Applications of partial differentiation

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Contents

Today we will cover. . .

- Applications of partial differentiation to:
- (i) Propagation of uncertainty.
- (ii) Rates of change of functions of multiple variables.

Propagation of uncertainty

We previously learned how to add or subtract independent normally distributed variables

In particular, if a new variable is created by adding or subtracting two independent random variables, then it's variance is the sum of the variances of the two constituent variables.

Using partial differentiation, we can extend this idea to much more complex composite variables.

In general, if:

$$y = f(x_1, x_2, \ldots, x_n)$$

where $x_1, \ldots, x_n$ are independent random variables with variances $\sigma_{x_1}^2, \ldots, \sigma_{x_n}^2$, then the variance of y is given by:

Propagation of uncertainty

$$\sigma_y^2 = \left(\frac{\partial y}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \cdots + \left(\frac{\partial y}{\partial x_n}\right)^2 \sigma_{x_n}^2$$

Evaluated at the mean values of $x_1, \ldots, x_n$.

Propagation of uncertainty - Example 1

Given that x and y are independent random variables with mean $\mu_x = 0$ and $\mu_y = 3$, and $z = 3x + \sin(x) + y^2$, determine a formula for the variance of z in terms of the variances of x and y:

$$\frac{\partial z}{\partial x} = 3 + \cos(x) \qquad \frac{\partial z}{\partial y} = 2y$$

Thus,

$$\begin{aligned}
\sigma_z^2 &= (3 + \cos(\mu_x))^2 \sigma_x^2 + (2\mu_y)^2 \sigma_y^2 \\
&= (\cos^2(\mu_x) + 6\cos(\mu_x) + 9)\sigma_x^2 + (4\mu_y^2)\sigma_y^2 \\
&= (\cos^2(0) + 6\cos(0) + 9)\sigma_x^2 + (4(3)^2)\sigma_y^2 \\
&= 16\sigma_x^2 + 36\sigma_y^2
\end{aligned}$$

Propagation of uncertainty - Example 2

Show that the sum of two independent random variables has a variance that is the sum of their variances.

Let x_1 and x_2 be independent random variables and let y be such that $y = x_1 + x_2$.

Then,

$$\frac{\partial y}{\partial x_1} = 1 \qquad \frac{\partial y}{\partial x_2} = 1$$

And so aplying the variance formula:

$$\begin{aligned} \sigma_y^2 &= (1)^2\sigma_{x_1}^2 + (1)^2\sigma_{x_2}^2 \\ &= \sigma_{x_1}^2 + \sigma_{x_2}^2 \end{aligned}$$

So we do indeed recover the previous result.

Propagation of uncertainty - Example 3 (I/IV)

A company manufactures roadside bollards by fitting a partially-sheared sphere of radius $t$cm and volume $\frac{5}{4}\pi t^3$ to a cylinder of height $h$cm and radius $r$cm. Thus, the cylinder has volume given by $\pi r^2 h$ and the overall volume of the bollard is:

$$V = \pi r^2 h + \frac{5}{4}\pi t^3$$

The components are manufactured with the variables h, r and t obeying approximately normal distributions with mean: $h = 50$cm, $r = 6$cm and $t = 10$cm, and variances:

$$\sigma_h^2 = 4cm^2 \qquad \sigma_r^2 = 0.5cm^2 \qquad \sigma_t^2 = 1cm^2$$

What is the mean and variance of the volume of the bollards?

Example 3 (II/IV)

The mean is simply:

$$\begin{aligned}\mu_V &= \pi\mu_r^2\mu_h + \frac{5}{4}\pi\mu_t^3 \\ &= \pi(6)^2(50) + \frac{5}{4}\pi(10)^3 \\ &= 9581.85759\ldots \\ &\approx 9580cm^3\end{aligned}$$

Calculating the partial derivatives:

$$\frac{\partial V}{\partial r} = 2\pi rh \qquad \frac{\partial V}{\partial h} = \pi r^2 \qquad \frac{\partial V}{\partial t} = \frac{15}{4}\pi t^2$$

Example 3 (III/IV)

Thus the variance of the volume is given by:

$$\begin{aligned}
\sigma_V^2 &= (2\pi\mu_r\mu_h)^2\sigma_r^2 + (\pi\mu_r^2)^2\sigma_h^2 + \left(\frac{15}{4}\pi\mu_t^2\right)^2\sigma_t^2 \\
&= 4\pi^2\mu_r^2\mu_h^2\sigma_r^2 + \pi^2\mu_r^4\sigma_h^2 + \frac{225}{16}\pi^2\mu_t^4\sigma_t^2 \\
&= 4\pi^2(6)^2(50)^2(0.5) + \pi^2(6)^4(4) + \frac{225}{16}\pi^2(10)^4(1) \\
&= 2327341.544\ldots \\
&\approx 2,330,000 cm^6
\end{aligned}$$

Example 3 (IV/IV))

As the variables are normally-distributed, we can then determine boundaries for 68% of the bollards produced - as they will lie within one standard deviation of the mean for a normally-distributed variable.

The standard deviation of the volume is:

$$\sigma_V = \sqrt{2327341.544} = 1525.563\cdots \approx 1530cm^3$$

And so 68% of all bollards produced will have a volume in the range:

$$9580 \pm 1530cm^3$$

Rates of change for multivariate functions

Sometimes it is necessary to solve problems in which different quantities have different rates of change - for this we make use of first order partial derivatives.

Second order partial derivatives are used in the solution of partial differential equations, for example in wave theory, thermodynamics (entropy, continuity theorem) and fluid mechanics. They are also used in optimisation problems.

Rates of change for multivariate functions

For a general multi-variate function z, it can be shown that the rate of change of z w.r.t. time t is given by:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial z}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t} + \frac{\partial z}{\partial x_3}\frac{\mathrm{d}x_3}{\mathrm{d}t} + \dots$$

where $x_1, x_2, x_3, \dots$ are variables that z depends on, and we have to consider how each of their rates of change contributes to the rate at which z changes.

Rates of change for multivariate functions

Functions of two variables:

If $z = f(x, y)$ and x and y are functions of t (that is, $x = x(t)$ and $y = y(t)$) then z is ultimately a function of t only, and:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Functions of three variables:

If $w = f(x, y, z)$ and $x = x(t)$, $y = y(t)$ and $z = z(t)$ then w is ultimately a function of t only, and:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

Multivariate rates of change - Example 1 (I/IV)

The height of a right circular cone is increasing at $3mms^{-1}$ and its radius is decreasing at $2mms^{-1}$. Determine (to 3 s.f.) the rate at which the volume is changing (in cm^3s^{-1}) when the height is 3.2 cm and the radius is 1.5 cm.

The volume of a right circular cone is given by $V = \frac{1}{3}\pi r^2 h$. From above, as this formula for V depends on two variables, r and h, the rate of change of volume is:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial V}{\partial h}\frac{\mathrm{d}h}{\mathrm{d}t}$$

Obtaining the partial derivatives of V wrt r and h:

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi rh, \quad \text{and} \quad \frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$$

Since the height is *increasing* at $3mms^{-1}$, i.e. $0.3cm/s$, then:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = +0.3$$

and since the radius is *decreasing* at $2mms^{-1}$, i.e. $0.2cm/s$, then:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -0.2$$

Multivariate rates of change - Example 1 (III/IV)

Substituting both of these into the rate-of-change equation gives:

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\partial V}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial V}{\partial h}\frac{\mathrm{d}h}{\mathrm{d}t} \\
&= \left(\frac{2}{3}\pi r h\right)(-0.2) + \left(\frac{1}{3}\pi r^2\right)(0.3) \\
&= \frac{-0.4}{3}\pi r h + 0.1\pi r^2
\end{aligned}$$

Then to find the rate of change of volume specifically when $h = 3.2cm$ and $r = 1.5cm$:

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{-0.4}{3}\pi rh + 0.1\pi r^2 \\ &= \frac{-0.4}{3}\pi \times 1.5 \times 3.2 + 0.1\pi(1.5)^2 \\ &= -1.304cm^3s^{-1}\end{aligned}$$

So the volume is *decreasing* at a rate of $1.30cm^3s^{-1}$ at that particular moment.

Multivariate rates of change - Example 2 (I/III)

A rectangular box has sides of length x cm, y cm and z cm. Sides x and z are expanding at rates of $3mms^{-1}$ and $5mms^{-1}$, and side y is contracting at a rate of $2mms^{-1}$. Determine the rate of change of volume when x is 3 cm, y is 1.5 cm and z is 6 cm.

The volume of a cuboid is given by:

$$V = xyz$$

Hence, the rate of change of volume is given by the formula:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial V}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial V}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

Multivariate rates of change - Example 2 (II/III)

Partially differentiating $V = xyz$ with respect to x, y and z then:

$$\frac{\partial V}{\partial x} = yz, \qquad \frac{\partial V}{\partial y} = xz, \qquad \frac{\partial V}{\partial z} = xy$$

We also know that the rates of change of x, y and z are:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.3, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -0.2, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = 0.5$$

Multivariate rates of change - Example 2 (III/III)

Substituting all of this and the required values of x, y and z into the formula for rate of change of volume yields:

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial V}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial V}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\
&= (yz)(0.3) + (xz)(-0.2) + (xy)(0.5) \\
&= (1.5 \times 6)(0.3) + (3 \times 6)(-0.2) + (3 \times 1.5)(0.5) \\
&= 1.35 cm^3 s^{-1}
\end{aligned}
$$