Partial differentiation

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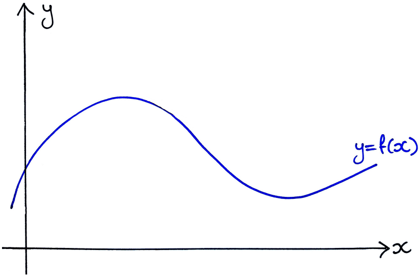
Contents

Today we will cover. . .

- Functions of multiple variables.
- Introduction to partial differentiation.
- Higher-order partial derivatives.

Functions of two variables

Previously, we considered $y = f(x)$, where y is a function of (i.e. its value depends on) only one variable x. This can be represented in two dimensions by a curve.



Examples:

$$y = x^3$$

$$y = 5x - \cos(3x)$$

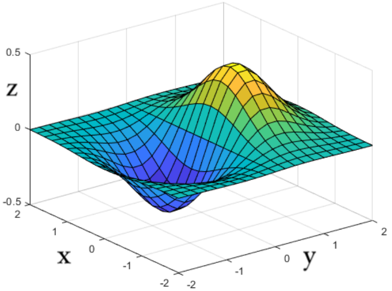
$$f(t) = 3\sin(t)$$

Functions of two variables

However, many physical modelling situations involve a function of multiple variables. This can be written as:

$$z = f(x, y)$$

and represented as a surface plot in three dimensions, where the height z depends on both the x and y co-ordinates.



Examples:

$$z = x^3 + 2y - 1$$

$$f(h, t) = 3\sin(h + t)$$

$$z(x, t) = 13t^{-1} - x^2 t^3$$

Functions of two variables

Physical Examples:

- The volume V of a cylinder is given by $V = \pi r^2 h$. The volume will change if either the radius r or the height h is changed.

- Time period of oscillation of a mass m on a spring:

  $$T = 2\sqrt{\frac{m}{k}}$$

  So $T = f(m, k)$, where m is the mass and k the spring constant.

- Pressure of an ideal gas: $p = \frac{mRT}{V}$, i.e. $p = f(T, V)$, where T is the temperature and V the volume.

To think about the gradient of this 3-d surface, or the rate of change of z as both x and y potentially change simultaneously, we use **partial differentiation**.

A "curly dee", ∂, is used to distinguish between partial and ordinary differentiation. Hence if $V = \pi r^2 h$, then:

- $\frac{\partial V}{\partial r}$ means the partial derivative of V with respect to r
- $\frac{\partial V}{\partial h}$ means the partial derivative of V w.r.t. h.

When differentiating a function of two or more variables, the other variables (which you are *not* differentiating w.r.t.) are **held fixed as though they were constants.**

Example 1

Given that $V = \pi r^2 h$, determine (i) $\frac{\partial V}{\partial r}$ and (ii) $\frac{\partial V}{\partial h}$:

(i) Since we are differentiating wrt r, we hold h constant:

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}\left(\pi r^2 h\right) = \pi h \frac{\partial}{\partial r}\left(r^2\right) = 2\pi r h$$

(ii) Since this time we are differentiating wrt h, we hold r constant:

$$\frac{\partial V}{\partial h} = \frac{\partial}{\partial h}\left(\pi r^2 h\right) = \pi r^2 \frac{\partial}{\partial h}\left(h\right) = \pi r^2$$

Example 2

If $z(x, y) = 5x^4 + 2x^3y^2 - 3y$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

To find $\frac{\partial z}{\partial x}$, we treat y as a constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(5x^4 + 2x^3y^2 - 3y\right) = 20x^3 + 6x^2y^2$$

To find $\frac{\partial z}{\partial y}$, we treat x as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(5x^4 + 2x^3y^2 - 3y\right) = 4x^3y - 3$$

Example 3

If $y(x,t) = 4\sin(3x)\cos(2t)$, find $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial t}$.

To find $\frac{\partial y}{\partial x}$, we treat t as a constant:

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}\left(4\sin(3x)\cos(2t)\right) = 12\cos(3x)\cos(2t)$$

To find $\frac{\partial y}{\partial t}$, we treat x as a constant:

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}\left(4\sin(3x)\cos(2t)\right) = -8\sin(3x)\sin(2t)$$

Example 4 (I/III)

If $f(x, y, z) = -7x\,e^{-3xy} + 8x^2z^3$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$.

This case features a function of *three* variables, so to find $\frac{\partial f}{\partial x}$, we treat both y and z as constants:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(- 7x\,e^{-3xy} + 8x^2z^3\right) \\ &= \frac{\partial}{\partial x}\left(- 7x\,e^{-3xy}\right) + 16xz^3\end{aligned}$$

For the first term, we will need to use the product rule.

Example 4 (II/III)

Let $u = -7x$ and $v = \mathrm{e}^{-3xy}$, then:

$$\frac{\partial u}{\partial x} = -7 \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\mathrm{e}^{-3xy}\right) = -3y\,\mathrm{e}^{-3xy}$$

So substituting these into the product rule:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \left(-7x\right) \times \left(-3y\,\mathrm{e}^{-3xy}\right) + \left(\mathrm{e}^{-3xy}\right) \times \left(-7\right) + 16xz^3 \\ &= 21xy\,\mathrm{e}^{-3xy} - 7\,\mathrm{e}^{-3xy} + 16xz^3\end{aligned}$$

Example 4 (III/III)

Similarly, to find $\frac{\partial f}{\partial y}$, we treat x and z as constants:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(-7x\,\mathrm{e}^{-3xy} + 8x^2z^3\right) = 21x^2\,\mathrm{e}^{-3xy}$$

To find $\frac{\partial f}{\partial z}$, we treat x and y as constants:

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(-7x\,\mathrm{e}^{-3xy} + 8x^2z^3\right) = 24x^2z^2$$

Second order partial derivatives

As with ordinary differentiation, a partial derivative may be differentiated partially again to give higher order partial derivatives.

If $f = f(x, y)$ is a function of two variables, then:

- Differentiating $\frac{\partial f}{\partial x}$ wrt x gives: $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$
- Differentiating $\frac{\partial f}{\partial y}$ wrt y gives: $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$
- Differentiating $\frac{\partial f}{\partial x}$ wrt y gives: $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$
- Differentiating $\frac{\partial f}{\partial y}$ wrt x gives: $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

Second order partial derivatives

It is important to note that these final two are equivalent. So there is no difference between partially differentiating f by x and then by y, than if instead you partially differentiated f by y and then by x:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

This is always true regardless of what the function f is.

Example 5 (I/III)

If $z(x, y) = 4x^2y^3 - 2x^3 + 7y^2$, then find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial^2 z}{\partial x \partial y}$.

To find $\frac{\partial^2 z}{\partial x^2}$, first determine the first partial derivative wrt x:

$$\frac{\partial z}{\partial x} = 8xy^3 - 6x^2$$

and then the second partial derivative wrt x can be determined:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(8xy^3 - 6x^2\right) = 8y^3 - 12x$$

Example 5 (II/III)

To find $\frac{\partial^2 z}{\partial y^2}$, first determine the first partial derivative wrt y:

$$\frac{\partial z}{\partial y} = 12x^2y^2 + 14y$$

and then the second partial derivative wrt y can be determined:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(12x^2y^2 + 14y\right) = 24x^2y + 14$$

Example 5 (III/III)

Then to determine $\frac{\partial^2 z}{\partial y \partial x}$:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(8xy^3 - 6x^2\right) = 24xy^2$$

and finally $\frac{\partial^2 z}{\partial x \partial y}$:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(12x^2y^2 + 14y\right) = 24xy^2$$

So we have confirmed that: $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

MATLAB

Partial differentiation is actually the same as regular differentiation in MATLAB, using the `diff` command with two arguments. The only difference is that you will need to remember to declare all variables as symobolic first. For example:

$$\frac{\partial}{\partial x}\left(xy^2 + 3y\sin(x)\right)$$

```
syms x y
diff( x*y*y+3*y*sin(x) , x )
```