MMaD: Lecture 8 handout

Functions of two variables and partial differentiation

Previously, we considered $y = f(x)$, where y is a function of (i.e. its value depends on) only one variable x. This can be represented in two dimensions by a curve.

However, a function can depend on **multiple** variables. This can be written as:

$$z = f(x, y)$$

and represented as a surface plot in three dimensions, where the height z depends on both the x and y co-ordinates.

To think about the gradient of this 3-d surface, we use **partial differentiation**.

A "curly dee", ∂, is used to distinguish between partial and ordinary differentiation.

Second order partial derivatives

A partial derivative may be differentiated partially again to give higher order partial derivatives.

If $f = f(x, y)$ is a function of two variables, then:

- Differentiating $\frac{\partial f}{\partial x}$ w.r.t. x gives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

- Differentiating $\frac{\partial f}{\partial y}$ w.r.t. y gives:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

- Differentiating $\frac{\partial f}{\partial x}$ w.r.t. y gives:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

- Differentiating $\frac{\partial f}{\partial y}$ w.r.t. x gives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Note that for any function f, it is always the case that:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Example 6 (Partial differentiation)

If

$$z(x,y) = 4x^2y^3 - 2x^3 + 7y^2$$

then find:

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y}$$