Revision of differentiation

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Contents

Today we will cover. . .

- Differentiation as gradient and rate-of-change.
- Standard rules of differentiation.
- Product rule.
- Chain rule.
- Higher-order derivatives.
- Maxima and minima of functions of one variable.

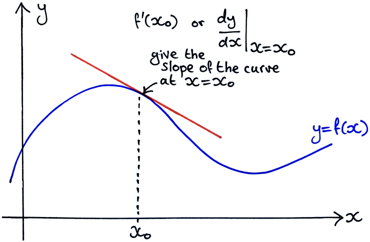
This will be a brief summary, as these topics should be familiar to all students. If you need extra practice, make use of the plentiful material and examples provided on the module Blackboard site.

Differentiation

Given a function $y = f(x)$, the "derivative of y with respect to x" can be written as:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{or} \quad y'$$

This yields the gradient (slope) of the function, which is equivalent to its rate of change as x changes.



The derivative with respect to time t, may also be expressed as $\dot{y}$.

Standard rules of differentiation

For constants a and n:

$$y = ax^n \implies \frac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}$$
$$y = ax \implies \frac{\mathrm{d}y}{\mathrm{d}x} = a$$
$$y = a \implies \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$y = \mathrm{e}^x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$$
$$y = \sin(x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x)$$
$$y = \cos(x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\sin(x)$$
$$y = \ln(x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

Standard rules of differentiation - Examples

Examples:

Differentiate:

$$y = 5x^2, \qquad y = -3t^{10}$$

$$x = \sin(t), \qquad y = 2\sqrt{x}$$

$$y = -\frac{4}{x^3}$$

Standard rules of differentiation - Solutions

$$\frac{\mathrm{d}}{\mathrm{d}x}(5x^2) = 5 \times 2x^{2-1} = 10x$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(-3t^{10}) = -3 \times 10t^{10-1} = -30t^9$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sin(t)) = \cos(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(2\sqrt{x}) = \frac{\mathrm{d}}{\mathrm{d}x}(2x^{1/2}) = 2 \times \frac{1}{2}x^{1/2-1} = 1x^{-1/2} = \frac{1}{\sqrt{x}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{4}{x^3}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(-4x^{-3}) = (-4) \times (-3)x^{-3-1} = 12x^{-4} = \frac{12}{x^4}$$

Linearity

Given two functions f and g,

$$y = f(x) \pm g(x) \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} \pm \frac{\mathrm{d}g}{\mathrm{d}x} = f'(x) \pm g'(x)$$

If a is a constant, then:

$$y = af(x) \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = a\frac{\mathrm{d}f}{\mathrm{d}x} = af'(x)$$

Example:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^4 + 13\sin(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(x^4) + 13\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = 4x^3 + 13\cos(x)$$

Gradient of a curve

Differentiating the formula for a curve yields a formula for the gradient of that curve at any point. In most cases, this gradient is dependent on x. Therefore, to find the gradient of a curve at a particular point, differentiate first and then substitute in the particular value of x to the formula obtained for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Example:

Find the gradient of the curve $y = x^2 + 4x - 7$ at the point $(2, 5)$.

Solution:

$$y'(x) = 2x + 4$$

$$y'(x = 2) = 2(2) + 4 = 8$$

The Product Rule

Suppose $f(x)$ is the product of two functions (i.e. the result of multiplying them together):

$$f(x) = u(x) \cdot v(x)$$

Then the derivative is given by the Product Rule.

Product rule

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = v(x) \cdot \frac{\mathrm{d}u(x)}{\mathrm{d}x} + u(x) \cdot \frac{\mathrm{d}v(x)}{\mathrm{d}x}$$

This may also be written as:

$$(uv)' = vu' + uv'$$

The Product Rule

Example:

Differentiate $y = x^2 \sin(x)$

Solution:

$$\text{Let } \; u = x^2 \; \text{ and } \; v = \sin(x)$$

Then

$$u' = 2x \; \text{ and } \; v' = \cos(x)$$

Thus

$$\begin{aligned} y' = (uv)' &= u \cdot v' + u' \cdot v \\ &= x^2 \cos(x) + 2x \sin(x) \end{aligned}$$

The Product Rule

Example:

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = e^x \cos(x)$

Solution:

$$\text{Let } u = e^x \text{ and } v = \cos(x)$$

Then

$$u' = e^x \text{ and } v' = -\sin(x)$$

Thus

$$\begin{aligned} y' = (uv)' &= u \cdot v' + u' \cdot v \\ &= e^x(-\sin(x)) + e^x \cos(x) \\ &= e^x \left(\cos(x) - \sin(x) \right) \end{aligned}$$

The Chain Rule

A composite function is defined as a "function of a function". For example, $h(x)$ is a composite function when:

$$h(x) = g(f(x))$$

where f and g are functions. In this case, to calculate the output of h, x is the input to function f and the output is then taken as the input for function g.

Examples of composite functions:

$$y = \sin(3x + 1), \qquad y = e^{x^2+2}$$

$$y = (2x - 5)^4, \qquad y = \cos(\sin(x))$$

The Chain Rule

To differentiate composite functions, we use the Chain Rule.

If $y = g(f(x))$, then we write $u = f(x)$ and so $y = g(u)$. Then...

Chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

Always state final answers in terms of the original variables - in this case, x not u.

The Chain Rule

Example:

If $y = \sin(3t^2 + 5)$, find y'

Solution:

$$\text{Let} \quad u = 3t^2 + 5 \quad \text{then} \quad y = \sin(u)$$

$$\text{Then} \quad \frac{\mathrm{d}u}{\mathrm{d}t} = 6t \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}u} = \cos(u)$$

Thus

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}t} = \cos(u) \cdot (6t) \\ &= 6t \cos\left(3t^2 + 5\right) \end{aligned}$$

The Chain Rule

Example:

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \mathrm{e}^{x^2+3x-1}$

Solution:

$$\text{Let} \quad u = x^2 + 3x - 1 \quad \text{then} \quad y = \mathrm{e}^u$$

$$\text{Then} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x + 3 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u$$

Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^u \cdot (2x + 3)$$

$$= (2x + 3)\,\mathrm{e}^{x^2+3x-1}$$

The Chain Rule

By applying the chain rule, we can derive some additional standard rules.

If a is a constant,

$$y = \sin(ax) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = a\cos(ax)$$

$$y = \cos(ax) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -a\sin(ax)$$

$$y = \tan(ax) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = a\sec^2(ax)$$

$$y = \mathrm{e}^{ax} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = a\,\mathrm{e}^{ax}$$

Second-order Derivatives

Differentiating a function $y = f(x)$ with respect to x yields the "first derivative" of y, denoted by y', $f'(x)$, or $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Differentiating the result (with respect to x) yields the "second derivative" of y, denoted by:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \quad \text{or} \quad y'' \quad \text{or} \quad f''(x)$$

This tells us the rate at which the gradient of y changes.

Examples

(1) $f(x) = 5x^3 \implies f'(x) = 15x^2 \implies f''(x) = 30x$

(2) $f(x) = 6\,\mathrm{e}^x \implies f'(x) = 6\,\mathrm{e}^x \implies f''(x) = 6\,\mathrm{e}^x$

Physical applications of differentiation as "rate of change":

Suppose an object moves in a straight line with its position along the line $x(t)$.

- Velocity is the rate of change of position, so:

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t}$$

- Acceleration is the rate of change of velocity, so:

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t}$$

  Therefore we also note that

$$a(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

Physical applications of differentiation as "rate of change":

Example

A car moves in a straight line from A to B. At any time t (in seconds), the displacement of the car from A is given by:

$$x(t) = t^3 + 2t^2 \quad \text{metres}$$

What is the acceleration of the car after 4s?

Solution:

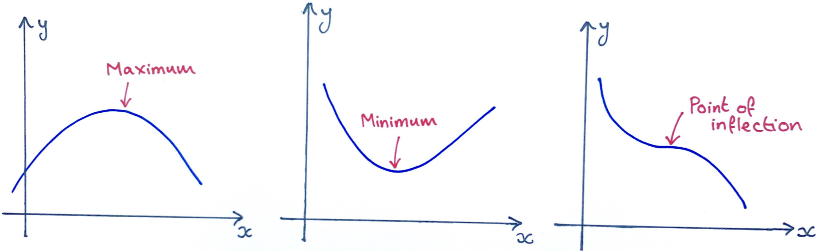
$$a(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t^3 + 2t^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(3t^2 + 4t\right) = 6t + 4$$

$$\therefore a(t = 4) = 6(4) + 4 = 28 \; ms^{-2}$$

Maximising and Minimising Functions

We may want to find extreme values of a function f in a range of x. This occurs where the gradient is zero (or at boundaries).

Stationary Points: If $f'(x) = 0$ when $x = a$ for some value a, then a is called a stationary point or turning point of the function f. There are three types:



Maximising and Minimising Functions

To find the maximum and minimum points of a curve $y = f(x)$:

1. Calculate the first derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$
2. Solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for x. This tells us the location of points where the gradient is zero (i.e. the stationary points).
3. Calculate the second derivative $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$
4. Determine the value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at each stationary point, and apply the "Second Derivative Test":

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0 \implies \text{local min.} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0 \implies \text{local max.}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \implies \text{no conclusion}$$

Maximising and Minimising Functions

Example:

Find and classify the stationary point(s) of

$$y = x^2 + 4x - 1$$

Solution:
Obtaining the first derivative:

$$y' = 2x + 4$$

Finding the values of x where gradient is zero:

$$y' = 0 \implies 2x + 4 = 0 \implies x = -2$$

Then obtaining the y-coordinate:

$$y(x = -2) = (-2)^2 + 4(-2) - 1 = -5$$

And applying the second derivative test:

$$y'' = 2 \implies y''(x = -2) = 2 > 0$$

Hence, $(-2, -5)$ is a minimum turning point.

MATLAB

Differentiation uses the `diff` command, which requires two arguments: the function to be differentiated, and then the symbolic variable you are differentiating with respect to. For example:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(2x)\right)$$

```
syms x
diff( sin(2*x), x )
```

This can be combined with the `solve` function to determine the location of stationary points:

```
syms x
solve( diff( sin(2*x), x ) = 0, x )
```