MMaD: Lecture 9 handout

Propagation of uncertainty

In general, if

$$y = f(x_1, x_2, \ldots, x_n)$$

where $x_1, \ldots, x_n$ are independent random variables with variances $\sigma^2_{x_1}, \ldots, \sigma^2_{x_n}$, then the variance of y is given by:

$$\sigma^2_y = \left(\frac{\partial y}{\partial x_1}\right)^2 \sigma^2_{x_1} + \cdots + \left(\frac{\partial y}{\partial x_n}\right)^2 \sigma^2_{x_n}$$

Evaluated at the mean values of $x_1, \ldots, x_n$.

Example 3 (Propagation of uncertainty)

A company manufactures roadside bollards by fitting a partially-sheared sphere of radius $t$cm and volume $\frac{5}{4}\pi t^3$ to a cylinder of height $h$cm and radius $r$cm. Thus, the cylinder has volume given by $\pi r^2 h$ and the overall volume of the bollard is:

$$V = \pi r^2 h + \frac{5}{4}\pi t^3$$

The components are manufactured with the variables h, r and t obeying approximately normal distributions with mean: $h = 50$cm, $r = 6$cm and $t = 10$cm, and variances:

$$\sigma^2_h = 4cm^2 \qquad \sigma^2_r = 0.5cm^2 \qquad \sigma^2_t = 1cm^2$$

What is the mean and variance of the volume of the bollards?

Rates of change for multivariate functions

Functions of **two** variables:

If $z = f(x, y)$ and x and y are functions of t (that is, $x = x(t)$ and $y = y(t)$) then z is ultimately a function of t only, and:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Functions of **three** variables:

If $w = f(x, y, z)$ and $x = x(t)$, $y = y(t)$ and $z = z(t)$ then w is ultimately a function of t only, and:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

Example 1 (rates of change)

The height of a right circular cone is increasing at $3mms^{-1}$ and its radius is decreasing at $2mms^{-1}$. Determine (to 3 s.f.) the rate at which the volume is changing (in cm^3s^{-1}) when the height is 3.2 cm and the radius is 1.5 cm.

Note that the volume of a right circular cone is given by:

$$V = \frac{1}{3}\pi r^2 h$$

Example 2 (rates of change)

A rectangular box has sides of length x cm, y cm and z cm. Sides x and z are expanding at rates of $3mms^{-1}$ and $5mms^{-1}$, respectively and side y is contracting at a rate of $2mms^{-1}$. Determine the rate of change of volume when x is 3 cm, y is 1.5 cm and z is 6 cm.

Note that the volume of a cuboid is given by:

$$V = xyz$$