineering to encounter quantities that vary in a sinusoidal (sine-like) fashion over time t. If y was such a quantity, we could say that:

General wave equation/sinusoidal function:

$$y = A\sin(\omega t + \phi) + B$$

where A is the **amplitude**, B is the mean value (shift along the y-axis), ω is the **angular frequency** and ϕ is the **phase shift**.

What do these mean?

- **Amplitude** is the maximum displacement of the wave from equilibrium. As $y = \sin(x) + 3$ varies between 2 and 4, the amplitude is exactly 1.
- B is how much the wave is shifted up by. You can find it by locating the vertical mean value. As $y = \sin(x) + 3$ varies between 2 and 4, the vertical shift is 3.
- The **phase shift** is how much the graph is shifted to the left by. $y = \sin(x + 0.1)$ crosses the x-axis at $x = -0.1$, so the phase shift is 0.1 and the entire curve is shifted 0.1 to the left.

The **angular frequency** is the number of times the wave repeats in a distance of 2π along the horizontal axis. If the period (wavelength) is T the angular frequency is given by:

$$\omega = \frac{2\pi}{T}$$

The frequency is the number of complete wavelengths in a unit of 1 along the horizontal axis:

$$f = \frac{1}{T}$$

If the horizontal axis is time (seconds), then the frequency f units are Hertz (cycles/second).

Combining these formulae:

Frequency and period formulae:

$$T = \frac{2\pi}{\omega} \qquad T = \frac{1}{f} \qquad \omega = 2\pi f$$

The phase shift ϕ moves the graph horizontally. In particular, it moves it right by an amount $-\phi/\omega$

Of course, since the cosine function is also sinusoidal, we can write a general wave equation in the form:

$$y = A\cos(\omega t + \phi) + B,$$

where A, B, ω and ϕ all have the same meaning.

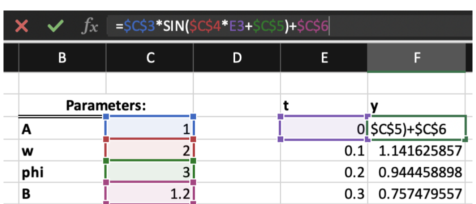
9 Trigonometric functions and equations

9.1 Learning Outcomes

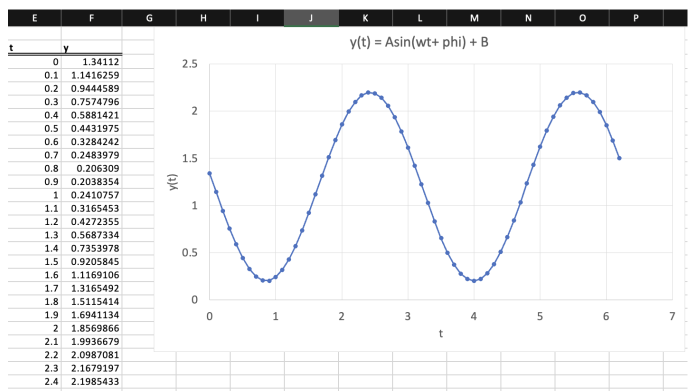
- Plot general trigonometric functions using EXCEL.
- Identify the formula of a general trigonometric function from a plot.
- Solve equations involving trigonometric functions.

9.2 Plotting with EXCEL

We can use EXCEL to easily plot a general trig. function with any values of the parameters A, ω, ϕ, B:



(a) Formula



(b) Plot

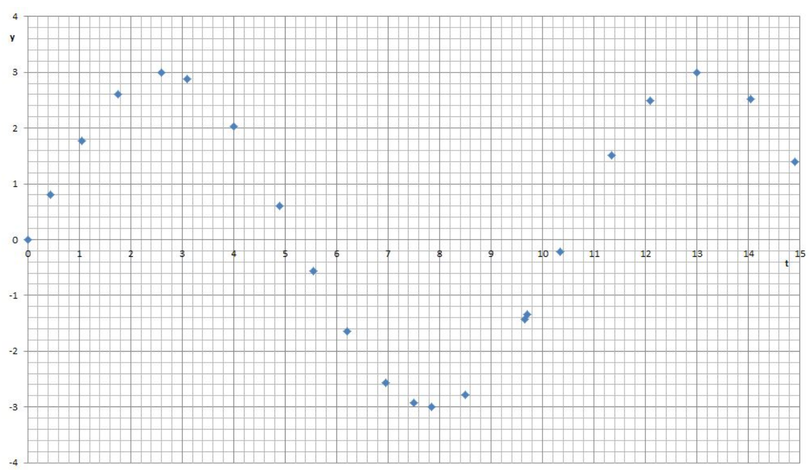
9.3 Determining the equation of a graph

Engineers may encounter sampled data from a periodic signal such as a current or audio signal, and wish to determine the sinusoidal function that matches it. In practice, this means determining the best values of the parameters A, ω, ϕ, B in the general form $y(t) = A\sin(\omega t + \phi) + B$. To do this:

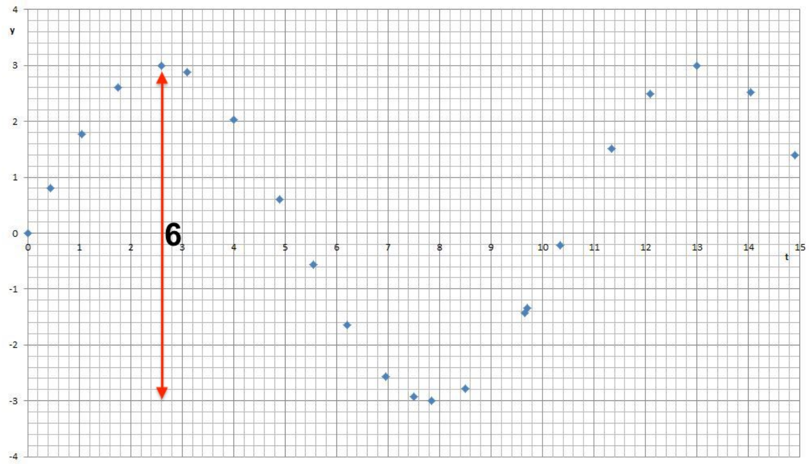
1. Find the maximum and minimum values of the wave. The amplitude A is half the distance between these, and the vertical shift B is their average.
2. Measure the period T, and determine angular frequency from $\omega = 2\pi/T$.
3. Substitute in the values of a point on the plot and solve for ϕ.

Example:

Determine the equation of the curve:



Note: deal with A first, then ω, and finally ϕ .

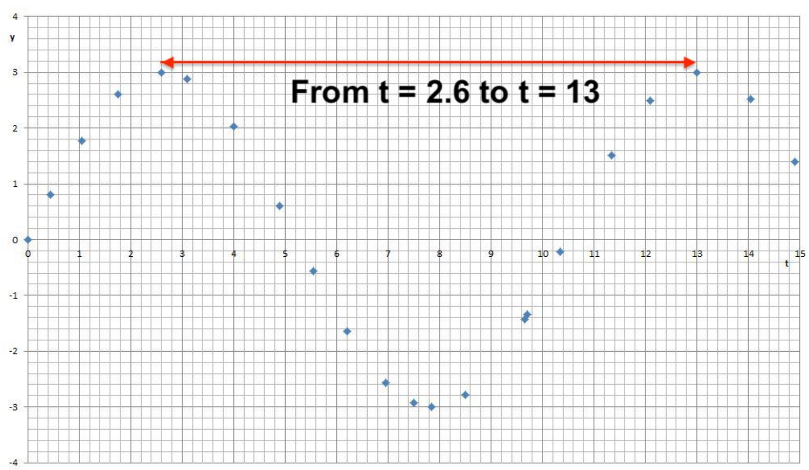


The total vertical displacement between two peaks is:

$$3 - (-3) = 6$$

so the amplitude is half this: $A = 3$

As the wave oscillates between -3 and 3, the mean height is zero and so the vertical shift is $B = 0$.

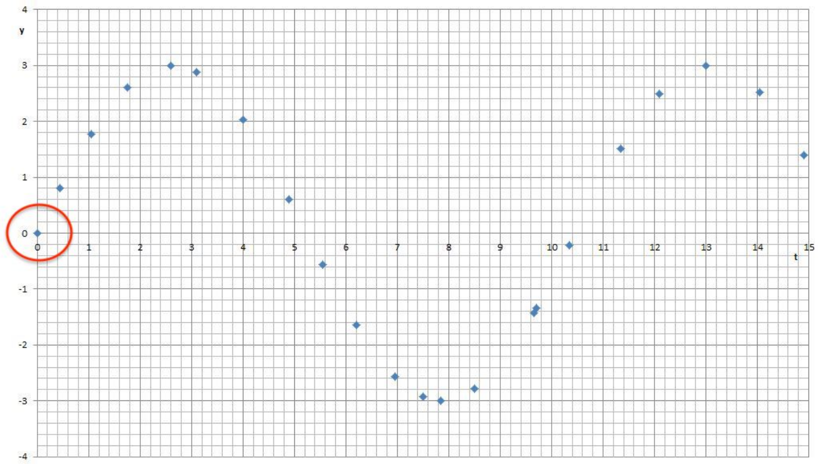


The horizontal distance between two neighbouring peaks is:

$$13 - 2.6 = 10.4$$

So the period is $T = 10.4$, and the angular frequency is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{10.4} = 0.60 \ \ (2\,\text{d.p.})$$



The graph goes through the point $(0, 0)$, so substituting this in to the equation so far to find ϕ:

$$y = 3\sin(0.60t + \phi)$$

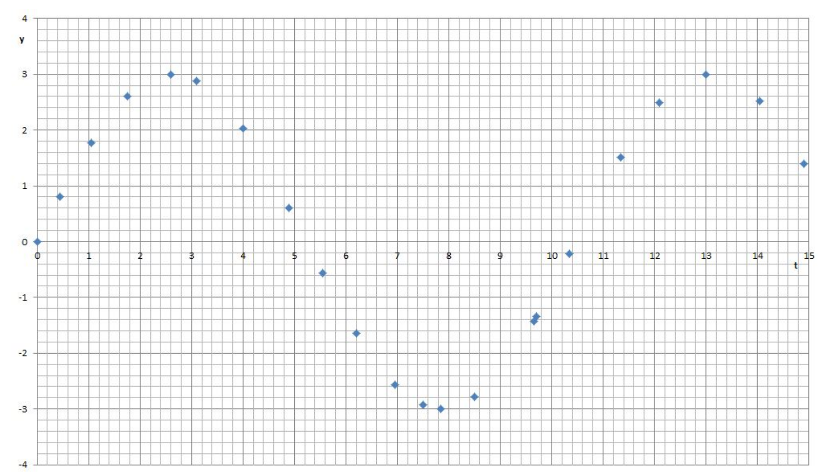
we obtain:

$$0 = 3\sin(0.60 \times 0 + \phi)$$

$$0 = \sin(\phi)$$

$$\phi = \sin^{-1}(0)$$

$$\phi = 0$$



So the equation of this sinusoidal wave is:

$$y = 3\sin(0.60t)$$

9.4 Solving Trigonometric Equations of the form $A\sin(\omega x + \phi) + B = C$

We have seen how to solve equations involving many kinds of functions (quadratics, logarithms, etc.). How do we solve equations involving trigonometric functions?

Due to the periodicity of sine and cosine, an equation of the type $\sin(x) = c$ will either have no solutions (as sine only takes a certain range of values) or infinitely-many solutions, unless we specify a restricted range of x that we are interested in.

For a general trigonometric equation $A\sin(\omega x + \phi) + B = C$, we follow a procedure to locate all solutions for x within a specified range:

1. Define a new variable $u = \omega x + \phi$ to simplify the the trig. function to $\sin(u) = c$, where $c = (C - B)/A$.
2. Calculate the new range in terms of u, by substituting the limits of x into this formula.
3. Determine the set of solutions for u:
   - Use the inverse trigonometric function on your calculator to obtain the **principal value**:

     $$u_0 = \sin^{-1}(c)$$

     This is the first solution, and the value closest to the y-axis.
   - For sine and cosine, use the symmetry of the graph to locate the **other** solution that occurs within the first cycle. This often takes the form $u_1 = \pi - u_0$ for sine, and $u_1 = -u_0$ for cosine.
   - To find all of the other solutions for u, then:
     - For sine and cosine, add and subtract integer multiples of 2π to **both** u_0 and u_1 until we are outside of the stated range.
     - For tangent, add and subtract multiplies of π to u_0.
     - Use high precision when calculating each solution, as errors may compound as we use u_0 to determine u_1 and then use that determine subsequent solutions.

4. Convert solutions for u back to the corresponding solutions for x using:

$$x = \frac{u - \phi}{\omega}$$

5. Verify the final solutions by substituting back into $A\sin(\omega x + \phi) + B$ and evaluating.

Example:

Solve

$$\sin(3x + 0.2) = 0.5$$

for $-\pi \leq x \leq \pi$.

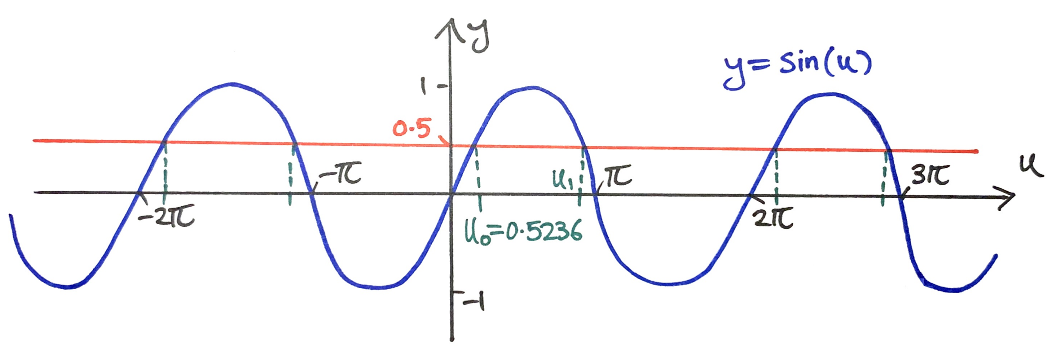
Solution:

Let $u = 3x + 0.2$, then the range is $-3\pi + 0.2 \leq u \leq 3\pi + 0.2$, or:

$$-9.2248 \leq u \leq 9.6248$$

Now the problem has been converted to:

"Solve $\sin(u) = 0.5$ for $-9.2248 \leq u \leq 9.6248$."



Obtain the principal value:

$$u_0 = \sin^{-1}(0.5) = \frac{\pi}{6} = 0.5236$$

From the symmetry of the graph, the other solution in the first period is:

$$u_1 = \pi - 0.5236 = 2.6180$$

Adding and subtracting multiples of 2π, we find six solutions for u in the acceptable range. Then convert these back to solutions for x using:

$$x = \frac{u - 0.2}{3}$$

u	In Range?	$x = \dfrac{u-0.2}{3}$
$u_0 = 0.5236$	Yes	$(0.5236 - 0.2)/3 = 0.1079$
$u_0 + 2\pi = 0.5236 + 2\pi = 6.8058$	Yes	2.2023
$u_0 + 4\pi = 0.5236 + 4\pi = 13.090$	No	-
$u_0 - 2\pi = 0.5236 - 2\pi = -5.7596$	Yes	-1.9865
$u_0 - 2\pi = 0.5236 - 4\pi = -12.043$	No	-
$u_1 = 2.6180$	Yes	0.8060
$u_1 + 2\pi = 2.6180 + 2\pi = 8.9012$	Yes	2.9004
$u_1 - 2\pi = 2.6180 - 2\pi = -3.6652$	Yes	-1.2884
$u_1 - 4\pi = 2.6180 - 4\pi = -9.9484$	No	-

So we have six valid solutions: $x = -1.288, \;\; -1.987, \;\; 0.108, \;\; 0.806, \;\; 2.202, \;\; 2.900$

9.5 Sketching Sinusoidal functions

Sketch the sinusoids defined by

$$(1) \qquad y = 5\sin\left(0.5t + \frac{\pi}{2}\right)$$

and

$$(2) \qquad y = -0.2\cos(2t - \pi)$$

Note: deal with A first, then ω, and finally ϕ.

9.5.1 Examples: Sketching Sinusoids - Solution 1

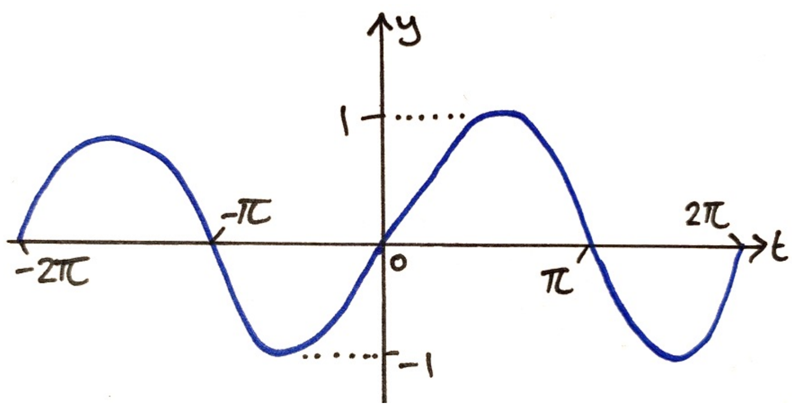


Figure 1: $y = \sin(t)$

Start by drawing the regular sine wave: $y = \sin(t)$

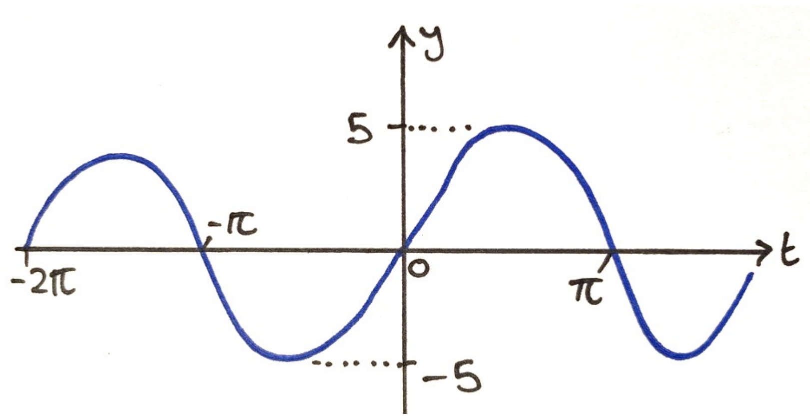


Figure 2: $y = 5\sin(t)$

The amplitude is $A = 5$, so rescale the y-axis and draw $y = 5\sin(t)$

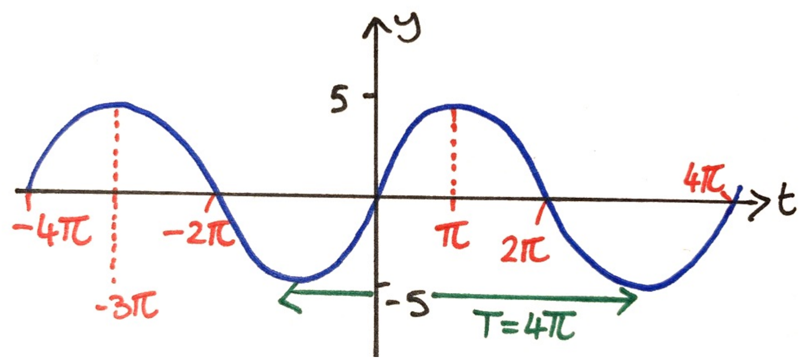


Figure 3: $y = \sin(0.5t)$

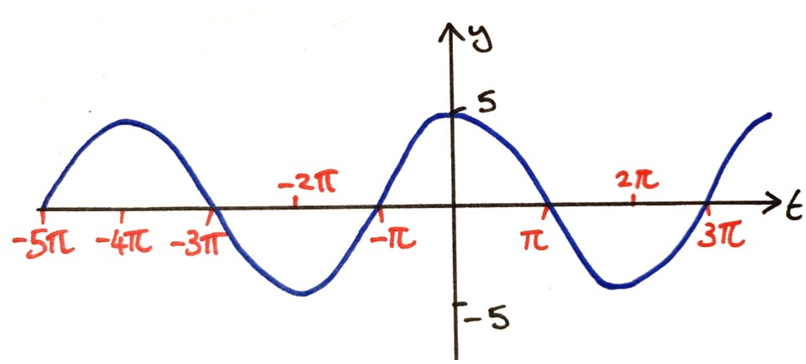


Figure 4: $y = 5\sin\left(0.5t + \frac{\pi}{2}\right)$

The angular frequency is $\omega = 0.5$, so compress the x-axis by a factor of 0.5 (or stretch by 2), so that the new period is:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{0.5} = 4\pi$$

Finally, shift right by

$$-\frac{\phi}{\omega} = -\frac{\pi/2}{0.5} = -\pi$$

or left by π radians.

9.5.2 Examples: Sketching Sinusoids - Solution 2

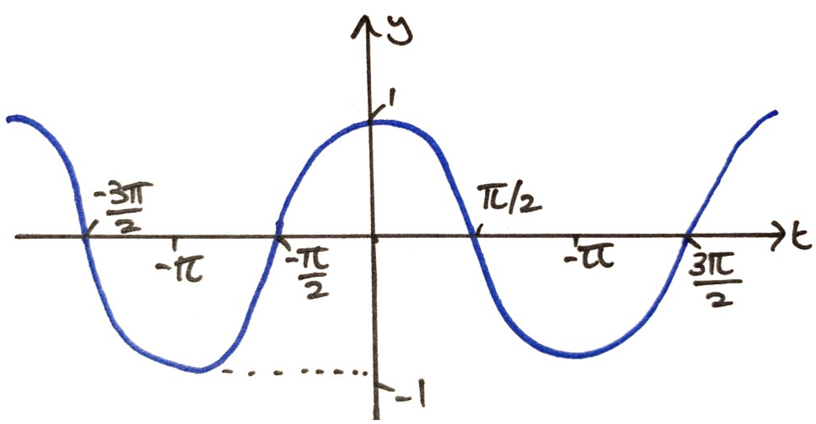


Figure 5: $y = \cos(t)$

Start by drawing the regular cosine wave: $y = \cos(t)$

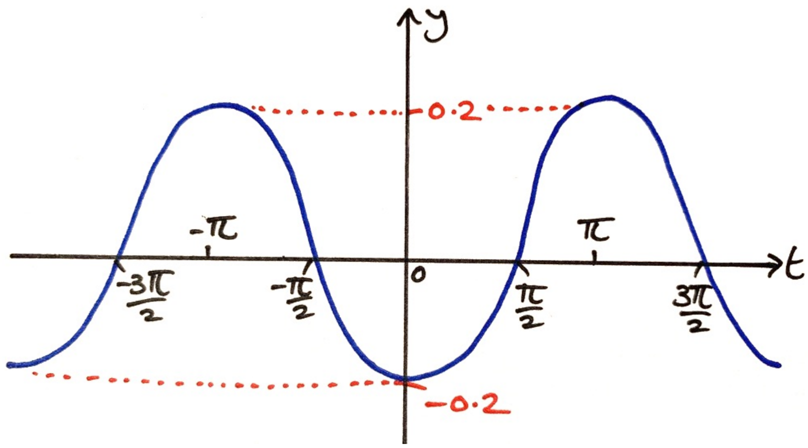


Figure 6: $y = -0.2\cos(t)$

The amplitude is $A = -0.2$, so rescale the y-axis and draw $y = -0.2\cos(t)$. The negative amplitude means that the curve is flipped upside-down (reflected in the x-axis).

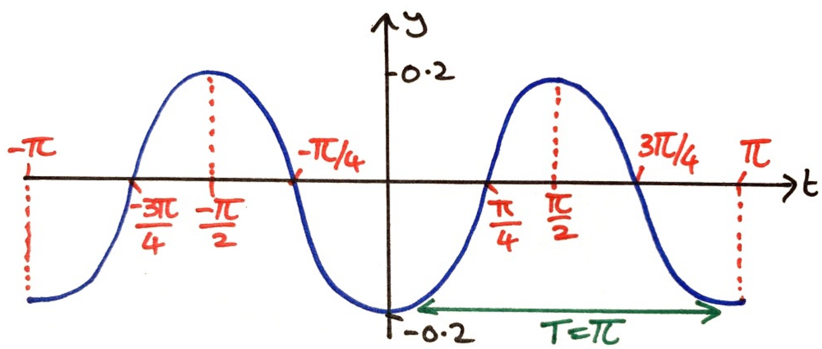


Figure 7: $y = -0.2\cos(2t)$

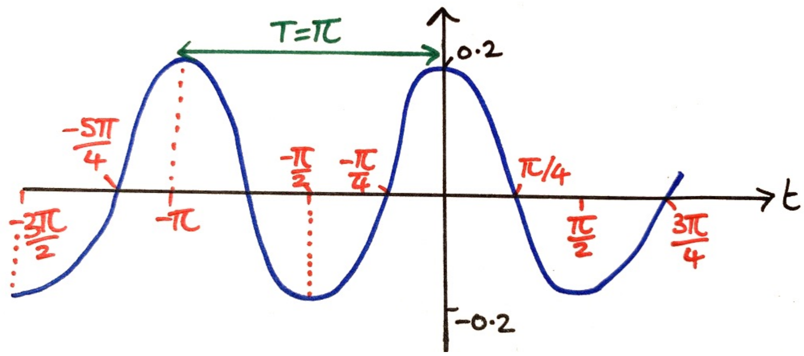


Figure 8: $y = -0.2\cos(2t - \pi)$

The angular frequency is $\omega = 2$, so compress the x-axis by a factor of 2, so that the new period is:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$$

Finally, shift right by

$$-\frac{\phi}{\omega} = -\frac{-\pi}{2} = \frac{\pi}{2}$$

radians.

9.6 Extra resources

The module Blackboard site has an additional video walkthrough for an example of each of these two kinds of questions.

When you get stuck on the relevant tutorial questions, watch these after revisiting the lecture material.

10 Introduction to differentiation

10.1 Learning Outcomes

- State what is meant by the gradient of a curve at a point.

- Differentiate functions to obtain their derivatives.

10.2 Motivation

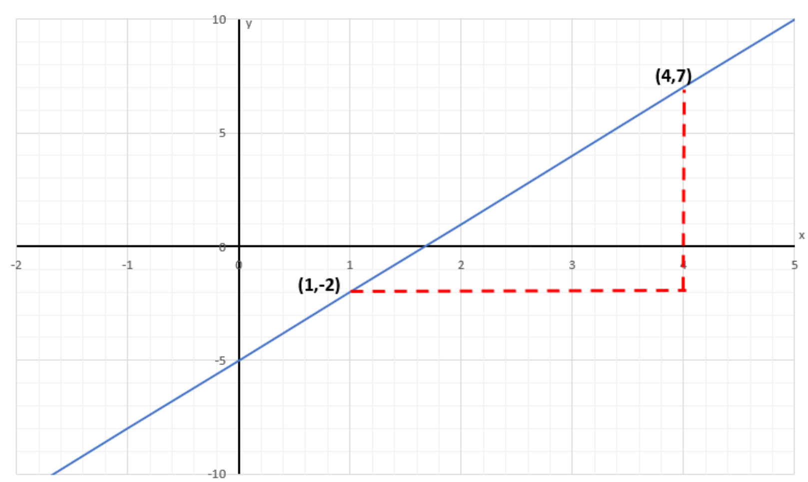
Differentiation allows us to calculate the **gradient** of a curve or, more specifically, a **rate of change** of one variable with respect to another variable.

Examples of rates of change:

- velocity (rate of change of displacement with respect to time)
- acceleration (rate of change of velocity w.r.t. time)
- power (rate of low of energy w.r.t. time)

10.3 Gradients

We have already seen how to calculate the gradient (steepness of the slope) of linear lines:



Here the gradient is:

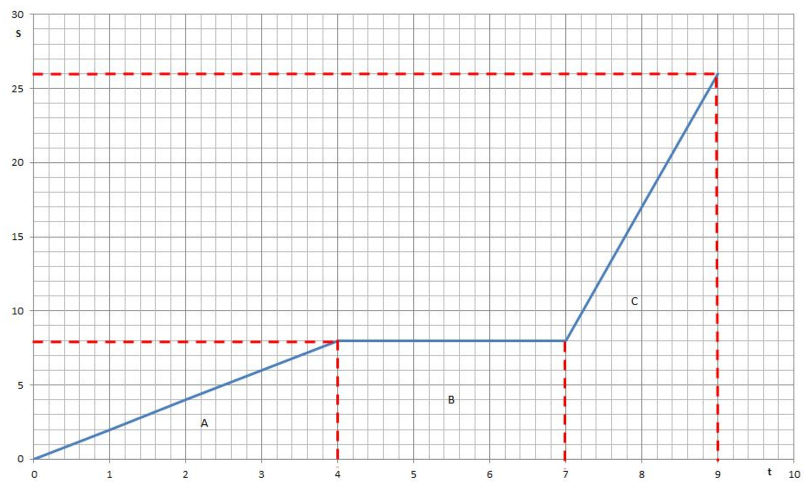
$$m = \frac{\Delta y}{\Delta x} = \frac{7-(-2)}{4-1} = \frac{9}{3} = 3$$

where Δ is the change, or difference in, y or x.

Note that with linear lines the gradient is constant throughout, i.e. no dependence on x.

10.3.1 Physical Example

Consider this piece-wise graph which illustrates the displacement S (m) of an object over time t (s).



From the graph we can see that the object is moving in regions A and C and is stationary in region B.

Calculating the gradient in region A:

$$m = \frac{\Delta S}{\Delta t} = \frac{8}{4} = 2$$

Consider the units:

$$\frac{\Delta S}{\Delta t} = \frac{\text{m}}{\text{s}}$$

which is m/s. So the gradient of a displacement-time curve gives a velocity (rate of change of displacement w.r.t. time). As the graph is a straight line, in A the object has a constant velocity of 2 m/s.

Calculating the gradient in region B:

$$m = \frac{\Delta S}{\Delta t} = \frac{0\,\text{m}}{3\,\text{s}} = 0\,\text{m/s}$$

This indicates that the object is travelling at 0 m/s, i.e. it is stationary.

Looking back at the graph: at 4 seconds the object is at the 8 metre mark and at 7 seconds the object is still at the 8 metre mark, so cannot be moving.

The physical example has provided us with two important results:

1) Consider the equation of the line in region A: it has form $y = mx + c$ and specifically $y = 2x$ (think of x and y rather than t and S). The gradient here was simply 2. If the equation of the line was $y = 5x$, then the gradient would be 5, etc.

Therefore, if:

$$y = ax, \text{ then gradient: } \; m = \frac{\Delta y}{\Delta x} = \frac{\text{difference in } y}{\text{difference in } x} = \frac{dy}{dx} = a$$

Similarly for region B the equation of the line is of the form $y = mx + c$ and specifically $y = 8$.

2) The gradient here was simply 0. If the equation of the line was $y = 9$, then the gradient would also be 0, as it is a straight horizontal line and has no steepness.

Therefore, if:

$$y = a, \text{ then gradient: } \; m = \frac{\Delta y}{\Delta x} = \frac{\text{difference in } y}{\text{difference in } x} = \frac{dy}{dx} = 0$$

10.3.2 Gradients of linear or constant functions

y	$\dfrac{dy}{dx}$
a (any constant)	0
ax	a

The functions in the right-hand column of this table are known as the derivatives of the functions in the left-hand column.

We obtain the derivative of a function by differentiation.

10.4 Notation

$\dfrac{dy}{dx}$ represents the gradient/derivative of a curve $y = f(x)$.

y' and $f'(x)$ are common alternatives to the symbol $\dfrac{dy}{dx}$. They all mean gradient/derivative/rate of change, where $f(x)$ is another way of writing that y is a function f of x.

$\dot{y}$ is also another way to represent the derivative, but in the case specifically w.r.t. **time**, i.e. $\dot{y} = \dfrac{dy}{dt}$.

If, instead of $y = f(x)$, we have different function and variable names such as $r = f(t)$ then the derivative of r is written as $\dfrac{dr}{dt}$.

10.5 Exercise

Determine the gradients of the following lines:

1) $y = 6x$ $\qquad \dfrac{dy}{dx} = 6$

2) $x = 9.7t$ $\qquad \dfrac{dx}{dt} = 9.7$

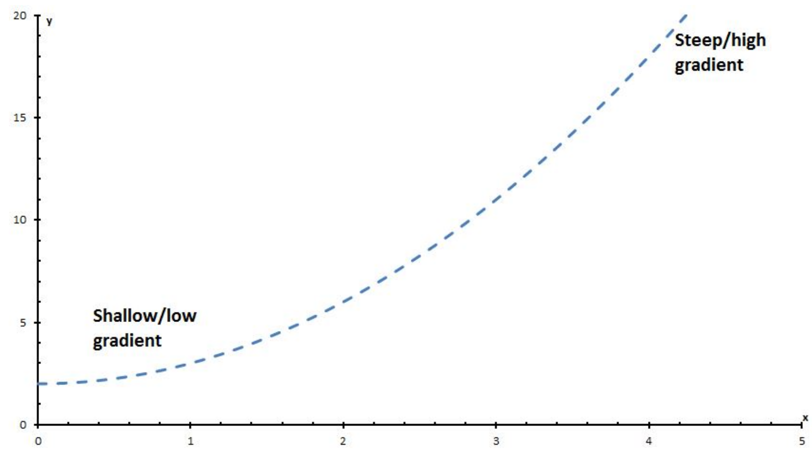
3) $r = \dfrac{3}{5}\theta$ $\qquad \dfrac{dr}{d\theta} = \dfrac{3}{5}$

4) $y = -12$ $\qquad y' = 0$

5) $P = \dfrac{7}{8}$ $\qquad P' = 0$

10.6 Differentiation

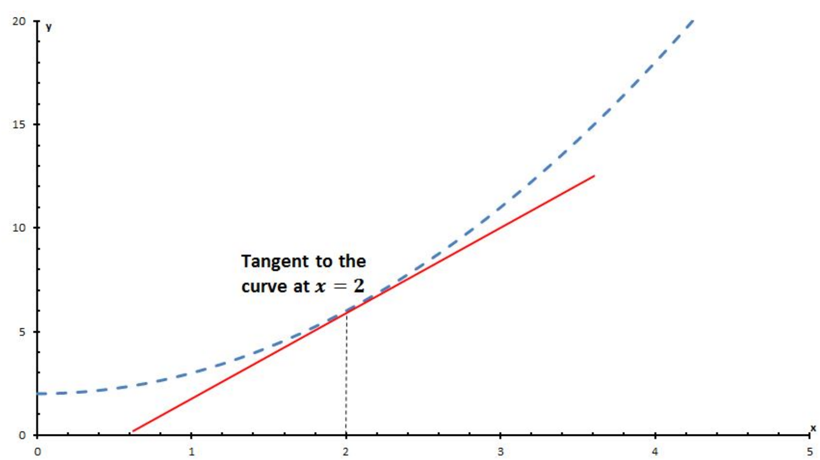
Calculating the gradient of a **curve** is harder as the gradient varies along the curve, i.e. it depends on x.



First, we would need a rigorous way of defining the gradient of a curve at a particular point:

> **The gradient of a curve at a point is equal to the gradient of the tangent line at that point.**

A tangent line is a straight line that only just touches the curve at exactly that particular point. So we could draw such a line at the point we were interested in. . .



Then we would set up a triangle (as in linear cases) to calculate the gradient of the tangent.

But how could we consistently draw perfect tangents to the curve at all infinitely-many points?

10.6.1 Differentiation: Standard rules

See Formulae booklet!

For standard functions, formulae for the derivatives have been proven using a general version of this process. We can use these rules (so we will never need to draw tangents!). For constant a, n:

y	$\frac{dy}{dx}$
a (any constant)	0
ax	a
ax^n	$n \times ax^{n-1}$
ae^{nx}	$n \times ae^{nx}$
$a \ln nx$	$\frac{a}{x}$
$a \sin nx$	$n \times a \cos nx$
$a \cos nx$	$-n \times a \sin nx$
$a \sinh nx$	$n \times a \cosh nx$
$a \cosh nx$	$n \times a \sinh nx$

10.6.2 Example 1

Calculate an expression for the gradient of $y = 7x^3$.

Looking in the left-hand-side of the table, we can see that this is in the form ax^n, where $a = 7$ and $n = 3$. The corresponding right-hand column instructs us on how to differentiate it:

If $y = ax^n$, then $\frac{dy}{dx} = n \times ax^{n-1}$

Therefore, in the case of $y = 7x^3$

$$\frac{dy}{dx} = 3 \times 7x^{3-1}$$
$$= 21x^2$$

10.6.3 Example 2

Calculate an expression for the gradient of $y = 4x^2$.

Again, this is in the form ax^n, where $a = 4$ and $n = 2$. Therefore

$$\frac{dy}{dx} = 2 \times 4x^{2-1}$$
$$= 8x$$

Note that this is an expression for the gradient, which is dependent upon x. If we wanted to calculate the gradient at a *specific point*, say $x = 5$, then we simply substitute this value into the gradient expression:

$$\left.\frac{dy}{dx}\right|_{x=5} = 8 \times 5 = 40$$

10.6.4 Exercise

Determine expressions for the gradients of the following curves:

1) $y = 3x^4$

2) $y = -7x^9$

3) $x = 9t^{-2}$

4) $y = \frac{7}{2}x^3$, and calculate the gradient at the point $x = 4$.

5) $y = \frac{2}{\phi^5}$, and calculate the gradient at the point $\phi = -2.4$.

6) $y = 4\sin(5x)$. This has the form $a\sin(nx)$. What are a and n?

Solutions:

1) $\dfrac{dy}{dx} = \dfrac{d}{dx}\big(3x^4\big) = 3 \times 4x^{4-1} = 12x^3$

2) $\dfrac{dy}{dx} = \dfrac{d}{dx}\big(-7x^9\big) = -7 \times 9x^{9-1} = -63x^8$

3) $\dfrac{dx}{dt} = \dfrac{d}{dt}\big(9t^{-2}\big) = 9 \times (-2)t^{-2-1} = -18t^{-3}$

4) $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\tfrac{7}{2}x^3\right) = \tfrac{7}{2} \times 3x^{3-1} = \dfrac{21}{2}x^2$

Hence, at $x = 4$, the gradient is:

$$\left.\frac{dy}{dx}\right|_{x=4} = \frac{21}{2}(4)^2 = 168$$

5) First, the function must be rewritten in the form: $y = 2\phi^{-5}$

Then

$$\frac{dy}{d\phi} = \frac{d}{d\phi}\big(2\phi^{-5}\big) = 2 \times (-5)\phi^{-5-1} = -10\phi^{-6}$$

At $\phi = -2.4$, we have:

$$\left.\frac{dy}{d\phi}\right|_{\phi=-2.4} = -10(-2.4)^{-6} = -0.05$$

6) $a = 4$ and $n = 5$, then the derivative is:

$$\frac{dy}{dx} = \frac{d}{dx}\big(4\sin(5x)\big) = 4 \times 5\cos(5x) = 20\cos(5x)$$

10.6.5 Example 3

Calculate an expression for the gradient of:

$$y = 3x^2 + 7x - 3 + 2e^{5x}$$

When we have a sum of multiple terms, in order to differentiate this we simply differentiate each term and sum their gradients in the same way (this property of differentiation is called *linearity*).

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\big(3x^2\big) + \frac{d}{dx}\big(7x\big) - \frac{d}{dx}\big(3\big) + \frac{d}{dx}\big(2e^{5x}\big) \\
&= \big(2 \times 3x^{2-1}\big) + \big(7\big) - \big(0\big) + \big(5 \times 2e^{5x}\big) \\
&= 6x + 7 + 10e^{5x}
\end{aligned}$$

11 Differentiation - the chain rule

11.1 Learning Outcomes

- Recognise "functions of functions".

- Apply the chain rule to differentiate these.

11.2 Motivation

It is not possible to differentiate every function using the rules covered earlier. For example, if we wished to differentiate:

$$y = 7x^3 \sin(5x),$$

there is no formula in the table for the precise form $ax^n \sin(mx)$.

Similarly we can't (yet) differentiate:

$$y = \frac{8e^{-6x} + 3x}{\cos(2x)} \text{ and } y = 9(2x-4)^3,$$

We will be learning additional rules to cover cases like these.

11.3 The Chain Rule

To differentiate a function of a function $y = f(g(x))$ (i.e. one function inside another function), we must use the chain rule.

The Chain Rule:

If $y = f(u)$ and $u = g(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{dy}{du} \times \frac{du}{\mathrm{d}x}$$

f is the "outer" function, and g is the "inner" function, which we designate as a new variable u.

We would use the chain rule for functions that look like:

$y = 5(3x - 8)^4$ where the inner function $3x - 8$ lies within the outer function $5(\mathrm{X})^4$

$y = -2\cos(4x + 7)$ where the function $4x + 7$ lies within the outer function $-2\cos(\mathrm{X})$

$y = 7e^{5x^2}$ where the function $5x^2$ lies within the function $7e^{\mathrm{X}}$

11.3.1 Example 1

To determine the derivative of

$$y = 3(5x-7)^4$$

we must first recognise that we have one function $5x-7$ inside another function $3\mathrm{X}^4$.

We make a substitution u, usually for the "thing" inside the brackets. Thus, if we let $u = 5x-7$, we can re-write the original equation (the outer function) as:

$$y = 3u^4$$

By introducing u, we have separated the original ""function of a function" into two "simple" functions: $y = 3u^4$ and $u = 5x-7$

The chain rule formula requires y to be differentiated w.r.t. u and u to be differentiated w.r.t. x:

$$u = 5x-7 \quad \Longrightarrow \quad \frac{du}{\mathrm{d}x} = 5$$

$$y = 3u^4 \quad \Longrightarrow \quad \frac{\mathrm{d}y}{du} = 12u^3$$

Substituting these into the rule:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{dy}{du} \times \frac{du}{\mathrm{d}x} \\
&= (12u^3) \times (5) \\
&= 60u^3
\end{aligned}$$

However, this is not the final answer, as we must now substitute $u = 5x-7$ back into the answer:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 60u^3 = 60(5x-7)^3$$

Always state the final answer in terms of the original variables (in this case x) and not u, which we introduced during the process of solving the problem.

11.3.2 Example 2

Determine the derivative of

$$y = -5\cos(2x + 3)$$

First, substitute the inner function: $u = 2x + 3$.

Second, re-write the original equation: $y = -5\cos(u)$.

Now calculate the derivatives of $u = g(x)$ and $y = f(u)$:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}u} = 5\sin(u)$$

Now, substitute both results into the chain rule formula:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= (5\sin u) \times (2) \\ &= 10\sin(u) \end{aligned}$$

Finally substitute $u = 2x + 3$ back into the answer:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 10\sin(2x + 3)$$

11.3.3 Example 3

Determine the derivative of

$$y = 3e^{(5x^2-3x+1)}$$

First, substitute the inner function $u = 5x^2 - 3x + 1$

Then re-write the original equation (the outer function): $y = 3e^u$

Now calculate the derivatives of u $(= g(x))$ and y $(= f(u))$:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 10x - 3 \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}u} = 3e^u$$

Substitute both results into the chain rule formula:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= (3e^u) \times (10x - 3) \\
&= 3(10x - 3)e^u
\end{aligned}$$

Finally substitute $u = 5x^2 - 3x + 1$ back in to obtain the answer in terms of x only:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3(10x - 3)e^{(5x^2-3x+1)}$$

12 Differentiation - the product and quotient rules

12.1 Learning Outcomes

- Apply the product and quotient rules to differentiate more complicated functions.

12.2 The Product Rule

The function

$$y = 9x^2 e^{7x}$$

is comprised of one function $9x^2$ **multiplied** by another function e^{7x}. This is more complicated than any of our standard functions, but it also isn't a "function of a function" (there is no obvious inner part), so the chain rule cannot help either.

In order to differentiate this, we need to use the **product rule**.

The product rule tells us how to differentiate a function that is the product (multiple) of two functions.

Product Rule:

If $y = u \cdot \nu$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}\nu}{\mathrm{d}x} + \nu\frac{\mathrm{d}u}{\mathrm{d}x}$$

This formula is made up of two functions, u and ν. Note that these are two elements of the overall function y that we want to differentiate. To determine $\mathrm{d}\nu/\mathrm{d}x$ we must differentiate ν w.r.t. x and similarly to determine $\mathrm{d}u/\mathrm{d}x$ we must differentiate u w.r.t. x.

Let us now return to the original example.

In this example:

$u = 9x^2 \quad \therefore \quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = 18x$

$\nu = e^{7x} \quad \therefore \quad \frac{\mathrm{d}\nu}{\mathrm{d}x} = 7e^{7x}$

Substituting these values into the product rule gives:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= u\frac{\mathrm{d}\nu}{\mathrm{d}x} + \nu\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= (9x^2) \times (7e^{7x}) + (e^{7x}) \times (18x) \\
&= 63x^2e^{7x} + 18xe^{7x}
\end{aligned}$$

12.2.1 Example 2

Differentiate:

$$y = -5x^4 \sin(3x)$$

Let $u = -5x^4 \quad \therefore \quad \frac{\mathrm{d}u}{\mathrm{d}x} = -20x^3$

and $\nu = \sin(3x) \quad \therefore \quad \frac{\mathrm{d}\nu}{\mathrm{d}x} = 3\cos(3x)$

Substituting these values into the product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}\nu}{\mathrm{d}x} + \nu\frac{\mathrm{d}u}{\mathrm{d}x} = (-5x^4) \times (3\cos(3x)) + (\sin(3x)) \times (-20x^3)$$

This should be simplified as much as possible:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= (-5x^4) \times (3\cos(3x)) + (\sin(3x)) \times (-20x^3) \\
&= -15x^4\cos(3x) - 20x^3\sin(3x)
\end{aligned}$$

This could be further simplified by factorisation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -5x^3(3x\cos(3x) + 4\sin(3x))$$

12.3 The Quotient Rule

The equation

$$y = \frac{9\cos(3x)}{5x^4}$$

is comprised of one function $9\cos(3x)$ **divided** by another function $5x^4$. Again, none of our existing rules are able to handle this[1] so in order to differentiate this function we need to use the **quotient rule**.

The quotient rule tells us how to differentiate a function that is a fraction (quotient) of two functions.

Quotient Rule:

If $y = \dfrac{u}{\nu}$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\nu\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}\nu}{\mathrm{d}x}}{\nu^2}$$

This is very similar to the product rule method, but we substitute the four terms into a different equation. Note that it is essential that u is the numerator, and ν the denominator.

Let us now return to the original example.

In this example:

$$u = 9\cos(3x) \quad \therefore \quad \frac{\mathrm{d}u}{\mathrm{d}x} = -27\sin(3x)$$

$$\nu = 5x^4 \quad \therefore \quad \frac{\mathrm{d}\nu}{\mathrm{d}x} = 20x^3$$

Substituting these values into the quotient rule gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\nu\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}\nu}{\mathrm{d}x}}{\nu^2}$$

[1] Can you think of a way to re-write this function so that we could use the product rule?

Thus,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(5x^4) \times (-27\sin(3x)) - (9\cos(3x)) \times (20x^3)}{(5x^4)^2} \\
&= \frac{-135x^4\sin(3x) - 180x^3\cos(3x)}{25x^8} \\
&= \frac{-45x^3}{25x^8}\big(3x\sin(3x) + 4\cos(3x)\big) \\
&= \frac{-9}{5x^5}\big(3x\sin(3x) + 4\cos(3x)\big)
\end{aligned}$$

12.3.1 Example 4

Differentiate:

$$y = \frac{9x^3}{2\sin(5x)}$$

Let $u = 9x^3 \quad \therefore \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 27x^2$

and $\nu = 2\sin(5x) \quad \therefore \quad \frac{\mathrm{d}\nu}{\mathrm{d}x} = 10\cos(5x)$

Substituting these values into the quotient rule gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\nu\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}\nu}{\mathrm{d}x}}{\nu^2}$$

$$= \frac{(2\sin(5x)) \times (27x^2) - (9x^3) \times (10\cos(5x))}{(2\sin(5x))^2}$$

$$= \frac{54x^2\sin(5x) - 90x^3\cos(5x)}{4\sin^2(5x)}$$

13 Differentiation - applications

13.1 Learning Outcomes

- Determine higher order derivatives.
- Locate stationary points and classify their nature.
- Use the principles of differential calculus to solve engineering problems.

13.2 Rates of change

We have seen that differentiation allows us to calculate the gradient of a curve at any point, indicating how quickly the variable on the y-axis is changing as a result of change in the variable on the x-axis. Therefore, as mentioned previously, gradients represent **rates of change**.

Physical examples:

x-axis	y-axis	Gradient
Time: t	Displacement: S	Velocity: $\nu = \dfrac{\mathrm{d}S}{\mathrm{d}t}$
	Velocity: ν	Acceleration: $a = \dfrac{\mathrm{d}\nu}{\mathrm{d}t}$
	Energy: E	Power: $P = \dfrac{\mathrm{d}E}{\mathrm{d}t}$
	Charge: q	Current: $I = \dfrac{\mathrm{d}q}{\mathrm{d}t}$
	Momentum: p	Force: $F = \dfrac{\mathrm{d}p}{\mathrm{d}t}$
	Angular displacement: θ	Angular velocity: $\omega = \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$

13.2.1 Example 1

A projectile is thrown directly upwards such that its vertical displacement S m changes over time t s in accordance with the formula:

$$S = 2.4t - 4.9t^2$$

Determine a formula for its velocity and, hence, the velocity of the projectile after 4 seconds.

Solution:

First, we differentiate to obtain a formula for the velocity ν at any time t (*do* ***not*** *substitute in* $t = 4$ *until this is done!*)

$$\begin{aligned}
\nu(t) &= \frac{\mathrm{d}S}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(2.4t - 4.9t^2\right) \\
&= 2.4 - 9.8t
\end{aligned}$$

Then evaluate this at $t = 4$ to determine the velocity at that time:

$$\nu(t = 4) = 2.4 - 9.8 \times 4 = -36.8 \text{ m/s}$$

13.2.2 Example 2

When charging up, the charge q (C) held by a capacitor varies with time t (s) such that

$$q = 10^{-7}\left(1 - e^{-50t}\right)$$

Determine the current flow in the circuit at $t = 8$ s.

Solution:

Current is the rate of change of charge, so differentiate w.r.t. time:

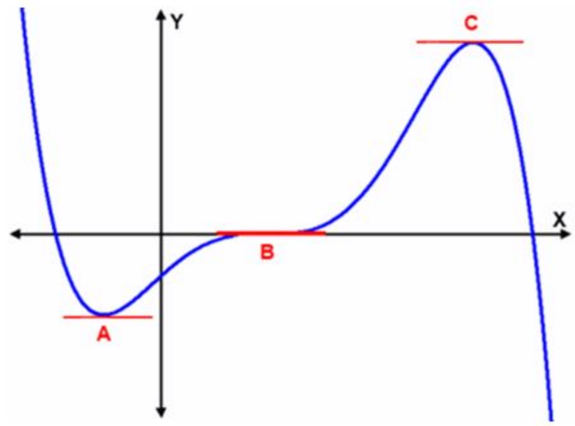
$$\begin{aligned}
I(t) &= \frac{\mathrm{d}q}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(10^{-7}\left(1 - e^{-50t}\right)\right) \\
&= 10^{-7} \times 50e^{-50t} \\
&= 5 \times 10^{-6} e^{-50t}
\end{aligned}$$

Evaluate at $t = 8$:

$$I(t = 8) = 5 \times 10^{-6} e^{-50\times 8} = 9.58 \times 10^{-180} \approx 0$$

13.3 Stationary Points

Consider the curve:



Points A, B and C are all points on the curve where the gradient is zero:

> **Stationary points:**
>
> $$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

- A is a minimum
- B is a point of inflection
- C is a maximum

Optimisation:

- So **extreme values** (maxima and minima) occur at **stationary points** (or at the edge of the range under consideration).
- Finding values of x that provide a maximum or a minimum value of y may be relevant to an optimisation problems, e.g. what number of check-out staff will maximise profits?
- We can find these by differentiating the function and solving for "**where is the derivative equal to zero?**"

13.3.1 Example 3

The approximate annual cost C (in £100's) of carrying out maintenance on a machine part at a frequency of f (per year) is given by:

$$C = 5e^{-0.5f} + 0.6f$$

Determine the maintenance frequency f that will incur the lowest overall cost, i.e. the optimal maintenance frequency.

Solution:

As the cost C is the quantity to optimise, we must obtain a formula for its derivative w.r.t. f:

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}f} &= \frac{\mathrm{d}}{\mathrm{d}f}\big(5e^{-0.5f} + 0.6f\big) \\ &= -2.5e^{-0.5f} + 0.6\end{aligned}$$

Now set this equal to zero, and solve for f:

$$\frac{\mathrm{d}C}{\mathrm{d}f} = 0$$

Thus,

$$\therefore\ -2.5e^{-0.5f} + 0.6 = 0$$

$$\therefore\ e^{-0.5f} = \frac{0.6}{2.5} = \frac{6}{25} = 0.24$$

$$\therefore\ -0.5f = \ln(0.24)$$

$$\therefore\ f = \frac{1}{-0.5}\ln(0.24) = -2\ln(0.24) = 2.85 \quad \text{to (2 d.p.)}$$

So 2.85 times per year, which incurs a cost of

$$C = 5e^{-0.5\times 2.85} + 0.6 \times 2.85 = 2.91 \implies £291$$

This is the only value of f that gives an extreme value of C, but how can we be sure that it is a *minimum* specifically?

13.4 Higher Order Derivatives

We can differentiate $y = 2x^3$ once to get the **first derivative**:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2$$

We can differentiate again to obtain the **second derivative**:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(6x^2\right) = 12x$$

We could also differentiate for a third and fourth time, etc.:

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 12 \quad \text{and} \quad \frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 0$$

For an engineering application, acceleration $a(t)$ is the 2^{nd} order derivative of displacement $S(t)$, since:

$$\nu = \frac{\mathrm{d}S}{\mathrm{d}t} \quad \text{and} \quad a = \frac{\mathrm{d}\nu}{\mathrm{d}t}$$

hence,

$$a = \frac{\mathrm{d}}{\mathrm{d}t}(\nu) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}S}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2S}{\mathrm{d}t^2}$$

2^{nd} order derivatives are used to **classify stationary points.**

Second derivative test:

If $y = f(x)$ has a stationary point at $x = a$ then:

- if $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ at $x = a$ then it is a **maximum** point at a.
- if $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ at $x = a$ then it is a **minimum** point at a.
- if $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = a$ then the nature is unknown and needs further investigation.

13.5 Example 4

For a limited speed range, the torque-speed relationship for an AC induction motor is approximated by the formula:

$$\tau = -0.0016\omega^3 + 0.17\omega^2 - 3.4\omega + 250$$

where τ is the torque generated as a percentage of full-load torque and ω is (angular) speed as a percentage of synchronous (angular) speed.

Find the *maximum* and *minimum* torque points and check the result with a plot of τ against ω.

Solution:

To obtain extreme values of τ, we first differentiate it w.r.t. ω:

$$\begin{aligned}\frac{\mathrm{d}\tau}{\mathrm{d}\omega} &= \frac{\mathrm{d}}{\mathrm{d}\omega}\big(-0.0016\omega^3 + 0.17\omega^2 - 3.4\omega + 250\big)\\ &= -0.0048\omega^2 + 0.34\omega - 3.4\end{aligned}$$

Set $\frac{d\tau}{d\omega} = 0$ and solve for ω:

$$-0.0048\omega^2 + 0.34\omega - 3.4 = 0$$

$$\therefore \quad -48\omega^2 + 3400\omega - 34000 = 0 \quad \text{simplifying...}$$

$$\therefore \quad 6\omega^2 - 425\omega + 4250 = 0$$

Now use the quadratic formula with $a = 6$, $b = -425$, $c = 4250$:

$$\begin{aligned}
\omega &= \frac{-(-425) \pm \sqrt{(-425)^2 - 4 \times 6 \times 4250}}{2 \times 6} \\
&= \frac{425 \pm \sqrt{78625}}{12} \\
&= \frac{425 \pm 280.4015}{12} \\
&= 12.0499 \quad \text{or} \quad 58.7835
\end{aligned}$$

So these values of ω are the "locations" of the extreme values of τ

Substitute these values into the original function to determine the extreme values of τ that occur at these points:

$$\begin{aligned}
\tau(\omega = 12.0499) &= -0.0016(12.0499)^3 + 0.17(12.0499)^2 \\
&\qquad -3.4(12.0499) + 250 \\
&= 230.9149\ldots
\end{aligned}$$

and

$$\begin{aligned}
\tau(\omega = 58.7835) &= -0.0016(58.7835)^3 + 0.17(58.7835)^2 \\
&\qquad -3.4(58.7835) + 250 \\
&= 312.5869\ldots
\end{aligned}$$

To confirm the classifications, find the second derivative:

$$\begin{aligned}
\frac{\mathrm{d}^2\tau}{\mathrm{d}\omega^2} &= \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\omega}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\omega}\big(-0.0048\omega^2 + 0.34\omega - 3.4\big) \\
&= -0.0096\omega + 0.34
\end{aligned}$$

And evaluate this at each stationary point.

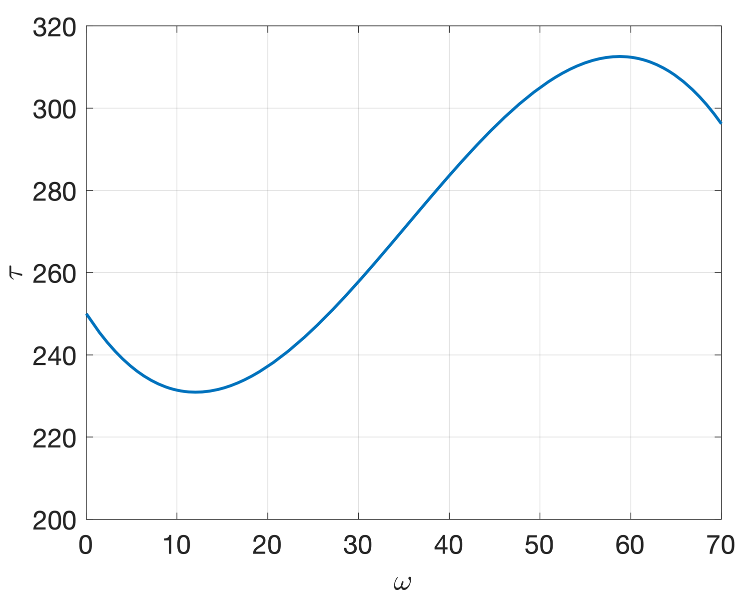
$$\left.\frac{\mathrm{d}^2\tau}{\mathrm{d}\omega^2}\right|_{\omega=12.0499} = -0.0096 \times 12.0499 + 0.34 = +0.2243 > 0$$

So at $\omega = 12.05$ there is a **minimum** of $\tau = 230.91$.

$$\left.\frac{\mathrm{d}^2\tau}{\mathrm{d}\omega^2}\right|_{\omega=58.7835} = -0.0096 \times 58.7835 + 0.34 = -0.2243 < 0$$

Confirming that at $\omega = 58.78$ there is a **maximum** of $\tau = 312.59$.

Plotting this function in Excel or MATLAB to check our results:



13.6 Applications of calculus to motion

In this part of the course we have been learning about how the displacement $s(t)$, velocity $v(t)$ and acceleration $a(t)$ at time t of a body are related to each other through calculus (i.e. through differentiation and integration). In particular, the key theory to remember is:

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \text{and} \quad v(t) = \frac{\mathrm{d}s}{\mathrm{d}t}, \quad \text{so also} \quad a(t) = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \tag{10}$$

in (other) words, velocity is the rate of change of displacement, and acceleration is the rate of change of velocity.

Going the other way using integrals instead:

$$s(t) = \int v(t)\,\mathrm{d}t \quad \text{and} \quad v(t) = \int a(t)\,\mathrm{d}t$$

What about "speed is distance over time"?

"But wait!", you may say, "isn't speed just distance divided by time, . . . and isn't acceleration just the change in velocity divided by the time passed?"

You may indeed be familiar with these ideas, expressed mathematically as:

$$v(t) = \frac{\Delta s}{\Delta t} \quad \text{and} \quad a(t) = \frac{\Delta v}{\Delta t} \tag{11}$$

where Δ (pronounced "delta") means "the change" of that particular quantity. Look at these formulae in 11 again. Then look at the new rules in 10, and back again. Do you notice anything? They actually look pretty similar except that instead of the Δ we have d in the new calculus rules! That's because the rules in 11 are just special cases of these more general rules that we are now learning. If velocity is **constant** then the rule $v = \Delta s/\Delta t$ is actually *the same* as $v = \mathrm{d}s/\mathrm{d}t$, while if velocity is not constant then it is only an approximation of the true relationship. Similarly, if acceleration is constant then $a = \Delta v/\Delta t$ is actually equivalent to $a = \mathrm{d}v/\mathrm{d}t$, while if acceleration is not constant it is only an approximation.

But what about the equations of motion?

These old rules in 11 are themselves just informal ways of stating some of the equations of motion that you may also have encountered. These include rules such as:

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

and a few others, where u is the initial speed, v is the final speed, a is the acceleration and s is the displacement during this time t.

These equations are still true, they haven't somehow become false, **but** once again what we need to recognise is that the equations of motion **only describe situations where acceleration is constant**. They do **not** apply to any other situation where the acceleration is not constant. That's why we learn this new application of differentiation and integration: these new rules in 10 supersede any previous ones you may have learned which were really just special cases of this more general set of rules!

To see this, let's consider that first equation of motion:

$$v = u + at$$

Here, the initial speed u is a constant, as is the acceleration a. So an example might be $v = 15 + 3t$ if $u = 15$ and $a = 3$. In other words, this is just a linear relationship (a straight line) between the final speed and the amount of time passed. Every second the speed increases at a constant rate of 3m/s^2. Now if we different this equation with respect to time, we get:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(u + at\big) = a$$

so even in the special case where these equations of motion apply, the more general truth still holds that acceleration is the derivative w.r.t time of velocity.

Conclusion

So what do you need to take away from this?

- It is **always** true that:

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \text{and} \quad v(t) = \frac{\mathrm{d}s}{\mathrm{d}t}, \quad \text{so also} \quad a(t) = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

  and

$$s(t) = \int v(t)\,\mathrm{d}t \quad \text{and} \quad v(t) = \int a(t)\,\mathrm{d}t$$

  In general, you should use these relationships when trying to solve problems about motion on this module.

- You can **only** use:

$$v(t) = \frac{\Delta s}{\Delta t}$$

  when you know for a fact that velocity is **constant**. Otherwise this relationship is **not true!**

- You can **only** use:

$$a(t) = \frac{\Delta v}{\Delta t}$$

  and the equations of motion when you know for a fact that acceleration (or deceleration) is **constant**. Otherwise this relationship is **not true!**

14 Introduction to integration

14.1 Learning Outcomes

- Understand what integration is, in terms of (i) the inverse of differentiation, and (ii) finding the area under a curve.

- Learn the standard rules for integration, using the tables in the formula booklet.

- Understand the difference between **definite** and **indefinite** integration.

14.2 Motivation

Integration is used to solve many problems, such as determining how much of a quantity has accumulated over time:

- Integrate power over time to determine total energy.

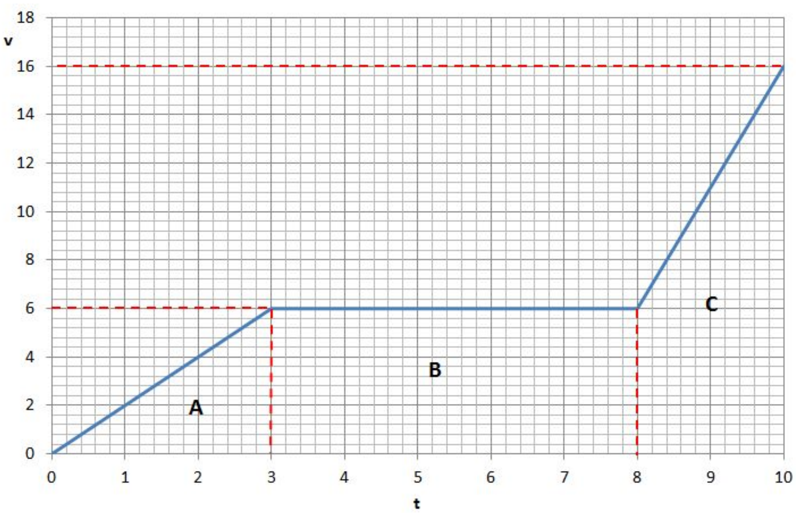
- Integrate force over distance to determine energy spent (valuable in potential energy problems).

- Integrate flow rate over time to determine accumulation of the flowing quantity e.g. mass, volume, charge etc.

Integration concerns calculating the **area underneath a curve.**

14.2.1 Example

Consider again this piece-wise graph which illustrates the velocity v (m/s) of an object over time t (s).



Considering region B we can determine that the object is travelling at 6 m/s between $t = 3$ s and $t = 8$ s.

So in this region the object has travelled 6 m after 1 second, 12 m after 2 seconds and 30 m after 5 seconds.

We could also calculate the total distance travelled in region B by calculating the **area underneath the curve:**

$$\text{Area} = \text{Width} \times \text{Height}$$

$$= 5\ (\text{s}) \times 6\ (\text{ms}^{-1})$$

$$= 30\ \text{m}$$

14.2.2 Integration as the opposite of differentiation

We could also calculate the displacement in regions A and C, using the formulae for the area of a trapezium.

However, calculating the area underneath a **curve** is non-trivial.

So, just as we calculated gradients by referring to our table of derivatives, we will now look at integrating functions by referring to a table of integrals.

Integration is the **inverse of differentiation**, just as division is the inverse of multiplication and subtraction is the inverse of addition. This means that many of the rules will be familiar, but reversed!

14.3 Standard rules of integration:

Find this in the formula booklet!

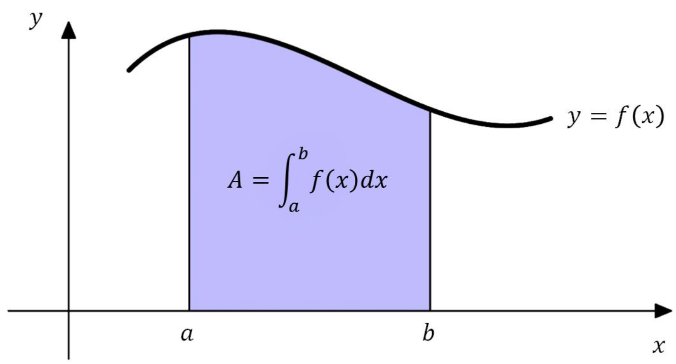
y	$\int y\,dx$
a (any constant)	$ax + C$
ax^n	$\frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$
ae^{nx}	$\frac{ae^{nx}}{n} + C$
$\frac{a}{x}$	$a \ln x + C$
$\frac{a}{nx+b}$	$\frac{a\ln(nx+b)}{n} + C$
$a \sin nx$	$\frac{-a \cos nx}{n} + C$
$a \cos nx$	$\frac{a \sin nx}{n} + C$
$a \sinh nx$	$\frac{a \cosh nx}{n} + C$
$a \cosh nx$	$\frac{a \sinh nx}{n} + C$

14.4 Definite Integration

Integration can be either **definite** or **indefinite**.

Definite integration allows us to calculate the exact area enclosed between:

- a curve,
- two vertical lines and
- the x-axis.



Area A is given by the definite integral of function $f(x)$ between the limits $x = a$ and $x = b$.

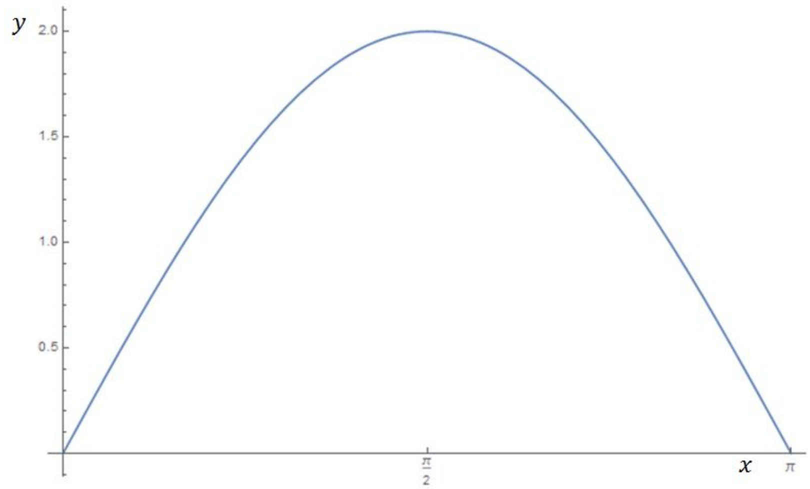
14.4.1 Definite Integration Example 1

Using formal integration, let's confirm the answer from the earlier example. In region B, between $x = 3$ and $x = 8$, the function was a constant: $y = 6$

$$
\begin{aligned}
\text{disp} &= \int_3^8 6 \, \mathrm{d}x \\
&= \left[6x\right]_3^8 \quad \text{Now sub. in upper and lower limits:} \\
&= (6 \times 8) - (6 \times 3) \quad \text{(upper) - (lower)} \\
&= 48 - 18 \\
&= 30\text{m} \qquad \text{as expected.}
\end{aligned}
$$

14.4.2 Definite Integration Example 2

Determine the area under the curve $y = 2\sin x$ between the limits of $x = 0$ and $x = \pi$. Note that we will always use radians for trigonometric functions in this context unless stated otherwise.



Solution:

$$
\begin{aligned}
\text{Area} &= \int_0^{\pi} 2\sin x \, \mathrm{d}x \\
&= \Big[-2\cos x \Big]_0^{\pi} \\
&= \big(-2\cos(\pi) \big) - \big(-2\cos(0) \big) \qquad \text{(use radians!)} \\
&= 2 + 2 \\
&= 4 \text{ units squared}
\end{aligned}
$$

14.4.3 Physical Example 3

Determine the displacement, S m, of an object between the times $t = 2$ s and $t = 5$ s given that the expression for its velocity is:

$$\mathrm{v} = 3t^2 - 6t + 7$$

Solution:

$$\begin{aligned}
S &= \int_2^5 \mathrm{v}\,\mathrm{d}t \\
&= \int_2^5 3t^2 - 6t + 7\,\mathrm{d}t \\
&= \left[\frac{3t^3}{3} - \frac{6t^2}{2} + 7t\right]_2^5 \\
&= \left[t^3 - 3t^2 + 7t\right]_2^5 \\
&= (5^3 - 3 \times 5^2 + 7 \times 5) - (2^3 - 3 \times 2^2 + 7 \times 2) \\
&= 85 - 10 \quad = \quad 75\,\mathrm{m}
\end{aligned}$$

14.5 Indefinite Integration

The alternative is **indefinite integration**, where we do not know the upper and lower limits. The solutions will always contain the constant of integration, C, accounting for this uncertainty.

Example 4: Integrate the function $5x^2 + 7x - 2$ with respect to x:

$$\int 5x^2 + 7x - 2 \,\mathrm{d}x \quad = \quad \frac{5x^3}{3} + \frac{7x^2}{2} - 2x + C$$

Note:

- only one $+C$ is required at the end of the answer, as if there were multiple unknown constants they could be merged.
- $+C$ could appear in definite integration, but it would disappear when subtracting the lower limit from the upper.

14.5.1 Indefinite Integration Example 5

Integrate:

$$y = \int 4x^3 - 7e^{3x} \,\mathrm{d}x$$

Solution:

$$\begin{aligned} y &= \int 4x^3 - 7e^{3x} \,\mathrm{d}x \\ &= \frac{4x^4}{4} - \frac{7e^{3x}}{3} + C \\ &= x^4 - \frac{7}{3}e^{3x} + C \end{aligned}$$

We can only determine the constant of integration if we are given additional information about the curve or the physical problem, such as a coordinate, or an initial condition.

For example, if we were also told that $y = 2$ when $x = 0$, then substitute this in and solve for C:

$$y = x^4 - \frac{7e^{3x}}{3} + C$$

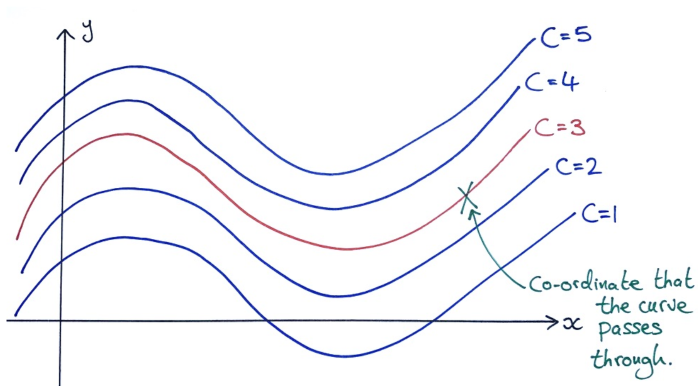
$$2 = 0^4 - \frac{7e^{3\times 0}}{3} + C$$

$$2 = -\frac{7}{3} + C$$

$$\therefore \qquad C = 2 + \frac{7}{3} = \frac{13}{3}$$

Therefore, the particular solution is: $y = x^4 - \frac{7e^{3x}}{3} + \frac{13}{3}$

How does a condition help us find C? Remember that integration is the inverse of differentiation, so indefinite integrate shows us the *set* of curves whose gradient all obey the same function. Thus, they are all parallel and vertically-shifted depending on the value of C.



A co-ordinate assigns a value to C, selecting a particular curve.

14.6 Exercises

Integrate the following:

1) $\int 6\sin(3x) + 5x^{-2}\,\mathrm{d}x$

2) $\displaystyle\int \frac{7x^2}{3} + e^{-2x} + \frac{5}{x}\,\mathrm{d}x$

3) $\displaystyle\int_1^3 \frac{2}{x^3} + 2x + 8\,\mathrm{d}x$

4) Determine the particular solution of $y = \int 3 - 7x + 12x^2\,\mathrm{d}x$, given that when $x = -2$, $y = 0.5$.

Solutions:

1) Indefinite integration:

$$\begin{aligned}\int 6\sin(3x) + 5x^{-2}\,\mathrm{d}x &= -\frac{6}{3}\cos(3x) + \frac{5}{-1}x^{-1} + C \\ &= -2\cos(3x) - \frac{5}{x} + C\end{aligned}$$

2) Indefinite integration:

$$\begin{aligned}\int \frac{7x^2}{3} + e^{-2x} + \frac{5}{x}\,\mathrm{d}x &= \frac{7}{9}x^3 + \frac{1}{-2}e^{-2x} + 5\ln(x) + C \\ &= \frac{7}{9}x^3 - \frac{1}{2}e^{-2x} + 5\ln(x) + C\end{aligned}$$

3) Definite integration:

$$\begin{aligned}
\int_1^3 \frac{2}{x^3} + 2x + 8\,\mathrm{d}x &= \int_1^3 2x^{-3} + 2x + 8\,\mathrm{d}x \\
&= \left[\frac{2}{-2}x^{-2} + \frac{2}{2}x^2 + 8x\right]_1^3 \\
&= \big[-x^{-2} + x^2 + 8x\big]_1^3 \\
&= \left(\frac{-1}{9} + 9 + 24\right) - \left(\frac{-1}{1} + 1 + 8\right) \\
&= 32\,\frac{8}{9} - 8 \quad = \quad 24\,\frac{8}{9}
\end{aligned}$$

4) First, the indefinite integral:

$$\begin{aligned}
y &= \int 3 - 7x + 2x^2\,\mathrm{d}x \\
&= 3x - \frac{7}{2}x^2 + \frac{2}{3}x^3 + C
\end{aligned}$$

Then substitute in the condition $x = -2$ and $y = 0.5$:

$$0.5 = 3(-2) - \frac{7}{2}(-2)^2 + \frac{2}{3}(-2)^3 + C$$

15 Integration - by substitution

15.1 Learning Outcomes

- Solve more challenging integrals using the technique **integration by substitution**.

15.2 Integration Using Substitution

Integrations of the form:

$$\int (5-x)^3 \, \mathrm{d}x$$

$$\int (6x+7)(3x^2+7x-8)^5 \, \mathrm{d}x$$

$$\int \frac{8x}{4x^2-3} \, \mathrm{d}x$$

require a substitution to be made in order to integrate them.

This is somewhat equivalent to the *chain rule* in differentiation.

Basically, we choose the interior component and call that a new variable (usually u unless that has already been used in the problem). We then convert *everything in the integral* to be in terms of this new variable *only*.

If this was the right technique to employ, the new version of the integral will be something we know how to solve.

15.2.1 Example 1

Determine $\int 7(2-6x)^5 \, \mathrm{d}x$ using substitution.

First, let $u = 2-6x$ (usually the object inside the brackets).

Next, we rewrite as: $\int 7u^5 \, \mathrm{d}x$

The problem here is that we are trying to integrate the expression $7u^5$ with respect to x, so we must next deal with the $\mathrm{d}x$ part.

As $u = 2-6x$, we can differentiate this to give $\frac{\mathrm{d}u}{\mathrm{d}x} = -6$. Rearranging this gives $\frac{\mathrm{d}u}{-6} = \mathrm{d}x$.

To summarise, we are trying to determine $\int 7(2-6x)^5 \, \mathrm{d}x$, and have $u = 2-6x$ and $\frac{\mathrm{d}u}{-6} = \mathrm{d}x$.

Substituting both in:

$$\int 7u^5 \, \frac{\mathrm{d}u}{-6} \quad \text{or rather,} \quad \int -\frac{7}{6}u^5 \, \mathrm{d}u$$

This is now a simple integral *entirely in terms of* u (no x left over!) and we can evaluate it.

$$\begin{aligned}
\int -\frac{7}{6}u^5 \, \mathrm{d}u &= -\frac{7}{6}\frac{u^6}{6} + C \\
&= -\frac{7u^6}{36} + C
\end{aligned}$$

To finish the integral, substitute the u back to obtain the final answer in terms of the original variable x:

$$\int 7(2-6x)^5 \, \mathrm{d}x = -\frac{7(2-6x)^6}{36} + C$$

15.2.2 Example 2

Evaluate the following definite integral using substitution:

$$\int_0^1 (5-x)^3 \, \mathrm{d}x$$

Be careful with the limits!

Solution:

First, we make the following substitution:

$$u = 5 - x$$

Then differentiate it:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -1 \quad \text{so, rearranging gives:} \quad \mathrm{d}x = -\mathrm{d}u$$

The limits are in terms of x, so they *also* need to converted to corresponding limits for u:

$$x = 0 \implies u = 5 - (0) = 5$$

$$x = 1 \implies u = 5 - (1) = 4$$

Now substitute all of these into the integral:

$$\int_{x=0}^{x=1} (5-x)^3 \, \mathrm{d}x = \int_{u=5}^{u=4} u^3(-1) \, \mathrm{d}u = -\int_{u=5}^{u=4} u^3 \, \mathrm{d}u$$

So now this is entirely in terms of u, and it is soluble:

$$\begin{aligned} -\int_{u=5}^{u=4} u^3 \, \mathrm{d}u &= -\left[\frac{1}{4}u^4\right]_5^4 \\ &= -\left\{\left(\frac{1}{4}(4)^4\right) - \left(\frac{1}{4}(5)^4\right)\right\} \\ &= 92.25 \end{aligned}$$

15.3 Integration Using Substitution - Extra rules

We can make use of the following general results:

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C$$

and

$$\int f'(x)\,[f(x)]^n\,\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} + C$$

and

$$\int f'(x)\,g\left(f(x)\right)\,\mathrm{d}x = \int g(u)\,\mathrm{d}u$$

15.3.1 Example 3

For example, one could determine:

$$\int \frac{14x+3}{7x^2+3x-2}\,\mathrm{d}x$$

by recognising that it is of the form:

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C$$

$$\therefore \quad \int \frac{14x+3}{7x^2+3x-2}\,\mathrm{d}x = \ln|7x^2+3x-2| + C$$

Check your answer using the full substitution method.

15.4 Exercises

Integrate the following:

1) $\displaystyle\int \frac{\tau}{3}\sin\left(\tau^2\right)\,\mathrm{d}\tau$

2) $\displaystyle\int (6x+7)(3x^2+7x-8)^5\,\mathrm{d}x$

3) $\displaystyle\int \frac{8x}{4x^2-3}\,\mathrm{d}x$

Solution:

1) $\displaystyle\int \frac{\tau}{3}\sin\left(\tau^2\right)\,\mathrm{d}\tau$

Let:

$$u = \tau^2$$

Then differentiate it:

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = 2\tau \quad \text{so, rearranging:} \quad \mathrm{d}\tau = \frac{\mathrm{d}u}{2\tau}$$

Substituting both in, notice that the extra τ's cancel. If this didn't happen, we could not proceed!

$$\int \frac{\tau}{3}\sin\left(\tau^2\right)\,\mathrm{d}\tau \;=\; \int \frac{\tau}{3}\sin(u)\,\frac{\mathrm{d}u}{2\tau} \;=\; \frac{1}{6}\int \sin(u)\,\mathrm{d}u$$

Due to the cancellation, this is now a simple integral in terms of u only:

$$\begin{aligned}\frac{1}{6}\int \sin(u)\,\mathrm{d}u &= \frac{1}{6}\big(-\cos(u)\big) + C \\ &= -\frac{1}{6}\cos(u) + C \\ &= -\frac{1}{6}\cos\left(\tau^2\right) + C\end{aligned}$$

Being sure to give the final answer in terms of τ and not u.

2) $\displaystyle\int (6x+7)(3x^2+7x-8)^5 \, \mathrm{d}x$

It's not always obvious, but let's try the "biggest" internal function:

$$u = 3x^2 + 7x - 8$$

Then differentiate it:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 6x + 7 \quad \text{so, rearranging:} \quad \mathrm{d}x = \frac{\mathrm{d}u}{6x+7}$$

Notice that, again, when we make the substitutions the remaining x term is perfectly cancelled away! If this didn't happen, we would have to reconsider our method - either the wrong choice of u, or a different approach may be required altogether.

$$\begin{aligned}
\int (6x+7)(3x^2+7x-8)^5 \, \mathrm{d}x &= \int (6x+7)u^5 \, \frac{\mathrm{d}u}{6x+7} \\
&= \int u^5 \, \mathrm{d}u \\
&= \frac{1}{6}u^6 + C \\
&= \frac{1}{6}\big(3x^2+7x-8\big)^6 + C
\end{aligned}$$

3) $\displaystyle\int \frac{8x}{4x^2-3}\,\mathrm{d}x$

Usually if there are fractions, try substituting the denominator:

Let $\quad u = 4x^2 - 3$

Then differentiate it:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 8x \quad \text{so, rearranging:} \quad \mathrm{d}x = \frac{\mathrm{d}u}{8x}$$

Yet again, this will cancel the x on the numerator, so this is definitely the right method.

$$\begin{aligned}
\int \frac{8x}{4x^2-3}\,\mathrm{d}x &= \int \frac{8x}{u} \cdot \frac{1}{8x}\,\mathrm{d}u \\
&= \int \frac{1}{u}\,\mathrm{d}u \\
&= \ln|u| + C \\
&= \ln|4x^2-3| + C
\end{aligned}$$

Or we could have solved this instantly (but still explaining our working) by recognising that, with $f(x) = 4x^2 - 3$, this has form:

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C$$

16 Integration - by parts

16.1 Learning Outcomes

- Use the "by parts" technique to evaluate complicated integrals that consist of two functions multiplied together.

16.2 Why do we need this rule?

Consider the following integrals. They are too complicated to use the standard rules, but it does not look like we can use the substitution method either - there is no "inner" function. Instead, these all appear to be the product of two small functions.

$$\int 3x \ln(x)\, \mathrm{d}x$$

$$\int x^4 \sin(x)\, \mathrm{d}x$$

$$\int \mathrm{e}^x \ln(5x)\, \mathrm{d}x$$

$$\int \cos(2x)\big(5x + x^3\big)\, \mathrm{d}x$$

These are all cases where we are required to integrate products of functions in the form $g(x)f(x)$. In these instances we must use the by parts formula.

16.3 Integration by Parts

The "by parts" rule:

$$\int u \frac{\mathrm{d}\mathrm{v}}{\mathrm{d}x}\, \mathrm{d}x \;=\; u\mathrm{v} - \int \mathrm{v} \frac{\mathrm{d}u}{\mathrm{d}x}\, \mathrm{d}x$$

Integration by parts is the integral equivalent of the product rule. It is obtained by re-arranging the product rule and integrating it.

16.3.1 Example 1

Evaluate:

$$\int 2x\, e^{3x}\, \mathrm{d}x$$

Let $u = 2x$ and $\dfrac{\mathrm{dv}}{\mathrm{d}x} = e^{3x}$ (we will see later how to decide this)

We need to know all four terms: $u, \frac{\mathrm{d}u}{\mathrm{d}x}, v, \frac{\mathrm{dv}}{\mathrm{d}x}$

Therefore, differentiate u:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\big(2x\big) = 2$$

and integrate the other:

$$\mathrm{v} = \int \frac{\mathrm{dv}}{\mathrm{d}x}\, \mathrm{d}x = \frac{1}{3}e^{3x} \qquad \text{(ignore } +c \text{ for now)}$$

Substituting these values into the integration by parts rule:

$$\begin{aligned}
\int u\, \frac{\mathrm{dv}}{\mathrm{d}x}\, \mathrm{d}x &= u\mathrm{v} - \int \mathrm{v}\, \frac{\mathrm{d}u}{\mathrm{d}x}\, \mathrm{d}x \\
\int 2xe^{3x}\, \mathrm{d}x &= (2x) \times \left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right) \times (2)\, \mathrm{d}x \\
&= \frac{2}{3}xe^{3x} - \int \frac{2}{3}e^{3x}\, \mathrm{d}x \\
&= \frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + C
\end{aligned}$$

16.4 L-A-T-E

How do we decide which function should be u and which should be the derivative of v? It is best to choose u using the following order of priority (LATE) from highest to lowest priority:

Logarithmic $\ln(x)$

Algebraic x^n

Trigonometric $\sin(x)$ or $\cos(x)$

Exponential e^{nx}

16.4.1 Example 2

Evaluate:

$$5x\sin(7x)\,\mathrm{d}x$$

Using LATE, choose $u = 5x$ and $\dfrac{\mathrm{dv}}{\mathrm{d}x} = \sin(7x)$

Therefore, differentiate u:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(5x) = 5$$

and integrate the other to obtain v:

$$\mathrm{v} = \int \frac{\mathrm{dv}}{\mathrm{d}x}\,\mathrm{d}x = \int \sin(7x)\,\mathrm{d}x = -\frac{1}{7}\cos(7x)$$

Substituting these values into the integration by parts rule:

$$\begin{aligned}
\int u\frac{\mathrm{dv}}{\mathrm{d}x}\,\mathrm{d}x &= u\mathrm{v} - \int \mathrm{v}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \\
\int 5x\sin(7x)\,\mathrm{d}x &= (5x)\Big(-\frac{1}{7}\cos(7x)\Big) - \int\Big(-\frac{1}{7}\cos(7x)\Big)(5)\,\mathrm{d}x \\
&= -\frac{5}{7}x\cos(7x) + \int\frac{5}{7}\cos(7x)\,\mathrm{d}x \\
&= -\frac{5}{7}x\cos(7x) + \frac{5}{49}\sin(7x) + C
\end{aligned}$$

16.5 Definite Integration by Parts

The "by parts" rule for definite integrals:

$$\int_a^b u \frac{\mathrm{dv}}{\mathrm{d}x}\,\mathrm{d}x \;=\; \big[u\mathrm{v}\big]_a^b - \int_a^b \mathrm{v}\,\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Both the first term, and the second following the integral, will need to be evaluated at the upper and lower limits.

16.5.1 Example 3

Evaluate:

$$\int_{-1}^{2} -2xe^{4x}\,\mathrm{d}x$$

Using LATE, choose $u = -2x$ and $\dfrac{\mathrm{dv}}{\mathrm{d}x} = e^{4x}$

We obtain the other terms as normal:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\big(-2x\big) = -2$$

and to obtain v:

$$\mathrm{v} = \int \frac{\mathrm{dv}}{\mathrm{d}x}\,\mathrm{d}x = \int e^{4x}\,\mathrm{d}x = \frac{1}{4}e^{4x}$$

Substituting these values into the integration by parts rule:

$$\int_a^b u \frac{\mathrm{dv}}{\mathrm{d}x}\,\mathrm{d}x \;=\; \big[u\mathrm{v}\big]_a^b - \int \mathrm{v}\,\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

We obtain:

$$
\begin{aligned}
\int_{-1}^{2} -2xe^{4x}\,\mathrm{d}x &= \left[(-2x)\left(\frac{1}{4}e^{4x}\right)\right]_{-1}^{2} - \int_{-1}^{2}\left(\frac{1}{4}e^{4x}\right)(-2)\,\mathrm{d}x \\
&= \left[-\frac{1}{2}xe^{4x}\right]_{-1}^{2} + \int_{-1}^{2}\frac{1}{2}e^{4x}\,\mathrm{d}x \\
&= \left[-\frac{1}{2}xe^{4x} + \frac{1}{8}e^{4x}\right]_{-1}^{2}
\end{aligned}
$$

Now we simply have the extra step of evaluating both terms at the upper and lower limits, and determine the difference:

$$
\int_{-1}^{2} -2xe^{4x}\,\mathrm{d}x = \left[-\frac{1}{2}xe^{4x} + \frac{1}{8}e^{4x}\right]_{-1}^{2}
$$

$$
= \left(-\frac{1}{2}(2)e^{4\times 2} + \frac{1}{8}e^{4\times 2}\right) - \left(-\frac{1}{2}(-1)e^{4\times -1} + \frac{1}{8}e^{4\times -1}\right)
$$

$$
= \left(-\frac{7}{8}e^{8}\right) - \left(\frac{5}{8}e^{-4}\right)
$$

$$
= -2608.3 \quad (1\text{ d.p.})
$$

16.6 Exercises

Integrate the following:

$$
\int 3x^2 \ln(x)\,\mathrm{d}x
$$

17 Integration - applications

17.1 Learning Outcomes

- Apply integration to problems in an engineering context.

- Determine the area between two different curves plotted in the same plane.

17.2 Integration in an Engineering Context

Both processes of calculus can be applied to consider how certain quantities are changing with respect to another (usually time).

Differentiation gives the rates of change of a quantity whose function is known.

Integration instead can be used to find the **total amount of a quantity that accumulates over time**, given that we know something about it's rate of change or flow.

17.2.1 Example 1

The volume of water V accumulated in a tank is given by:

$$\mathrm{V} = \int Q \, \mathrm{d}t$$

where $Q = (1-t)^2 + 16$ is the (volumetric) flow rate of water into the tank and t is time.

If $\mathrm{V} = 4$ when $t = 0$, determine the relationship between V and t.

Note that all quantities are in SI units.

Substitute in the formula:

$$\begin{aligned} \mathrm{V} &= \int (1-t)^2 + 16 \, \mathrm{d}t \\ &= \int 1 + t^2 - 2t + 16 \, \mathrm{d}t \\ &= \int t^2 - 2t + 17 \, \mathrm{d}t \\ &= \frac{1}{3}t^3 - t^2 + 17t + C \end{aligned}$$

Using the condition to determine C:

$$\mathrm{V}(t=0) = 4 \implies 4 = 0 + C \implies C = 4$$

$$\therefore \ \mathrm{V} = \frac{1}{3}t^3 - t^2 + 17t + 4$$

17.2.2 Example 2

The amount of charge q passing a point in a circuit during a period of time τ is governed by:

$$q = \int_0^\tau i(t)\,\mathrm{d}t$$

where $i(t)$ is the current flow (in μA) at time $0 \leq t \leq \tau$

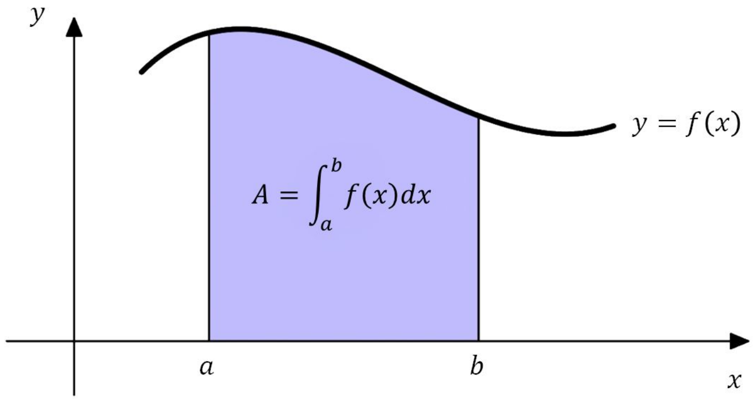
If $i(t) = 40e^{-0.1t}$, find a relationship between the variables q and t.

Substitute in the formula and evaluate the integral:

$$\begin{aligned}
q(\tau) &= \int_0^\tau 40e^{-0.1t}\,\mathrm{d}t \\
&= \left[\frac{40}{-0.1}e^{-0.1t}\right]_0^\tau \\
&= \big[-400e^{-0.1t}\big]_0^\tau \\
&= \big(-400e^{-0.1\times\tau}\big) - \big(-400e^{-0.1\times 0}\big) \\
&= 400\big(1 - e^{-0.1\tau}\big)
\end{aligned}$$

17.3 Areas Between Curves

Consider a curve $y = f(x)$ which is above the x-axis in the region $a < x < b$. Suppose A is the area bounded by the curve $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$:



Then A is called the area under the curve between $x = a$ and $x = b$.

The definite integral of the function in a region where the curve is above the x-axis yields a positive value, which is exactly this area. Hence:

$$A = \int_a^b f(x)\, \mathrm{d}x$$

However, we can extend this concept and determine areas that are defined in a more complex manner.

17.3.1 Example 3

Find the area enclosed between the curve $y = -x^2 - x + 6$ and the x-axis.

Solution:

No limits are given, so we shall have to determine appropriate limits by considering the shape of this curve.

As this is a $\cap$-shaped parabola, if there are two real roots then they will define the limits of the region "contained" between the curve and the x-axis. Thus, we find them by solving for $y = 0$:

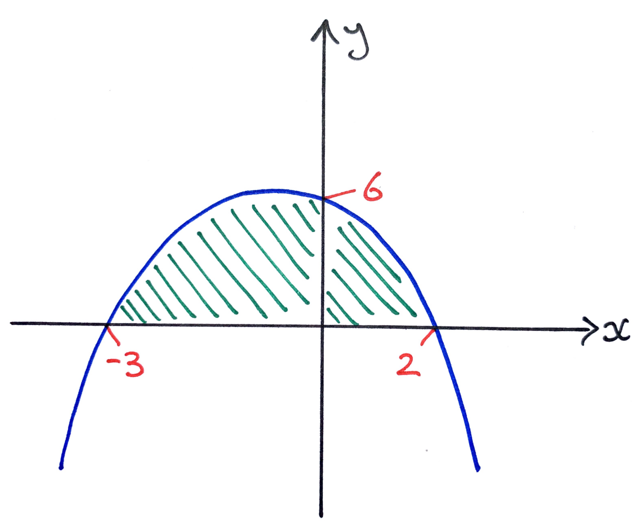
$$-x^2 - x + 6 = 0$$

$$\therefore\ x^2 + x - 6 = 0$$

$$\therefore\ (x+3)(x-2) = 0$$

$$\therefore\ x = -3 \quad \text{and} \quad x = 2$$

Or we could have used the quadratic formula to obtain these roots.



Now conduct the definite integral between these limits:

$$\int_{-3}^{2} -x^2 - x + 6\,\mathrm{d}x = \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x \right]_{-3}^{2}$$

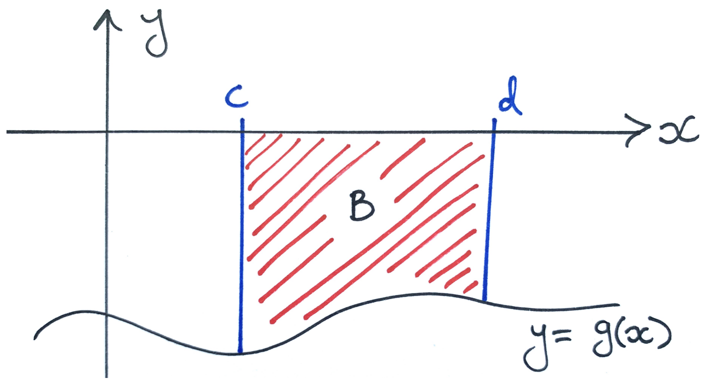
$$= \left(-\frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 + 6(2) \right) - \left(-\frac{1}{3}(-3)^3 - \frac{1}{2}(-3)^2 + 6(-3) \right)$$

$$= \left(-\frac{8}{3} - \frac{4}{2} + 12 \right) - \left(\frac{27}{3} - \frac{9}{4} - 18 \right)$$

$$= \frac{223}{12}$$

$$= 18.583\ldots$$

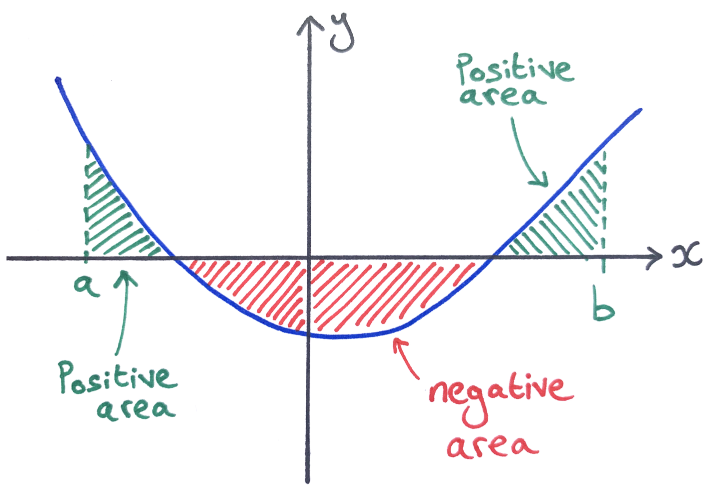
However, integrating the function of a curve over a range of x where it is *below* the x-axis gives a negative value, which is precisely $-1\times$ the area between the curve and the x-axis.



Thus, we have:

$$-B = \int_c^d g(x)\,\mathrm{d}x \quad \text{or} \quad B = \left| \int_c^d g(x)\,\mathrm{d}x \right|$$

So what if we wish to calculate the area enclosed by a curve that is both above and below the x-axis in different regions?

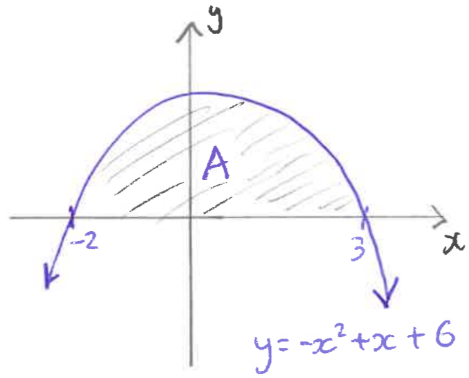


In this case, when asked to calculate the total *area* we must determine this positive value by separately calculating the integrals for regions where the curve is above and below the x-axis, and then summing their magnitudes/absolute values.

Therefore, we must begin by solving an equation to find where the curve crosses the x-axis. That is, we must check if the **roots** lie within the range of interest.

17.3.2 Example 4

Find the area between the curve $y = -x^2 + x + 6$ and the x-axis.



Solution

1. Either by using the quadratic equation, or factorising to $y = -(x^2 - x + 6) = -(x - 3)(x + 2)$, find the roots at $x = -2$ and $x = 3$.

2. Draw a graph.

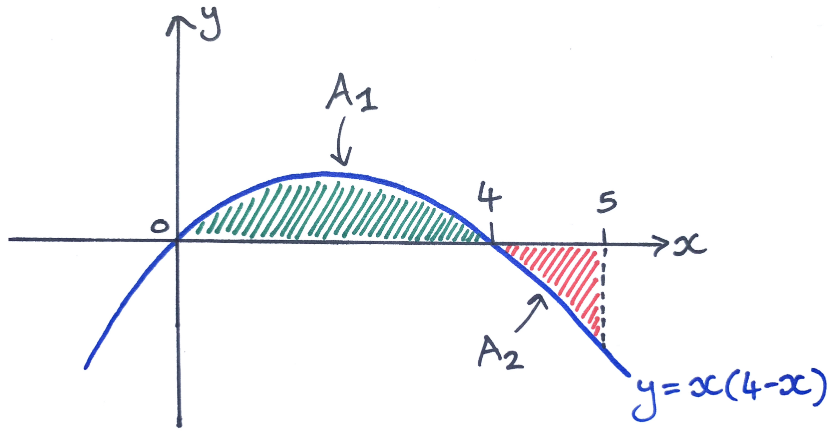
3. Formulate the definite integral:

$$A = \int_{-2}^{3} y(x)\,\mathrm{d}x \quad = \quad \int_{-2}^{3} -x^2 + x + 6\,\mathrm{d}x$$

4. Answer $= 125/6$ square units.

17.3.3 Example 5

Find the area between the curve $y = 4x - x^2$ and the x-axis from $x = 0$ to $x = 5$.



Solution

1. Either by using the quadratic equation, or factorising to $y = x(4 - x)$, deduce roots at $x = 0$ and $x = 4$.

2. Draw the graph.

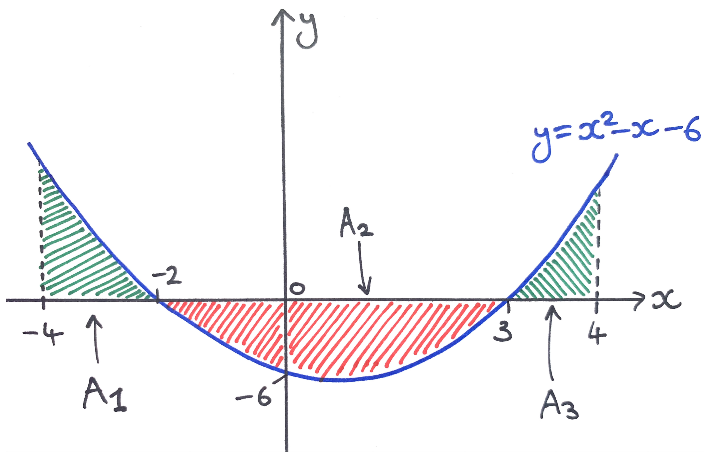
3. Since a root occurs in the range, the total area is split in two parts: above the x-axis in $0 < x < 4$, and below the x-axis in $4 < x < 5$. We must formulate these two integrals separately, then add their magnitudes:

$$\begin{aligned}
A &= A_1 + A_2 \\
&= \left| \int_0^4 4x - x^2 \,\mathrm{d}x \right| + \left| \int_4^5 4x - x^2 \,\mathrm{d}x \right| \\
&= \left| \left[2x^2 - \frac{1}{3}x^3 \right]_0^4 \right| + \left| \left[2x^2 - \frac{1}{3}x^3 \right]_4^5 \right| \\
&= \left| \frac{32}{3} \right| + \left| \frac{-7}{3} \right| \\
&= \frac{32}{3} + \frac{7}{3} \\
&= 13 \text{ square units}
\end{aligned}$$

17.3.4 Example 6

Find the area between the curve $y = x^2 - x - 6$ and the x-axis between $x = -4$ and $x = 4$, and compare this with the integral:

$$\int_{-4}^{4} x^2 - x - 6 \, \mathrm{d}x$$



Solution

1. Either by using the quadratic equation, or factorising to $y = (x-3)(x+2)$, deduce roots at $x = -2$ and $x = 3$.

2. Draw the graph.

3. This time two roots occur in the range, and so the graph shows three regions. The integrals over $-4 < x < -2$ and $3 < x < 4$ will give positive results, while the integral over $-2 < x < 3$ will be negative.

4. To find the total area we add the magnitudes of the three areas:

$$\begin{aligned}
A &= A_1 + A_2 + A_3 \\
&= \left|\int_{-4}^{-2} x^2 - x - 6 \, \mathrm{d}x\right| + \left|\int_{-2}^{3} x^2 - x - 6 \, \mathrm{d}x\right| + \left|\int_{3}^{4} x^2 - x - 6 \, \mathrm{d}x\right| \\
&= \left|\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x\right]_{-4}^{-2}\right| + \left|\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x\right]_{-2}^{3}\right| + \left|\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x\right]_{3}^{4}\right| \\
&= \left|\frac{38}{3}\right| + \left|\frac{-125}{6}\right| + \left|\frac{17}{6}\right| \\
&= \frac{38}{3} + \frac{125}{6} + \frac{17}{6} \\
&= \frac{109}{3} \text{ square units}
\end{aligned}$$

5. By comparison, the single integral (where the middle region adds a negative area) gives a smaller and negative result:

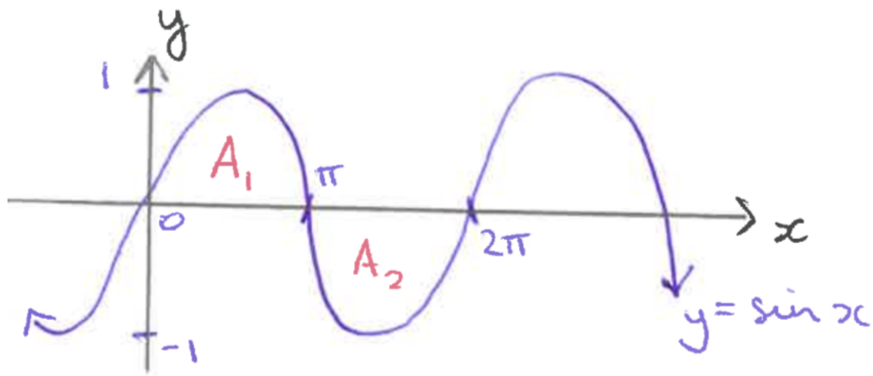
$$\frac{38}{3} - \frac{125}{6} + \frac{17}{6} = -\frac{16}{3}$$

This is the *net* area under the curve.

17.3.5 Example 7

What do you expect the answer of the following integral to be?

$$\int_0^{2\pi} \sin(x)\mathrm{d}x$$



Drawing a sketch, by symmetry we see that the area between $0 < x < \pi$ and the area between $\pi < x < 2\pi$ will cancel each other out. We confirm by calculation that the answer is zero:

$$\int_0^{2\pi} \sin(x)\,\mathrm{d}x \quad = \quad \Big[-\cos(x)\Big]_0^{2\pi} = (-1) - (-1) = 0$$

17.3.6 Example 8

Find the area enclosed **between** the curves

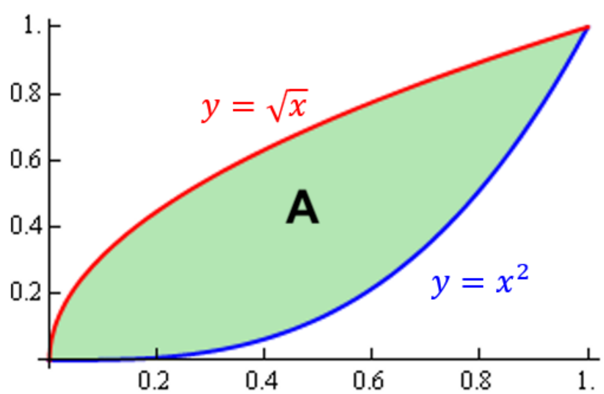
$$y = \sqrt{x}$$

and

$$y = x^2$$

Solution:

The trick here is to subtract the area under the lower curve from the area under the upper curve. The limits are found by solving the two equations simultaneously, to find their points of intersection.



Finding the values of x where the curves intersect:

$$\sqrt{x} = x^2 \quad \text{square both sides} \ldots$$

$$x = x^4$$

$$x^4 - x = 0 \quad \text{factorising} \ldots$$

$$x(x^3 - 1) = 0$$

$$x = 0 \quad \text{or} \quad x^3 - 1 = 0$$

If $x^3 - 1 = 0$, then $x^3 = 1$ and so:

$$x = \sqrt[3]{1} = 1$$

So the limits are $x = 0$ and $x = 1$

Evaluate the integral of the upper curve minus the lower curve with these limits:

$$\begin{aligned}
A &= \int_0^1 \sqrt{x} - x^2 \, \mathrm{d}x \;=\; \int_0^1 x^{\frac{1}{2}} - x^2 \, \mathrm{d}x \\
&= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1 \\
&= \left(\frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{3}(1)^3\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}} - \frac{1}{3}(0)^3\right) \\
&= \left(\frac{2}{3} \cdot 1 - \frac{1}{3} \cdot 1\right) - \big(0 - 0\big) \;=\; \frac{1}{3}
\end{aligned}$$

18 Introduction to complex numbers

18.1 Learning Outcomes

- Learn about the existence of j, the *imaginary number*.
- Recognise complex numbers.
- Add, subtract, multiply and divide pairs of complex numbers (in "Cartesian form").

18.2 Sets of Numbers

Numbers are understood to be organised in nested sets:

Set	Symbol	Examples
Natural numbers	$\mathbb{N}$	0, 1, 2, 3, ...
Integers (i.e. whole numbers)	$\mathbb{Z}$	$\mathbb{N}$ and -1, -2, -3, ...
Rational numbers	$\mathbb{Q}$	$\mathbb{Z}$ and all fractions
Real numbers	$\mathbb{R}$	$\mathbb{Q}$ and all irrationals, e.g. $\sqrt{2}$, π, e, etc

so

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Is $\mathbb{R}$ the biggest set that exists, or are there still more numbers?

Quite often we are required to solve equations, such as

$$x - 7 = 0$$

Here the solution is 7, which is a member of $\mathbb{N}$.

We have also solved equations such as

$$x + 4 = 0,$$

but if do not accept the existence of any numbers other than the members of $\mathbb{N}$, then you cannot solve this equation.

If we define the solution as being $x = -4$, this idea leads to an entirely new set of numbers, the integers: $\mathbb{Z}$.

Once we have accepted the set $\mathbb{Z}$, we can solve a wider variety of equations. For example:

$$x + 12 = 0$$

$$(x + 9)(x + 2) = 0,$$

etc.

Therefore we benefit from accepting the existence of negative numbers such as -1

18.3 Complex Numbers

18.3.1 The Imaginary Unit

Up until this point, we assumed that there is no solution to the square root of a negative number. So, for example, there is no solution to the equation:

$$x^2 + 1 = 0$$

However, in a similar manner to the previous slides, if we can accept that another number set exists outside of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, then we can solve this equation.

Thus mathematicians define the solution to be the imaginary number j, so that:

$$j^2 + 1 = 0$$

Let's examine the properties of our newly defined number.

If $j^2 + 1 = 0$ then:

Definition of the imaginary unit:

$$j^2 = -1$$

and

$$j = \sqrt{-1}$$

We call j the imaginary unit, as it is the unit of the imaginary numbers in the same way that 1 is the unit of the real numbers.

Hence, we can also express the square roots of the other negative numbers in terms of this unit j, for example:

$$\begin{aligned}\sqrt{-4} &= \sqrt{-1 \times 4} \\ &= \sqrt{-1} \times \sqrt{4} \\ &= j2\end{aligned}$$

and

$$
\begin{aligned}
\sqrt{-25} &= \sqrt{-1 \times 25} \\
&= \sqrt{-1} \times \sqrt{25} \\
&= j5
\end{aligned}
$$

18.3.2 Complex numbers

Any multiples of j, such as j, $j6$, $j0.3$, $j\dfrac{7}{3}$ and $j2\sqrt{3}$, are all imaginary numbers.

When we combine these with real numbers through addition/subtraction then we create **complex numbers**, e.g.

$$
6 - j5 \quad \text{and} \quad -3 + j
$$

Note: Mathematicians (and most software) use i, while engineers use j. Thus, an engineer would write $3 - j5$, while a mathematician would write $3 - 5i$.

18.4 Uses of Complex Numbers

Despite their apparent lack of physical meaning, complex numbers are essential for solving some real-world problems.

- Complex numbers can be used to represent certain engineering quantities, particularly in electronics:
  - AC currents and voltages
  - Impedances in AC circuits
- They are also used in:
  - The solution of differential equations
  - Aerodynamics (potential flow in two dimensions)
  - Control theory (stability analysis)
  - Signal processing (spectral analysis)

18.5 Cartesian form

The general (Cartesian/rectangular) form of a complex number is:

Standard Cartesian form of a complex number:

$$z = x + jy$$

where x and y are real numbers ($\mathbb{R}$) and $j = \sqrt{-1}$.

We say that the **real part** of z is x:

$$\mathrm{Re}(z) = x$$

and the **imaginary part** of z is y:

$$\mathrm{Im}(z) = y \qquad \text{(NOT } yj\text{)}$$

18.6 Complex Number Arithmetic

18.6.1 Addition/Subtraction

To add/subtract complex numbers we simply add/subtract the corresponding real and imaginary parts separately.

If $z_1 = 2 - j3, \quad z_2 = 6 - j2 \quad \text{and} \quad z_3 = -7 + j5,$

Calculate:

1) $z_1 + z_2$

2) $z_2 - z_1$

3) $z_2 - z_3$

4) $z_1 + z_3$

Solutions:

1)

$$\begin{aligned} z_1 + z_2 &= (2 - j3) + (6 - j2) \\ &= \big(2 + 6\big) + \big(-3 + (-2)\big)j \\ &= 8 - 5j \end{aligned}$$

2)

$$\begin{aligned} z_2 - z_1 &= (6 - j2) - (2 - j3) \\ &= 6 - j2 - 2 + j3 \\ &= (6 - 2) + (-j2 + j3) \\ &= 4 + j \end{aligned}$$

3)

$$\begin{aligned} z_2 - z_3 &= (6 - j2) - (-7 + j5) \\ &= \big(6 - (-7)\big) + \big(-2 - 5\big)j \\ &= 13 - 7j \end{aligned}$$

4)

$$\begin{aligned} z_1 + z_3 &= (2 - 3j) + (-7 + j5) \\ &= \big(2 + (-7)\big) + \big((-3) + 6\big)j \\ &= -5 + 2j \end{aligned}$$

18.6.2 Multiplication

To multiply complex numbers we simply expand the brackets; treating j just like any other constant.

Example:

Calculate:

1) $z_1 z_2$

2) $z_3 z_2$

Solutions:

1)

$$\begin{aligned} z_1 z_2 &= (2 - j3)(6 - j2) \\ &= 12 - j4 - j18 + j^2 6 \quad \text{(remember } j^2 = -1\text{)} \\ &= 12 - j22 + (-1)(6) \\ &= 6 - j22 \end{aligned}$$

2)

$$\begin{aligned} z_3 z_2 &= (-7 + j5)(6 - j2) \\ &= -42 + j14 + j30 - j^2 10 \\ &= -42 + j44 - (-1)(10) \\ &= -32 + j44 \end{aligned}$$

18.6.3 Division

To divide complex numbers, we first make the denominator *real*.

This can be achieved by multiplying both the numerator and denominator by the **complex conjugate** of the denominator.

> **Definition of complex conjugates:**
>
> For a complex number $z = x + jy$, the complex conjugate of z is:
>
> $$\overline{z} = x - jy$$

Furthermore it can be proven that

$$z\overline{z} = x^2 + y^2,$$

a result which is purely real and has no imaginary part.

Example:

Calculate:

1) $\dfrac{z_3}{z_2}$ 2) $\dfrac{z_1}{z_2}$

Solutions:

1)

$$\begin{aligned}
\frac{z_3}{z_2} &= \frac{-7+j5}{6-j2} = \frac{(-7+j5)(6+j2)}{(6-j2)(6+j2)} \\
&= \frac{-42-j14+j30+j^2 10}{36+j12-j12-j^2 4} \\
&= \frac{-42+j16-10}{36+4} \\
&= \frac{-52+j16}{40} = \frac{-13+j4}{10} \\
&= \frac{-13}{10}+j\frac{2}{5} = -1.3+j0.4
\end{aligned}$$

2)

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{2-j3}{6-j2} = \frac{(2-j3)(6+j2)}{(6-j2)(6+j2)} \\
&= \frac{12+j4-j18-j^2 6}{36+j12-j12-j^2 4} \\
&= \frac{12-j14+6}{36+4} \\
&= \frac{18-j14}{40} = \frac{9-j7}{20} \\
&= \frac{9}{20}-j\frac{7}{20} = 0.45-j0.35
\end{aligned}$$

19 Polar form of complex numbers

19.1 Learning Outcomes

- Represent complex numbers in an Argand diagram.

- Express complex numbers in rectangular/Cartesian and polar form, and convert between these.

19.2 Argand Diagrams

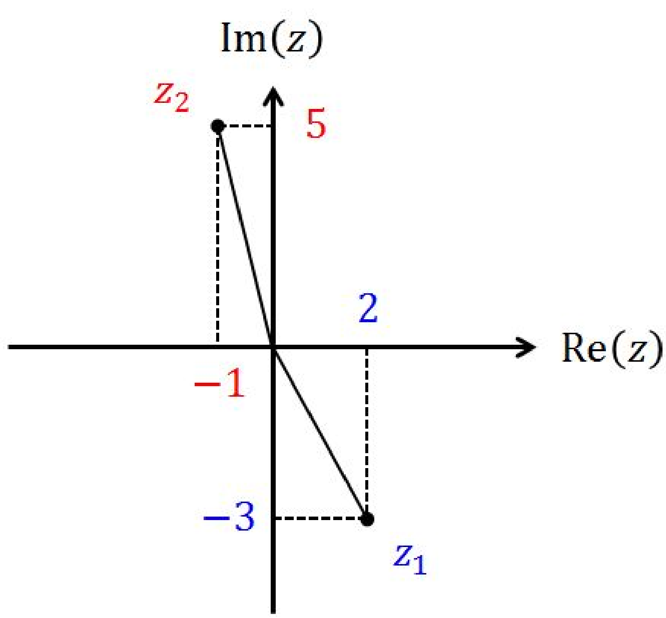
Complex numbers written in the form

$$z = x + jy$$

are said to be in rectangular form (also called Cartesian form).

In this form we can represent a complex number graphically using the co-ordinate (x, y) in an Argand diagram, where the x-axis is the Real part and the y-axis represents the Imaginary part.

Plotting $z_1 = 2 - j3$ and $z_2 = -1 + j5$ in an Argand diagram:



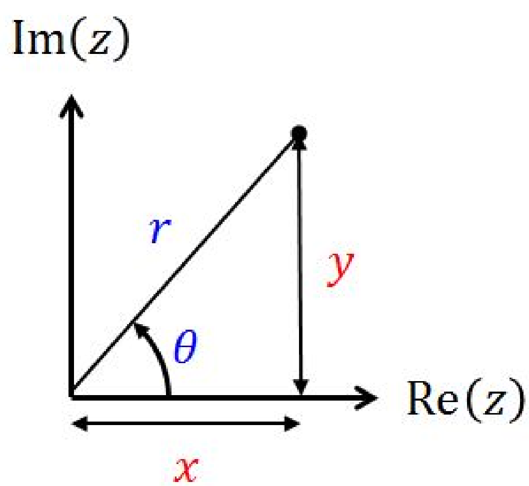
19.2.1 Modulus and Argument

The Argand diagram suggests an alternative way of representing complex numbers.

Instead of using the co-ordinates (x, y) to fix the position of the end of a line in the Argand diagram, we could define the line's position using the modulus r (length of the line) and the argument θ (angle relative to the positive real axis).

19.3 Polar Form

Given a complex number, $z = x + jy$, where both $x, y > 0$:



We can calculate the modulus using Pythagoras:

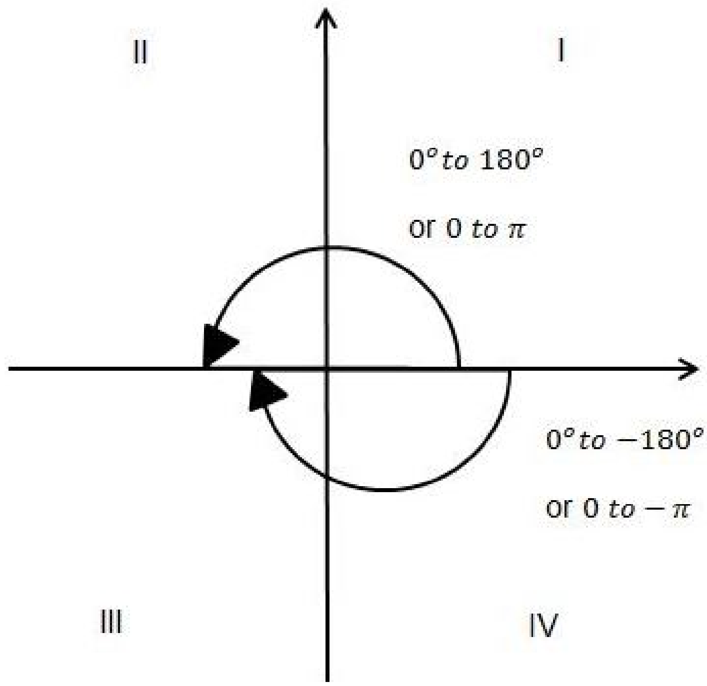
$$r = \sqrt{x^2 + y^2}$$

and the argument can be calculated using trigonometry:

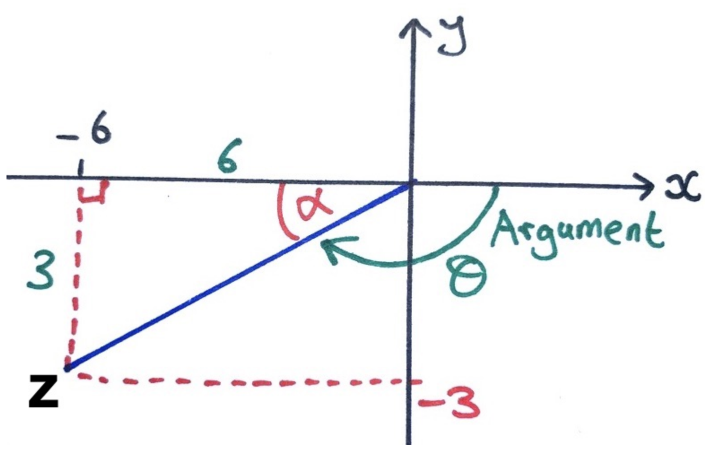
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

19.3.1 Measuring the Argument

If z is not in the first quadrant, we need to do more to find the argument, as it is measured **anti-clockwise** from the **positive real axis**. By convention, it should be in the range $-\pi < \theta \leq +\pi$:



Example:



This complex number is in the third quadrant.

Using right angle trig. we initially determine the angle α by:

$$\begin{aligned}\alpha &= \tan^{-1}\left(\frac{|y|}{|x|}\right) = \tan^{-1}\left(\frac{3}{6}\right) \\ &= 0.464\end{aligned}$$

But this is not the argument, rather:

$$\theta = \pi - \alpha = 2.678$$

and finally as it is rotating the "wrong" way, the argument is:

$$Arg(z) = -\theta = -2.678$$

19.3.2 Alternative method for measuring the argument

An alternative approach is to *always* calculate θ according to:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

and then:

- If the complex number lies in quadrant II on the Argand diagram, then we add 180^o or π to the result.

- If the complex number lies in quadrant III on the Argand diagram, then subtract 180^o or π from the result.

You may use this rule if preferred. Applying to the previous example:

$$Arg(z) = \tan^{-1}\left(\frac{-3}{-6}\right) - \pi = -2.678$$

19.3.3 Summary

So, if

$$z = x + jy$$

is a complex number written in Cartesian form, then

Polar form:

$$z = r\cos\theta + jr\sin\theta$$

where r is the modulus and θ is the argument

is the same complex number, but written in polar form. The shorthand form for this is:

$$z = r\angle\theta$$

19.3.4 Examples 1

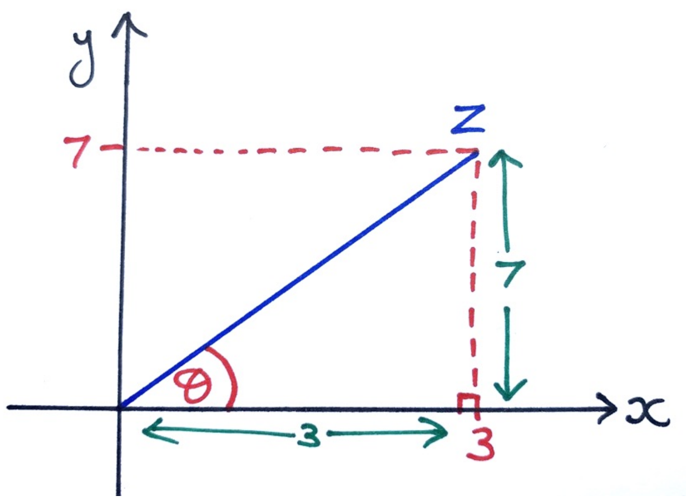
Express the following complex numbers in polar form:

1) $z = 3 + j7$

2) $z = -4 + j3$

Solution:

1)



Modulus:

$$\begin{aligned} r &= \sqrt{3^2 + 7^2} \\ &= \sqrt{58} \\ &= 7.616 \end{aligned}$$

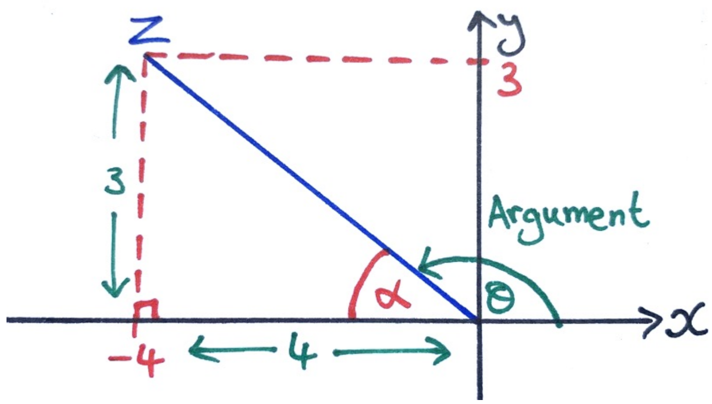
Argument:

$$\theta = \tan^{-1}\left(\frac{7}{3}\right) = 1.166$$

Hence,

$$z = 7.616\angle 1.166$$

2)



Modulus:

$$\begin{aligned} r &= \sqrt{3^2 + 4^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Argument:

$$\alpha = \tan^{-1}\left(\frac{3}{4}\right) = 0.644$$

$$\theta = \pi - 0.644 = 2.498$$

Hence, $z = 5\angle 2.498$

19.4 Converting to Rectangular Form

Given a complex number written in polar form:

$$z = r\cos\theta + jr\sin\theta$$

We are easily able to convert to rectangular form (by using trigonometry) by using the formulae:

$$\boxed{x = r\cos\theta \quad \text{and} \quad y = r\sin\theta}$$

Note that in this case in **does not matter which quadrant** the complex number lies in.

19.4.1 Examples 2

Express the following complex numbers in rectangular form:

1) $z = 8\angle 2.1$

2) $z = 5.3\angle -3$

Solutions:

1)

$$x = 8\cos(2.1) = -4.039 \quad \text{and} \quad y = 8\sin(2.1) = 6.906$$

$$\therefore\ z = -4.039 + j\,6.906$$

2)

$$x = 5.3\cos(-3) = -5.247 \quad \text{and} \quad y = 5.3\sin(-3) = -0.748$$

$$\therefore\ z = -5.247 - j\,0.748$$

19.5 Polar Form Arithmetic

Polar form can be useful since multiplications and division in polar form are much easier; as shown by the formulae:

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2)$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle(\theta_1 - \theta_2)$$

where

$$z_1 = r_1 \angle \theta_1 \quad \text{and} \quad z_2 = r_2 \angle \theta_2$$

19.5.1 Examples 3

Given that $z_1 = 5.3\angle 2.1$ and $z_2 = 2.7\angle - 0.3$, determine:

1) $z_1 z_2$

2) $\dfrac{z_1}{z_2}$

Solution:

1)

$$\begin{aligned} z_1 z_2 &= (5.3 \times 2.7)\angle\big(2.1 + (-0.3)\big) \\ &= 14.31\angle 1.8 \end{aligned}$$

2)

$$\begin{aligned} \frac{z_1}{z_2} &= \left(\frac{5.3}{2.7}\right)\angle\big(2.1 - (-0.3)\big) \\ &= 1.963\angle 2.4 \end{aligned}$$

20 Introduction to matrices

20.1 Learning Outcomes

- Identify properties of matrices.
- Perform matrix arithmetic, i.e. addition, subtraction and multiplication.

Matrices are used to handle many pieces of information at once, thereby lending themselves to the analysis of systems described by a set of similar equations. Such applications include:

- Circuit theory (sets of linear equations).
- Dynamical systems theory (sets of differential equations).
- Vector graphics and computer imaging.

20.2 Definitions and Notation

A matrix is a rectangular array of numbers, for example:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 3 & 0 \\ 2 & -8 \end{pmatrix}$$

and are generally represented by a bold capital letter: **A**, **C**, **X** etc., or by underlining the letter, i.e. $\underline{A}$.

The dimensions (or "order") of a matrix are $m \times n$ where m is the number of rows and n is the number of columns. The matrix **A** is a 3×2 matrix.

20.2.1 Order

The **order** of a matrix is a description of its dimensions - the number of rows, then the number of columns.

$\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ This is a 2×1 matrix. It is also a *vector*.

$$\begin{pmatrix} 1 & -2 & 8 \\ 3 & 1 & 4 \end{pmatrix} \qquad \text{This is a } 2 \times 3 \text{ matrix.}$$

$$\begin{pmatrix} 2 & 0 & -1 & 6 \end{pmatrix} \qquad \text{This is a } 1 \times 4 \text{ matrix.}$$

The numbers that make up a matrix are called elements.

An element may be written a_{ij}.

The lowercase a indicates that this is an element of the matrix $\mathbf{A}$.

The subscripts i and j refer to the row and column, respectively, in which the element a_{ij} is to be found.

For example, if $\mathbf{C}$ is the 2×3 matrix:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -8 \\ -3 & 2 & 5 \end{pmatrix}$$

then $c_{12} = 0$ and $c_{23} = 5$.

If the 2×2 matrix $\mathbf{D}$ possessed unspecified elements, we could write it as:

$$\mathbf{D} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

20.3 Matrix Addition and Subtraction

20.3.1 Matrix Addition

It is only possible to add matrices together if they have *exactly* the same order, i.e. the same number of rows and the same number of columns.

We may therefore add a 4×2 matrix to another 4×2, but we cannot add it to a 2×2.

To perform matrix addition we simply add together the corresponding elements of each matrix.

For example, if:

$$\underline{E} = \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix}, \quad \underline{F} = \begin{pmatrix} -1 & 5 \\ 8 & -4 \end{pmatrix},$$

$$\underline{G} = \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix}, \quad \underline{H} = \begin{pmatrix} 0 & -7 & 2 \\ 9 & -1 & 6 \end{pmatrix}$$

then ...

$$\begin{aligned} \underline{E} + \underline{F} &= \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 5 \\ 8 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 5 + -1 & 3 + 5 \\ -7 + 8 & 2 + -4 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 8 \\ 1 & -2 \end{pmatrix} \end{aligned}$$

These could be added as both $\underline{E}$ and $\underline{F}$ are 2×2.

Exercises:

Determine:

1) $\underline{G} + \underline{H}$

2) $\underline{E} + \underline{H}$

Solutions:

1) Both $\underline{G}$ and $\underline{H}$ have the same order (2×3), so we can proceed:

$$\begin{aligned}
\underline{G} + \underline{H} &= \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix} + \begin{pmatrix} 0 & -7 & 2 \\ 9 & -1 & 6 \end{pmatrix} \\
&= \begin{pmatrix} -9+0 & -7+(-7) & 2+2 \\ 1+9 & 3+(-1) & -2+6 \end{pmatrix} \\
&= \begin{pmatrix} -9 & -14 & 4 \\ 10 & 2 & 4 \end{pmatrix}
\end{aligned}$$

2) $\underline{E}$ is a 2×2 matrix, while $\underline{H}$ is a 2×3 matrix, so this addition is an invalid operation.

20.3.2 Matrix Subtraction

The same general rules concerning the addition of matrices also applies to subtraction i.e. the two matrices involved must be of the same dimension.

Of course, when subtracting matrices, corresponding elements will undergo a subtraction rather than an addition.

Note, that just as in normal arithmetic, addition is **commutative** and subtraction is not, meaning that $\underline{A} + \underline{B} = \underline{B} + \underline{A}$, but it is not necessarily true that $\underline{A} - \underline{B} = \underline{B} - \underline{A}$

For example, using the matrices defined earlier (**E**, **F**, **G** and **H**):

$$\begin{aligned}
\underline{E} - \underline{F} &= \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 5 \\ 8 & -4 \end{pmatrix} \\
&= \begin{pmatrix} 5 - -1 & 3 - 5 \\ -7 - 8 & 2 - -4 \end{pmatrix} \\
&= \begin{pmatrix} 6 & -2 \\ -15 & 6 \end{pmatrix}
\end{aligned}$$

20.3.3 Summary

Only matrices with the same order can be added/subtracted.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \end{pmatrix} \quad \text{INVALID}$$

$$\begin{aligned}
\begin{pmatrix} 4 & -1 & -2 \\ -2 & 3 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 3 \\ 1 & -4 & -6 \end{pmatrix} &= \begin{pmatrix} 4-2 & -1-0 & -2-3 \\ -2-1 & 3-(-4) & 5-(-6) \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 & -5 \\ -3 & 7 & 11 \end{pmatrix}
\end{aligned}$$

20.4 Matrix Multiplication

There are three types of matrix multiplication:

- Multiplication of a matrix by a scalar.
- Multiplication of elements in one matrix by corresponding elements in another matrix.
- Multiplication of a matrix by another matrix.

20.4.1 Scalar Multiplication

To multiply a matrix by a scalar (a real or complex *number*, rather than a vector or matrix) simply by multiplying ("scaling") each element of the matrix by that scalar.

Scalar Multiplication

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

Example: $$-3 \begin{pmatrix} 2 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} -3 \times 2 \\ -3 \times 8 \\ -3 \times -5 \end{pmatrix} = \begin{pmatrix} -6 \\ -24 \\ 15 \end{pmatrix}$$

For a further example:

$$\begin{aligned} 5\underline{G} &= 5 \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 5 \times -9 & 5 \times -7 & 5 \times 2 \\ 5 \times 1 & 5 \times 3 & 5 \times -2 \end{pmatrix} \\ &= \begin{pmatrix} -45 & -35 & 10 \\ 5 & 15 & -10 \end{pmatrix} \end{aligned}$$

20.4.2 Element-wise Multiplication

Element-wise multiplication simply refers to the multiplying of corresponding elements in different matrices. Note that the matrices must be the same size. For example:

$$
\begin{aligned}
\underline{E} \,.\times \underline{F} &= \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} .\times \begin{pmatrix} -1 & 5 \\ 8 & -4 \end{pmatrix} \\
&= \begin{pmatrix} 5 \times -1 & 3 \times 5 \\ -7 \times 8 & 2 \times -4 \end{pmatrix} \\
&= \begin{pmatrix} -5 & 15 \\ -56 & -8 \end{pmatrix}
\end{aligned}
$$

20.4.3 Matrix Multiplication

Matrix multiplication is a **non-commutative** operation. This means that $\underline{A} \times \underline{B}$ is *not* equivalent to $\underline{B} \times \underline{A}$ and does not necessarily yield the same result.

> The direction of matrix multiplication *can not be changed*.

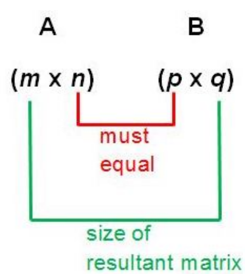
In fact, one direction might not even exist whilst the other does!

Thus, it is necessary to distinguish between pre-multiplying and post-multiplying, e.g. given the matrices $\underline{X}$ and $\underline{A}$:

- We can pre-multiply $\underline{X}$ by $\underline{A}$ to get $\underline{AX}$
- or post-multiply to get $\underline{XA}$

As with addition/subtraction, multiplication can only be performed if the two matrices involved have acceptable dimensions.

> **Criterion:** The number of columns in the first matrix must match the number of rows in the second.



If this is satisfied, the order of the result is given by the remaining dimensions - the same number of rows as the first matrix and columns as the second matrix.

Then to multiply the matrices, imagine setting the rows of the first upon the columns of the second. For example, when computing $\underline{AB}$, we find the element in the i^{th} row and the j^{th} column of $\underline{AB}$ by multiplying the i^{th} row of $\underline{A}$ by the j^{th} column of $\underline{B}$.

$$\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \quad \underset{2\times 1}{\begin{pmatrix} 5 \\ 6 \end{pmatrix}} = \begin{pmatrix} 1\times 5 + 2\times 6 \\ 3\times 5 + 4\times 6 \end{pmatrix} = \underset{2\times 1}{\begin{pmatrix} 17 \\ 39 \end{pmatrix}}$$

Example:

Calculate $\underline{E} \times \underline{G}$:

$$\begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} \times \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix}$$

2×2 $\qquad\qquad$ 2×3

The resultant matrix therefore exists and will be of the form:

$$\underline{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \end{pmatrix}$$

2×3

Hence,

$$\underline{E} \times \underline{G} = \begin{pmatrix} {\color{red}5} & {\color{red}3} \\ -7 & 2 \end{pmatrix} \begin{pmatrix} {\color{red}-9} & -7 & 2 \\ {\color{red}1} & 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} {\color{red}(5 \times -9) + (3 \times 1)} & & \\ & & \end{pmatrix}$$

$$\underline{E} \times \underline{G} = \begin{pmatrix} {\color{blue}5} & {\color{blue}3} \\ -7 & 2 \end{pmatrix} \begin{pmatrix} -9 & {\color{blue}-7} & 2 \\ 1 & {\color{blue}3} & -2 \end{pmatrix}$$

$$= \begin{pmatrix} {\color{red}(5 \times -9) + (3 \times 1)} & {\color{blue}(5 \times -7) + (3 \times 3)} & \\ & & \end{pmatrix}$$

$$\underline{E} \times \underline{G} = \begin{pmatrix} {\color{green}5} & {\color{green}3} \\ -7 & 2 \end{pmatrix} \begin{pmatrix} -9 & -7 & {\color{green}2} \\ 1 & 3 & {\color{green}-2} \end{pmatrix}$$

$$= \begin{pmatrix} {\color{red}(5 \times -9) + (3 \times 1)} & {\color{blue}(5 \times -7) + (3 \times 3)} & {\color{green}(5 \times 2) + (3 \times -2)} \\ & & \end{pmatrix}$$

$$\underline{E} \times \underline{G} = \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} (5 \times -9) + (3 \times 1) & (5 \times -7) + (3 \times 3) & (5 \times 2) + (3 \times -2) \\ (-7 \times -9) + (2 \times 1) & & \end{pmatrix}$$

$$\underline{E} \times \underline{G} = \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} (5 \times -9) + (3 \times 1) & (5 \times -7) + (3 \times 3) & (5 \times 2) + (3 \times -2) \\ (-7 \times -9) + (2 \times 1) & (-7 \times -7) + (2 \times 3) & \end{pmatrix}$$

$$\underline{E} \times \underline{G} = \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} (5 \times -9) + (3 \times 1) & (5 \times -7) + (3 \times 3) & (5 \times 2) + (3 \times -2) \\ (-7 \times -9) + (2 \times 1) & (-7 \times -7) + (2 \times 3) & (-7 \times 2) + (2 \times -2) \end{pmatrix}$$

$$\underline{E} \times \underline{G} = \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} (5 \times -9) + (3 \times 1) & (5 \times -7) + (3 \times 3) & (5 \times 2) + (3 \times -2) \\ (-7 \times -9) + (2 \times 1) & (-7 \times -7) + (2 \times 3) & (-7 \times 2) + (2 \times -2) \end{pmatrix}$$

$$= \begin{pmatrix} -42 & -26 & 4 \\ 65 & 55 & -18 \end{pmatrix}$$

20.4.4 Exercises

1) Let,

$$\underline{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \qquad \underline{C} = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$$

Calculate $\underline{BC}$ and $\underline{CB}$ if they exist.

2) Let,

$$\underline{A} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \qquad \underline{D} = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

Calculate $\underline{AD}$ if it exists.

Solutions:

1)

$$\underline{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \qquad \underline{C} = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$$

As $\underline{B}$ is a 2×1 and $\underline{C}$ is a 2×2 matrix, $\underline{B} \times \underline{C}$ does not exist as the columns of $\underline{B}$ do not match the number of rows of $\underline{C}$.

However, $\underline{C} \times \underline{B}$ does exist, and the result will be another 2×1 matrix:

$$\underline{CB} = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \times 3 + 2 \times -2 \\ 4 \times 3 + 5 \times -2 \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$$

2)

$$\underline{A} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \qquad \underline{D} = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

The result $\underline{A} \times \underline{D}$ will be another 2×2 matrix:

$$\begin{aligned} \underline{AD} &= \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 \times 3 + 0 \times 0 & 2 \times 1 + 0 \times (-2) \\ -1 \times 3 + 1 \times 0 & -1 \times 1 + 1 \times (-2) \end{pmatrix} \\ &= \begin{pmatrix} 6 + 0 & 2 + 0 \\ -3 + 0 & -1 - 2 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ -3 & -3 \end{pmatrix} \end{aligned}$$

21 Using matrices to solve systems of linear simultaneous equations

21.1 Learning Outcomes

- Perform additional matrix operations such as determining the inverse, transpose and determinant (if they exist).

- Solve pair of linear simultaneous linear equations using matrices.

21.2 Transpose of a Matrix

Taking the transpose of a matrix causes the 1^{st} row to become the first column, the 2^{nd} row to become the second column, etc.

For example, if

$$\underline{M} = \begin{pmatrix} 1 & -1 \\ 3 & 0 \\ 2 & -8 \end{pmatrix}$$

Then

$$\underline{M}^{\mathrm{T}} = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & -8 \end{pmatrix}$$

21.3 Determinant

Square matrices (with dimensions $n \times n$) have a property called the **determinant**.

The determinant of matrix $\underline{A}$ can be denoted by $det(\underline{A})$ or $|\underline{A}|$.

For a 2×2 matrix $\underline{A}$, the determinant is very simple to calculate by multiplying the diagonal entries:

Determinant of a 2×2 matrix:

$$det(\underline{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: determinant of a 2×2 matrix

Given the square matrix

$$\underline{B} = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$$

The determinant is given by:

$$\begin{aligned} det(\underline{B}) &= 3 \times 2 - (-1) \times 4 \\ &= 6 + 4 \\ &= 10 \end{aligned}$$

21.4 The Identity Matrix and Inverse Matrices

Identity matrices are square matrices in which all elements are zero except for the elements on the leading diagonal; these are all 1, e.g.

$$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;\; or \;\; \underline{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Pre/post-multiplying by $\underline{\mathrm{I}}$ has no impact, i.e.

$$\underline{A}\,\underline{I} = \underline{A} \;\; \text{and} \;\; \underline{I}\,\underline{A} = \underline{A}$$

For a **square** matrix $\underline{A}$, there may exist an inverse matrix $\underline{A}^{-1}$

Inverse Matrix:

$$\underline{AA}^{-1} = \underline{I} \quad \text{and} \quad \underline{A}^{-1}\underline{A} = \underline{I}$$

So an inverse matrix is analagous to the reciprocal of a number - it's what you multiply by to get back to 1 (or the identity):

$$5 \times \frac{1}{5} = 1$$

$$\underline{A} \times \underline{A}^{-1} = \underline{I}$$

21.4.1 Calculating the inverse of a 2×2 matrix

For a general 2×2 square matrix $\underline{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

Inverse of a 2×2 matrix

$$\underline{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{or} \quad \underline{A}^{-1} = \frac{1}{|\underline{A}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If the determinant of a square matrix is equal to zero, then that matrix has no inverse!

Example: Inverse of a 2×2 matrix:

To find (if it exists) the inverse of 2×2 square matrix $\underline{A}$:

$$\underline{A} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

First obtain the determinant:

$$det(\underline{A}) = (1)(2) - (-1)(0) = 2$$

Then as the determinant is non-zero, the inverse exists and is:

$$\underline{A}^{-1} = \frac{1}{det(\underline{A})} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}$$

21.4.2 Exercise: Inverse of a 2×2 matrix

For the following square matrices, find the determinant and the inverse matrix if it exists:

$$\underline{B} = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

$$\underline{C} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Solutions:

$$\underline{B} = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

$$\underline{B}^{-1} = \frac{1}{(1)(2) - (0)(-3)} \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3/2 & 1/2 \end{pmatrix}$$

$$\underline{C} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$det(\underline{C}) = (1)(-1) - (1)(-1) = 0$$

Hence $\underline{C}$ has zero determinant $\Longrightarrow$ its inverse does not exist.

21.5 Simultaneous Equations

21.5.1 Motivation

Many engineering problems can be modelled as a system of simultaneous equations.

For example, let's say that there are two materials $\underline{A}$ and $\underline{B}$, whose densities are unknown. You have two samples of different composites of these: one is 15% $\underline{A}$ and 85% $\underline{B}$ and has a density of $1kgm^{-3}$, while the other is 40% $\underline{A}$ and 60% $\underline{B}$ but twice as dense. This could be written as:

$$0.15\underline{A} + 0.85\underline{B} = 1$$

$$0.4\underline{A} + 0.6\underline{B} = 2$$

We wish to determine the densities of the constituents $\underline{A}$ and $\underline{B}$.

This is an example of a pair of simultaneous linear equations. Another example:

$$3x + 2y = 16$$

$$-x + 4y = 7$$

We will learn to solve them (i.e. find the unique values of x and y for which both equations are true) using a matrix method.

21.5.2 Matrix method for solution of simultaneous linear equations

Given a pair of simultaneous equations, ensure they are in this form first:

$$ax + by = p$$

$$cx + dy = q$$

1. Then write the pair of equations as a matrix equation:

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\underline{A}\underline{X} = \underline{B}$$

2. So the square matrix of coefficients is $\underline{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the vector $\underline{X}$ contains x and y which we want to find.

3. Calculate the inverse matrix $\underline{A}^{-1}$

4. Pre-multiply both sides by the inverse matrix to obtain $\underline{X}$:

$$\underline{A}\underline{X} = \underline{B} \quad \Longrightarrow \quad \underline{A}^{-1}\underline{A}\underline{X} = \underline{A}^{-1}\underline{B} \quad \Longrightarrow \quad \underline{X} = \underline{A}^{-1}\underline{B}$$

5. From the entries in vector $\underline{X}$, read off the values of x and y.

6. Substitute the values of x and y back into the original equations to verify solutions.

Example 1

Solve for x and y:

$$5x + 2y = 10$$

$$4x - 3y = 14$$

Re-writing this as a matrix equation,

$$\begin{pmatrix} 5 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

so we have $\underline{AX} = \underline{B}$, where

$$\underline{A} = \begin{pmatrix} 5 & 2 \\ 4 & -3 \end{pmatrix}, \quad \underline{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

Then,

$$\underline{A}^{-1} = \frac{1}{(5)(-3) - (2)(4)} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix} = \frac{-1}{23} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix}$$

and so

$$\underline{X} = \underline{A}^{-1}\underline{B} = \frac{-1}{23} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 10 \\ 14 \end{pmatrix} = \begin{pmatrix} 58/23 \\ -30/23 \end{pmatrix}$$

Thus we find $x = 58/23$ and $y = -30/23$.

Example 2:

Solve for x and y:

$$3x = 7 + 5y$$

$$4y + 2x = 20$$

First, re-write both of these in a consistent format:

$$3x - 5y = 7$$

$$2x + 4y = 20$$

Re-writing this as a matrix equation,

$$\begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 20 \end{pmatrix}$$

so we have $\underline{AX} = \underline{B}$, where

$$\underline{A} = \begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix}, \qquad \underline{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \underline{B} = \begin{pmatrix} 7 \\ 20 \end{pmatrix}$$

Then,

$$\underline{A}^{-1} = \frac{1}{(3)(4) - (-5)(2)} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix}$$

and so

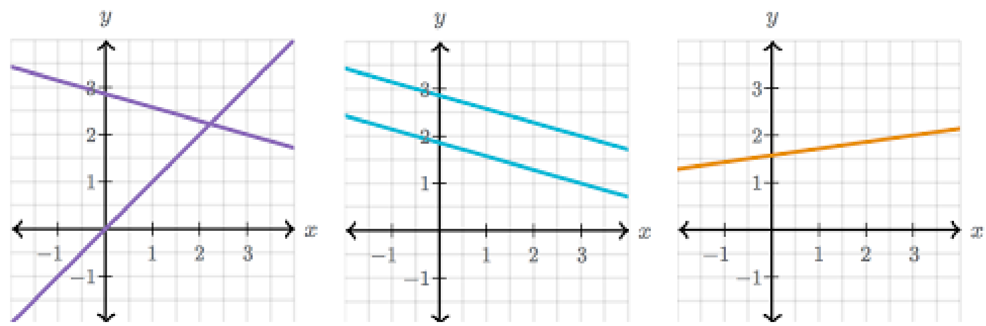
$$\underline{X} = \underline{A}^{-1}\underline{B} = \frac{1}{22} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ 20 \end{pmatrix} = \begin{pmatrix} 64/11 \\ 23/11 \end{pmatrix}$$

Thus we find $x = 64/11$ and $y = 23/11$.

21.5.3 Special cases

A linear equation $ax + by = d$ can be re-written in the form $y = mx + c$. In other words, we have been trying to find the co-ordinates of the point where two straight lines intersect.

What if the pair of lines are parallel or actually the same?



In these cases (zero solutions or infinitely many solutions), the matrix of coefficients will be **uninvertible** (its determinant $= 0$).

If the matrix of coefficients has determinant $= 0$, examine the two equations and determine if they are the same equation (infinitely-many solutions), or if they are contradictory (zero solutions).

$$x - 3y = 10$$

$$2x - 6y = 20$$

$$-2x + y = 3$$

$$4x - 2y = 17$$

The first pair are the **same**, and the second pair are **contradictory**.

21.5.4 Exercises

Use the matrix method to solve the following systems of simultaneous equations:

(a)
$$7x + 2y = 4$$
$$3x - 5y = 6$$

(b)
$$5x - 2y = 10$$
$$3x + 4y = 6$$

(c)
$$2x + 8y = 12$$
$$-3x - 2y = 12$$

22 Statistics I

22.1 Learning Outcomes

- Use EXCEL to obtain measures of central tendency for a data set: mean, median and mode.

- Use EXCEL to obtain measures of dispersion for a data set: range and standard deviation.

22.2 Introduction

Why study statistics?

Whenever information is gathered about a process, the results of an experiment, financial patterns, the characteristics of standardised machine parts, etc, then it will be necessary to perform statistical calculations on those data in order to be able to interpret their meaning and infer conclusions.

22.3 Types of data

- **Qualitative** - nonnumeric data such as "favourite colour", "hairstyle", "blood type".
- **Quantitative** - data that can be represented by a number. For example, "height", "number of family members".

Quantitative data are a collection of n measurements of a variable x, often written as $x_1, x_2, x_3, \ldots, x_n$. There are two types:

- **Discrete** – a variable that can be counted or that has a fixed set of values. For example, the number of visitors to a park (you can't have half or 0.2 of a person).
- **Continuous** – a variable that can be measured on a continuous scale. For example, "temperature" or "height".

22.4 Measures of Central Tendency

There are three measures of central tendency:

- Mean: what we usually refer to as the average. EXCEL command: `=AVERAGE(...)`

  $$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

- Median: this is the middle value in an ordered set of data: $\left(\frac{(n+1)}{2}\right)^{\text{th}}$ data point. EXCEL command: `=MEDIAN(...)`

- Mode: the most often occurring value. EXCEL: `=MODE(...)`

22.4.1 Example 1

The number of particles emitted by a radioactive source and detected by a Geiger counter in 40 consecutive period of 1 minute were measured/recorded as follows:

1	0	2	1	3	4	0	1	5	2
2	1	1	3	2	1	3	2	1	0
4	3	2	1	0	2	1	4	2	3
3	1	4	2	3	1	2	3	0	5

Summarise the data into a frequency table and find the mean, median and mode.

Solution:

x_i	Frequency f	fx_i	Cumulative frequency
0	5	0	5
1	11	11	16
2	10	20	26
3	8	24	34
4	4	16	38
5	2	10	40

The mean:

$$\bar{x} = \frac{\sum f x_i}{\sum f} = \frac{81}{40} = 2.025$$

The median is the 20.5^{th} value, thus 2 from the CF.

The mode is 1.

22.5 Measures of Dispersion

There are several ways to measure how spread out the data is around the average:

- Range
- Interquartile range (IQR)
- Standard deviation

22.5.1 Range and Quartiles

The range of the data is simply the difference between the largest and smallest values.

$$\text{Range} = \text{maximum value} - \text{minimum value}$$

EXCEL command: `=MAX(...)-MIN(...)`

22.5.2 Quartiles and IQR

Data can also be characterised by the upper and lower quartiles. Arrange the data values in increasing order, then ...

The lower quartile (L_{25}) is the median of the lower half of the data.
The upper quartile (U_{25}) is the median of the upper half.

The difference between them is the **interquartile range** (IQR):

$$IQR = U_{25} - L_{25}$$

EXCEL command: `=QUARTILE(...,3)-QUARTILE(...,1)`

22.5.3 Example: Range and Quartiles

Example:

$$1 \quad 1 \quad 2 \quad 3 \quad 3 \quad 3 \quad 4 \quad 5 \quad 5 \quad 7 \quad 7 \quad 8 \quad 10 \quad 19$$

The range is the largest value minus the lowest value: $19 - 1 = 18$

There are 14 data points, so the median of the first 7 is the 4th value. Thus: $L_{25} = 3$

The median of points 8-14 is the 11th value. Thus: $U_{25} = 7$

And so we have: $IQR = U_{25} - L_{25} = 7 - 3 = 4$

22.5.4 Population and sample

Population standard deviation: the data's average deviation from the mean (if one has access to **all** data).

$$\sigma = \sqrt{\frac{\sum(\overline{x} - x)^2}{n}} \quad \text{ungrouped data}$$

OR

$$\sigma = \sqrt{\frac{\sum f(\overline{x} - x)^2}{\sum f}} \quad \text{grouped data}$$

EXCEL command: `=STDEV.P(...)`

Sample standard deviation: the data's average deviation from the mean (if one has access to a **sample** of all data).

$$\sigma = \sqrt{\frac{\sum(\overline{x} - x)^2}{n-1}} \quad \text{ungrouped data}$$

OR

$$\sigma = \sqrt{\frac{\sum f(\overline{x} - x)^2}{\sum f - 1}} \quad \text{grouped data}$$

EXCEL command: `=STDEV(...)`

22.5.5 Return to Example 1

Determine the sample standard deviation for example 1 data using the step-by-step calculation and the in-built Excel function.

In this case:

$$n = 40$$

and we already determined that the mean of this sample is:

$$\bar{x} = 2.025$$

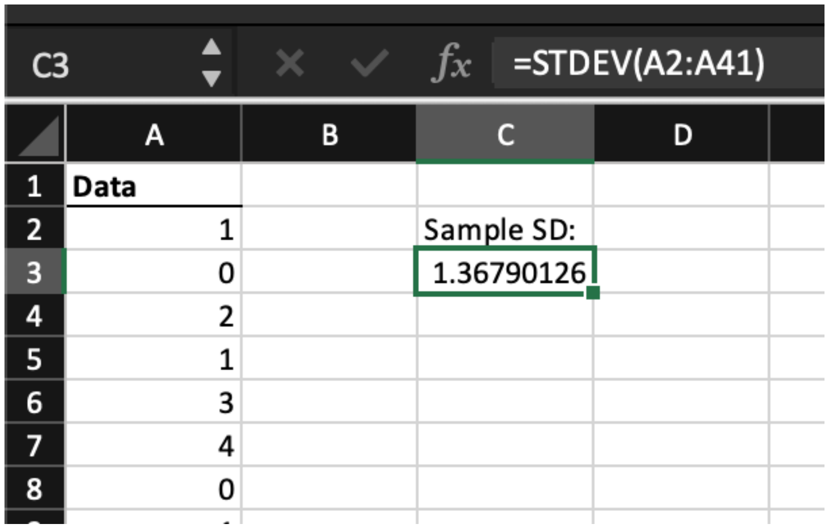
Hence:

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	f	$f(x_i - \bar{x})^2$
0	-2.025	4.1006	5	20.5030
1	-1.025	1.0506	11	11.5566
2	-0.025	0.0006	10	0.0060
3	0.975	0.9506	8	7.6048
4	1.975	3.9006	4	15.6024
5	2.975	8.8506	2	17.7013
				$\sum(x_i - \bar{x})^2 = 72.9741$

Then the sample standard deviation is:

$$\sigma = \sqrt{\frac{72.9741}{40 - 1}} = 1.368$$

In Excel:


C3
=STDEV(A2:A41)
A
B
C
D
1
Data
2
1
Sample SD:
3
0
1.36790126
4
2
5
1
6
3
7
4
8
0

23 Statistics II

23.1 Learning Outcomes

- Working with grouped data.
- Visualising grouped data using histograms in EXCEL.
- Fitting simple curves and trendlines using EXCEL.

23.2 Grouped Data

When there are many different measurements with few/no repetition or just a large number of data, then it is only possible to make any real sense of the data if they are grouped together in intervals.

In this case, the formulae for calculating values such as the mean are slightly different:

$$\bar{x} = \frac{\sum f x_i}{\sum f} \quad \text{(grouped data)}$$

23.2.1 Example 1: Grouped, continuous data

The heights (in cm) of 25 people of the same age were measured. The following table shows the data:

180.84	164.87	167.77	167.78	174.39
176.14	176.87	159.57	164.73	174.51
168.47	180.64	170.04	162.71	174.02
171.91	169.31	171.68	152.49	177.58
172.03	169.68	161.87	165.48	181.90

Summarise the data into a frequency table and find the mean.

Solution:

Group	Tally	Freq. f	Midpoint x_i	fx_i	C. f
150-155	I	1	152.5	152.5	1
155-160	I	1	157.5	157.5	2
160-165	IIII	4	162.5	650	6
165-170	~~IIII~~ I	6	167.5	1005	12
170-175	~~IIII~~ II	7	172.5	1207.5	19
175-180	III	3	177.5	532.5	22
180-185	III	3	182.5	547.5	25
		$\sum f = 25$		$\sum fx_i = 4257.5$	

$$\bar{x} = \frac{\sum fx_i}{\sum f} = \frac{4252.5}{25} = 170.1$$

The median is the 13^{th} value, thus 172.5 from the CF. The modal group is 170-175.

23.3 Visualising Data

Graphs can be used to quickly determine key characteristics of the data under analysis. Some possible graphs are:

- Bar charts and pie charts (discrete or qualitative data).
- Histograms/Frequency distributions (continuous, quantitative data).
- Frequency polygons.
- Cumulative frequency curves.

23.3.1 Creating a Histogram in EXCEL

1. Ensure that the Analysis Toolpak is enabled. In Windows 10, go to "File>Options>Add-ins" and ensure "Analysis Toolpak" is selected. For OSX, go to "Tools > Excel Add-ins . . . "

2. Choose a bin width suitable for your data, then create a column containing the upper limit of all of the bins.

3. Select the Data tab, then "Analysis" and choose "Histogram" from the list. A pop-up box will appear.

4. For "Input range", select *all* of the raw data points.

5. For "Bin range", select the cells containing your upper limits.

6. Click on "Ouput range" and choose an area of the worksheet that will not interfere with your raw data. Excel will create a frequency table (tally chart) here.

7. Make sure "Chart output" is ticked.

If we are given a set of ungrouped data, how many bins should we group it into for calculating statistics or when creating a histogram?

There are several rules that we can use to determine a sensible number of M bins for a set of N datapoints, such as:

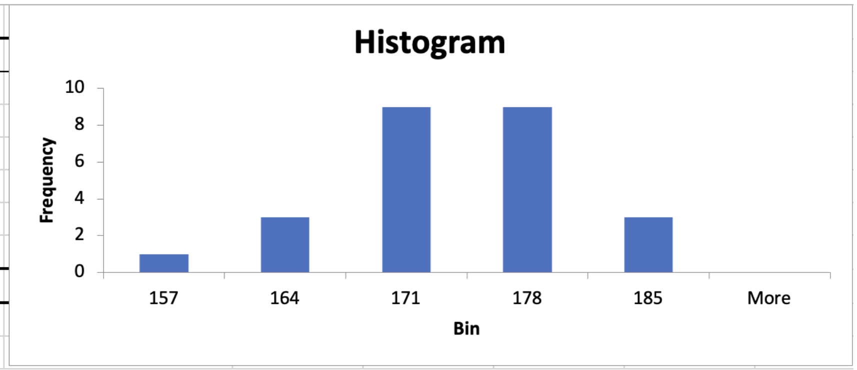
- The square root rule: $M = \lceil \sqrt{N} \rceil$
- Sturge's formula: $M = \lceil \log_2(N) + 1 \rceil$

In each case, the "ceiling" function $\lceil \quad \rceil$ indicates that the result should be rounded *up* to the nearest integer.

23.3.2 Example 2

Using the procedure outlined previously, we can produce a histogram for Example 1. There are $N = 25$ datapoints, so using the square root rule we group them in $M = 5$ bins.

Bin	Frequency
157	1
164	3
171	9
178	9
185	3
More	0



23.4 Curve Fitting

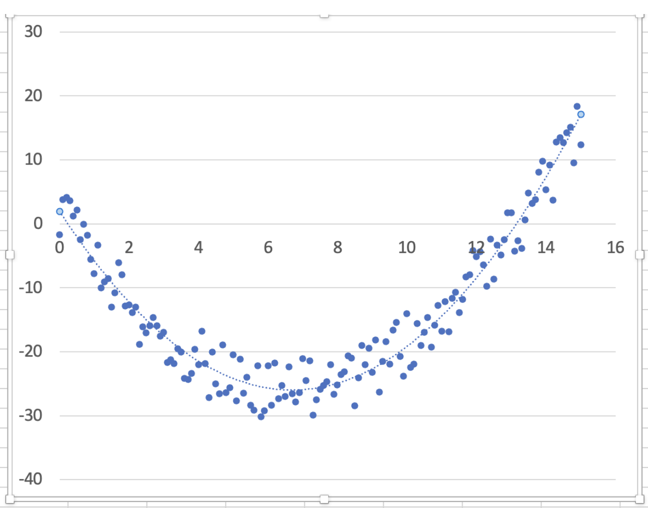
Often we can obtain a set of experimental data, and hypothesise that the relationship between the independent variable (that we control) and the dependent variable (that we measure) is described by some function. If we could determine the exact relationship, we could make further predictions by extrapolating the fitted curve.

Once we have decided on a general form of the relationship between our variables (e.g. linear, quadratic, exponential, power law), curve fitting is the process of **finding the set of parameter values** that best fits the set of experimental data.

23.4.1 Example 3

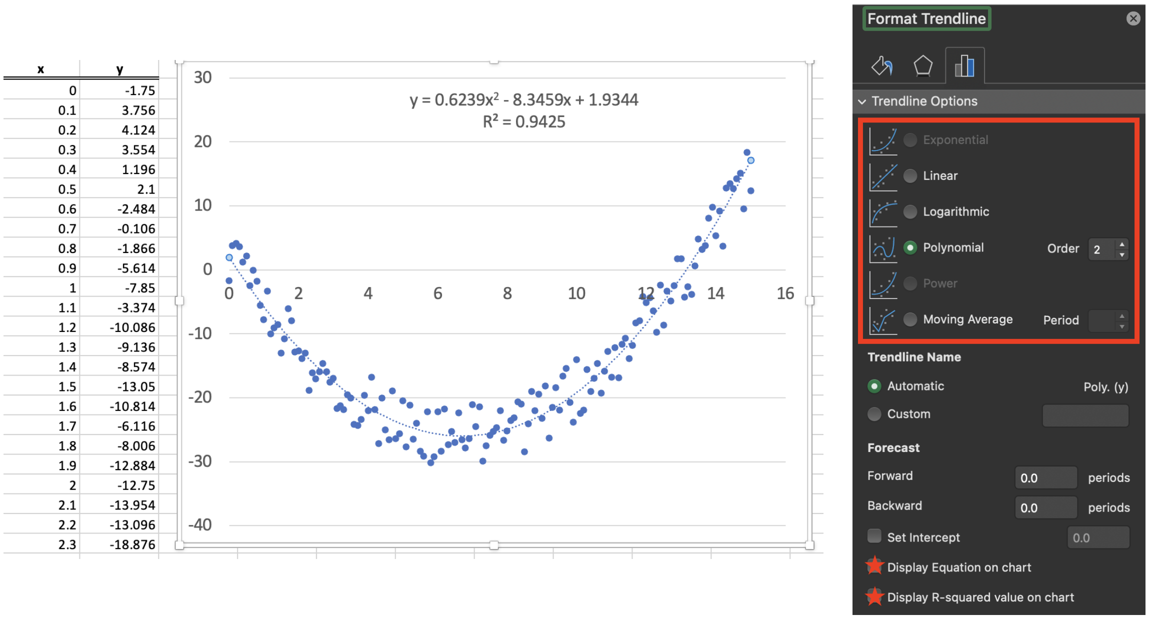
Suppose we have the set of data shown below:

x	y
0	-1.75
0.1	3.756
0.2	4.124
0.3	3.554
0.4	1.196
0.5	2.1
0.6	-2.484
0.7	-0.106
0.8	-1.866
0.9	-5.614
1	-7.85
1.1	-3.374
1.2	-10.086
1.3	-9.136
1.4	-8.574
1.5	-13.05
1.6	-10.814
1.7	-6.116
1.8	-8.006
1.9	-12.884
2	-12.75
2.1	-13.954
2.2	-13.096
2.3	-18.876



Perhaps a quadratic function $y = ax^2 + bx + c$ would fit. But what values of a, b and c

would result in the *best* fit?



x	y
0	-1.75
0.1	3.756
0.2	4.124
0.3	3.554
0.4	1.196
0.5	2.1
0.6	-2.484
0.7	-0.106
0.8	-1.866
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1.1	-3.374
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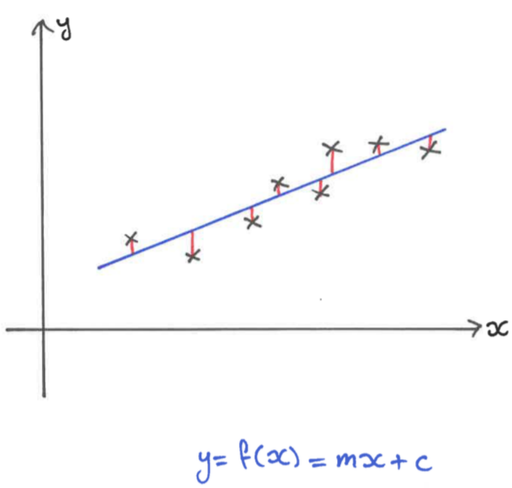
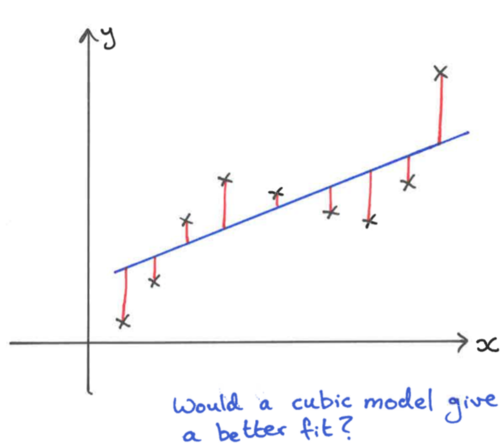
1.8	-8.006
1.9	-12.884
2	-12.75
2.1	-13.954
2.2	-13.096
2.3	-18.876

With the trendline tool we can specify fitting a 2nd-order polynomial (a quadratic function). The best such function is $y = 0.6239x^2 - 8.3459x + 1.9344$

23.4.2 R^2 and goodness-of-fit

In harder cases, we could choose several different models and fit the best parameter choices in each case.





How would we know which model described the data best by giving the closest fit?

To quantify the "goodness of fit" for each model, we can calculate the R^2 value, also called the coefficient of determination.

> Calculating R^2:
>
> For a set of N data points (x_i, y_i), to which a model is fitted given by $y = f(x)$, we calculate R^2 using:
>
> $$R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$$
>
> Where: $$SS_{res} = \sum_{i=1}^{N} \big(y_i - f(x_i)\big)^2 \quad \text{and} \quad SS_{tot} = \sum_{i=1}^{N} \big(y_i - \bar{y}\big)^2$$

How then can we interpret the resulting value of R^2?

> If R^2 is equal to 1, it means that the curve fits the data perfectly.
>
> A smaller value (nearer to zero) indicates a poorer fit.

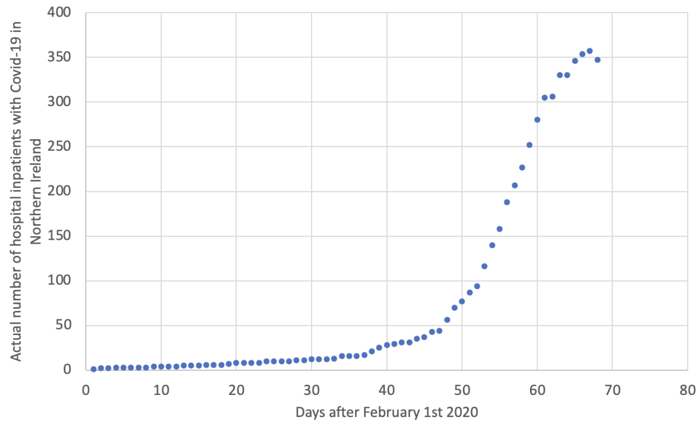
Given a data set, we can create a scatter plot, and undertake a curve-fitting procedure for each reasonable model to find the *best version of that model*. Then, compare the resulting R^2-values and determine which was the best overall best.

EXCEL's ability for curve fitting has limitations. Only certain simple functions can be fitted, and some cannot be fitted if there are zeros or negative values in the data.

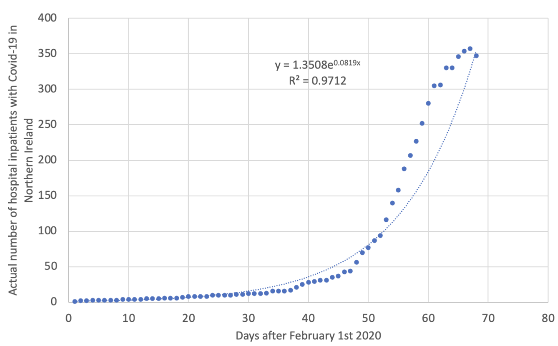
23.4.3 Example - curve fitting and forecasting

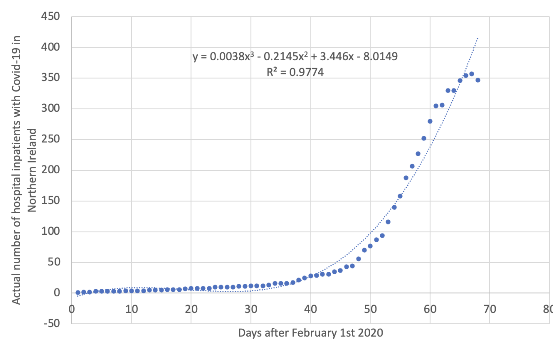
If we could use curve fitting to accurately fit a model to the data documenting the spread of coronavirus in the UK as it emerges, we may be able to estimate demand for healthcare and where and when to allocate resources.

Consider this data, which shows the number of people in hospital in Northern Ireland with Covid-19 between February 1st and April 7th 2020.



From the options available, the most reasonable (without using a very high-order polynomial) seem to be an exponential or a cubic function. They both fit the data very well ($R^2 \approx 0.97$).





Should we use these models to forecast hospitalisations in: (a) one week; (b) four months; (c) two years after the final datapoint?