Maths for Materials and Design

Differential Calculus and Rates of Change

Contents

1 Differentiation

1.1 In this section:

These notes briefly revise the following topics:

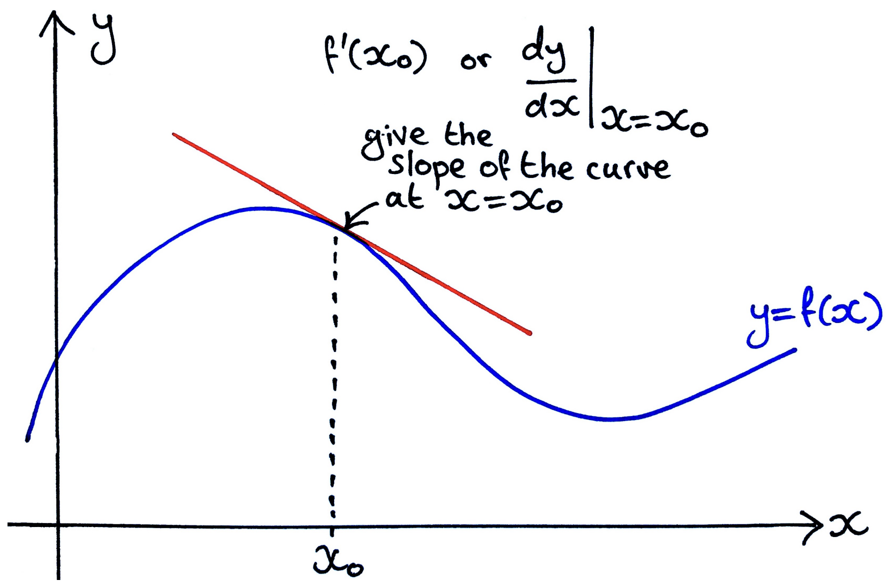
- Differentiation as gradient and rate-of-change
- Standard rules of differentiation
- Chain rule
- Product rule
- Higher-order derivatives
- Maxima and minima of functions of one variable

1.2 Introduction to Differentiation

Given a function of one variable $y = f(x)$, the "derivative of y with respect to x" can be written as:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{or} \quad y'$$

This yields the gradient (slope) of the function, which is equivalent to its rate of change as x changes.



The derivative of y with respect to time t, may also be expressed as $\dot{y}$.

1.2.1 Standard rules of differentiation

For constants a and n:

$$y = ax^n \implies \frac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}$$

$$y = ax \implies \frac{\mathrm{d}y}{\mathrm{d}x} = a$$

$$y = a \implies \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$y = \mathrm{e}^x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$$

$$y = \sin(x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x)$$

$$y = \cos(x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\sin(x)$$

$$y = \ln(x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

Example

Differentiate:

$$y = 5x^2, \qquad y = -3x^{10}$$

$$y = 9x, \qquad y = 2\sqrt{x}$$

$$y = -\frac{4}{x^3}, \qquad y = 100x^{0.5}$$

1.2.2 Linearity

Given two functions f and g,

$$y = f(x) \pm g(x) \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} \pm \frac{\mathrm{d}g}{\mathrm{d}x} = f'(x) \pm g'(x)$$

If a is a constant, then:

$$y = af(x) \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = a\frac{\mathrm{d}f}{\mathrm{d}x} = af'(x)$$

1.2.3 Gradient of a curve

Differentiating the formula for a curve yields a formula for the gradient of that curve at any point. In most cases, this gradient is dependent on x. Therefore, to find the gradient of a curve at a particular point, differentiate first and then substitute in the particular value of x to the formula obtained for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Example:

Find the gradient of the curve $y = x^2 + 4x - 7$ at the $(2, 5)$.

Solution:

$$y'(x) = 2x + 4$$

$$y'(x = 2) = 2(2) + 4 = 8$$

1.3 The Product Rule

Suppose $f(x)$ is the product of two functions (i.e. the result of multiplying them together):

$$f(x) = u(x) \cdot v(x)$$

Then the derivative is given by the Product Rule.

Product rule:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = v(x) \cdot \frac{\mathrm{d}u(x)}{\mathrm{d}x} + u(x) \cdot \frac{\mathrm{d}v(x)}{\mathrm{d}x}$$

This may also be written as:

$$(uv)' = vu' + uv'$$

Example:

Differentiate $y = x^2 \sin(x)$

Solution:

Let $u = x^2$ and $v = \sin(x)$

Then

$$u' = 2x \text{ and } v' = \cos(x)$$

Thus

$$\begin{aligned} y' = (uv)' &= u \cdot v' + u' \cdot v \\ &= x^2 \cos(x) + 2x \sin(x) \end{aligned}$$

Example:

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \mathrm{e}^x \cos(x)$

Solution:

$$\text{Let } \; u = \mathrm{e}^x \quad \text{and} \quad v = \cos(x)$$

Then

$$u' = \mathrm{e}^x \quad \text{and} \quad v' = -\sin(x)$$

Thus

$$\begin{aligned} y' = (uv)' &= u \cdot v' + u' \cdot v \\ &= \mathrm{e}^x(-\sin(x)) + \mathrm{e}^x \cos(x) \\ &= \mathrm{e}^x \big(\cos(x) - \sin(x)\big) \end{aligned}$$

1.4 The Chain Rule

A composite function is defined as a "function of a function". For example, $h(x)$ is a composite function when:

$$h(x) = g(f(x))$$

where f and g are functions. In this case, to calculate the output of h, x is the input to function f and the output is then taken as the input for function g.

Examples of composite functions:

$$y = \sin(3x+1), \qquad y = e^{x^2+2}$$

$$y = (2x-5)^4, \qquad y = \cos(\sin(x))$$

To differentiate composite functions, we use the Chain Rule.

If $y = g(f(x))$, then we write $u = f(x)$ and so $y = g(u)$. Then...

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Always state final answers in terms of the original variables - in this case, x not u.

Example:

If $y = \sin(3t^2+5)$, find y'

Solution:

Let $u = 3t^2 + 5$ then $y = \sin(u)$

Then $\dfrac{du}{dt} = 6t$ and $\dfrac{dy}{du} = \cos(u)$

Thus

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = \cos(u) \cdot (6t)$$

$$= 6t \cos\left(3t^2+5\right)$$

Example:

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \mathrm{e}^{x^2+3x-1}$

Solution:

Let $\quad u = x^2 + 3x - 1 \quad$ then $\quad y = \mathrm{e}^u$

Then $\quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x + 3 \quad$ and $\quad \dfrac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u$

Thus

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^u \cdot (2x+3) \\ &= (2x+3)\,\mathrm{e}^{x^2+3x-1} \end{aligned}$$

By applying the chain rule, we can derive some additional standard rules.

If a is a constant,

$$y = \sin(ax) \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = a\cos(ax)$$

$$y = \cos(ax) \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -a\sin(ax)$$

$$y = \tan(ax) \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = a\sec^2(ax)$$

$$y = \mathrm{e}^{ax} \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = a\,\mathrm{e}^{ax}$$

1.5 Second-order Derivatives

Differentiating a function $y = f(x)$ with respect to x yields the "first derivative" of y, denoted by y', $f'(x)$, or $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Differentiating the result (with respect to x) yields the "second derivative" of y, denoted by:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \quad \text{or} \quad y'' \quad \text{or} \quad f''(x)$$

This tells us the rate at which the gradient of y changes.

Examples

(1) $f(x) = 5x^3 \quad \Longrightarrow \quad f'(x) = 15x^2 \quad \Longrightarrow \quad f''(x) = 30x$

(2) $f(x) = 6\,\mathrm{e}^x \quad \Longrightarrow \quad f'(x) = 6\,\mathrm{e}^x \quad \Longrightarrow \quad f''(x) = 6\,\mathrm{e}^x$

1.6 Physical applications of differentiation as "rate of change":

Suppose an object moves in a straight line with its position along the line $x(t)$.

- Velocity is the rate of change of position, so:

  $$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t}$$

- Acceleration is the rate of change of velocity, so:

  $$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t}$$

  Therefore we also note that

  $$a(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

Example

A car moves in a straight line from A to B. At any time t (in seconds), the displacement of the car from A is given by:

$$x(t) = t^3 + 2t^2 \quad \text{metres}$$

What is the acceleration of the car after 4s?

Solution:

$$a(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(t^3 + 2t^2\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big(3t^2 + 4t\big) = 6t + 4$$

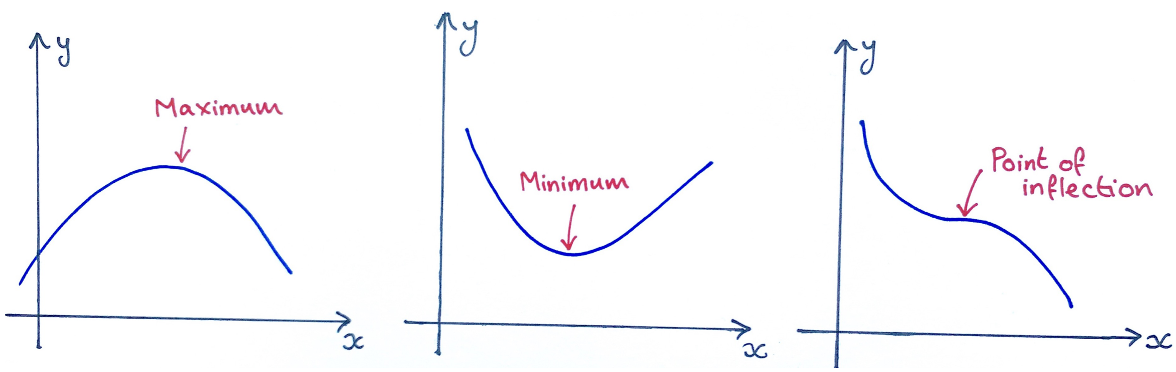
$$\therefore a(t = 4) = 6(4) + 4 = 28\ ms^{-2}$$

1.7 Maximising and Minimising Functions

We may want to find extreme values of a function f in a range of x. This occurs where the gradient is zero (or at boundaries).

Stationary Points: If $f'(x) = 0$ when $x = a$ for some value a, then a is called a stationary point or turning point of the function f.

There are three types:



To find the maximum and minimum points of a curve $y = f(x)$:

1. Calculate the first derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$.
2. Solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for x. This tells us the location of points where the gradient is zero (i.e. the stationary points).
3. Calculate the second derivative $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.
4. Determine the sign of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at each stationary point, and apply the "Second Derivative Test":

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0 \implies \text{local minimum}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0 \implies \text{local maximum}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \implies \text{no conclusion}$$

Example:

Find and classify the stationary point(s) of

$$y = x^2 + 4x - 1$$

Solution:

Obtaining the first derivative:

$$y' = 2x + 4$$

Finding the values of x where gradient is zero:

$$y' = 0 \implies 2x + 4 = 0 \implies x = -2$$

Then obtaining the y-coordinate:

$$y(x = -2) = (-2)^2 + 4(-2) - 1 = -5$$

And applying the second derivative test:

$$y'' = 2 \implies y''(x = -2) = 2 > 0$$

Hence, $(-2, -5)$ is a maximum turning point.

1.8 MATLAB

Differentiation uses the `diff` command, which requires two arguments: the function to be differentiated, and then the symbolic variable you are differentiating with respect to. For example:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\sin(2x)\big)$$

```
syms x
diff( sin(2*x), x )
```

This can be combined with the `solve` function to determine the location of stationary points:

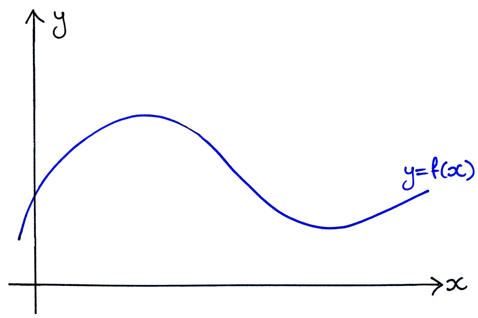
```
syms x
solve( diff( sin(2*x), x ) == 0, x )
```

But note that, despite there being an infinite set of solutions, MATLAB will only return the “simplest” one.

2 Partial differentiation

2.1 Functions of two variables

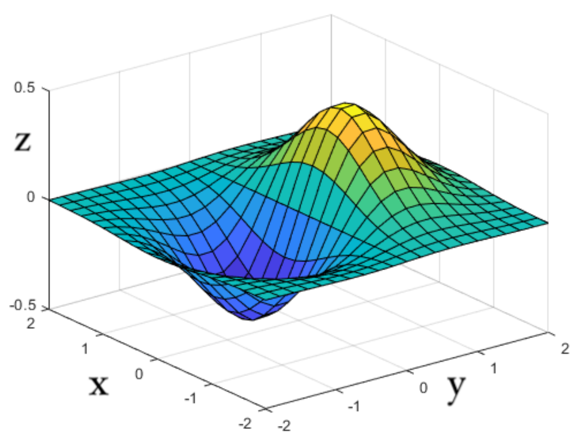
Previously, we considered $y = f(x)$, where y is a function of (i.e. its value depends on) only one variable x. This can be represented in two dimensions by a curve.



However, many physical modelling situations involve a function of multiple variables. This can be written as:

$$z = f(x, y)$$

and represented as a surface plot in three dimensions, where the height z depends on both the x and y co-ordinates.



Examples:

$$z = x^3 + 2y - 1, \qquad f(h, t) = 3\sin(h + t)$$

Physical Examples:

- The volume V of a cylinder is given by $V = \pi r^2 h$. The volume will change if either the radius r or the height h is changed. The formula may be stated mathematically as $V = f(r, h)$, which means that "V is some function of both r and h".

- Time period of oscillation of a mass m on a spring:

  $$T = 2\sqrt{\frac{m}{k}}$$

  i.e. $T = f(m, k)$, where m is the mass and k the spring constant.

- Pressure of an ideal gas:

  $$p = \frac{mRT}{V}$$

  i.e. $p = f(T, V)$, where T is the temperature and V the volume.

2.2 Introduction to Partial Differentiation

To think about the gradient of this 3-d surface, or the rate of change of z as both x and y potentially change simultaneously, we need to develop our theory of differentiation to include **partial differentiation**.

A “curly dee”, ∂, is used to distinguish between partial and ordinary differentiation. Hence if $V = \pi r^2 h$, then $\frac{\partial V}{\partial r}$ means the partial derivative of V with respect to r, and $\frac{\partial V}{\partial h}$ means the partial derivative of V w.r.t. h.

When differentiating a function of two or more variables, the variable with which the function is being differentiated wrt is differentiated, while the other variables are held fixed as if they were constants.

Example 1: Determine $\frac{\partial V}{\partial r}$ if $V = \pi r^2 h$.

Since we are differentiating wrt r, we hold h constant:

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}\big(\pi r^2 h\big) = \pi h \frac{\partial}{\partial r}\big(r^2\big) = 2\pi r h$$

Example 2: Determine $\frac{\partial V}{\partial h}$ if $V = \pi r^2 h$.

Since this time we are differentiating wrt h, we hold r constant:

$$\frac{\partial V}{\partial h} = \frac{\partial}{\partial h}\big(\pi r^2 h\big) = \pi r^2 \frac{\partial}{\partial h}\big(h\big) = \pi r^2$$

Example 3: If $z(x, y) = 5x^4 + 2x^3y^2 - 3y$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

To find $\frac{\partial z}{\partial x}$, we treat y as a constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\big(5x^4 + 2x^3y^2 - 3y\big) = 20x^3 + 6x^2y^2$$

To find $\frac{\partial z}{\partial y}$, we treat x as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\big(5x^4 + 2x^3y^2 - 3y\big) = 4x^3y - 3$$

Example 4: If $y(x,t) = 4\sin(3x)\cos(2t)$, find $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial t}$.

To find $\frac{\partial y}{\partial x}$, we treat t as a constant:

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}\big(4\sin(3x)\cos(2t)\big) = 12\cos(3x)\cos(2t)$$

To find $\frac{\partial y}{\partial t}$, we treat x as a constant:

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}\big(4\sin(3x)\cos(2t)\big) = -8\sin(3x)\sin(2t)$$

Example 5: If $f(x,y,z) = -7x\,\mathrm{e}^{-3xy} + 8x^2z^3$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.

To find $\frac{\partial f}{\partial x}$, we treat y and z as constants:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\big(-7x\,\mathrm{e}^{-3xy} + 8x^2z^3\big)\\ &= -7x \times (-3y)\,\mathrm{e}^{-3xy} + \mathrm{e}^{-3xy} \times (-7) + 16xz^3\\ &= 21xy\,\mathrm{e}^{-3xy} - 7\,\mathrm{e}^{-3xy} + 16xz^3\end{aligned}$$

where we have used the product rule in the first step and term.

To find $\frac{\partial f}{\partial y}$, we treat x and z as constants:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\big(-7x\,\mathrm{e}^{-3xy} + 8x^2z^3\big) = 21x^2\,\mathrm{e}^{-3xy}$$

To find $\frac{\partial f}{\partial z}$, we treat x and y as constants:

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\big(-7x\,\mathrm{e}^{-3xy} + 8x^2z^3\big) = 24x^2z^2$$

2.3 Second order Partial derivatives

As with ordinary differentiation, where a differential coefficient may be differentiated again, a partial derivative may be differentiated partially again to give higher order partial derivatives.

Example notation:

If $f = f(x, y)$ is a function of two variables, then:

- Differentiating $\frac{\partial f}{\partial x}$ wrt x gives: $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$
- Differentiating $\frac{\partial f}{\partial y}$ wrt y gives: $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$
- Differentiating $\frac{\partial f}{\partial x}$ wrt y gives: $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$
- Differentiating $\frac{\partial f}{\partial y}$ wrt x gives: $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

It is important to note that these final two are equivalent. So there is no difference between partially differentiating f by x and then by y, than if instead you partially differentiated f by y and then by x:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

This is true regardless of what f is.

Example 6: If $z(x,y) = 4x^2y^3 - 2x^3 + 7y^2$, then find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial^2 z}{\partial x \partial y}$.

To find $\frac{\partial^2 z}{\partial x^2}$, first determine the first partial derivative wrt x:

$$\frac{\partial z}{\partial x} = 8xy^3 - 6x^2$$

and then the second partial derivative wrt x can be determined:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\big(8xy^3 - 6x^2\big) = 8y^3 - 12x$$

To find $\frac{\partial^2 z}{\partial y^2}$, first determine the first partial derivative wrt y:

$$\frac{\partial z}{\partial y} = 12x^2y^2 + 14y$$

and then the second partial derivative wrt y can be determined:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\big(12x^2y^2 + 14y\big) = 24x^2y + 14$$

Then to determine $\frac{\partial^2 z}{\partial y \partial x}$:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\big(8xy^3 - 6x^2\big) = 24xy^2$$

and finally $\frac{\partial^2 z}{\partial x \partial y}$:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\big(12x^2y^2 + 14y\big) = 24xy^2$$

so we confirm that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

2.4 Propagation of Uncertainty

Earlier in this module, we learned how to add or subtract independent normally distributed variables. In particular, if a new variable is created by adding or subtracting two independent random variables, then it's variance is the sum of the variances of the two constituent variables.

Using partial differentiation, we can extend this idea to much more complex composite variables.

In general, if $y = f(x_1, x_2, \ldots, x_n)$, where $x_1, \ldots, x_n$ are independent random variables with variances $\sigma_{x_1}^2, \ldots, \sigma_{x_n}^2$, then the variance of y is given by:

$$\sigma_y^2 = \left(\frac{\partial y}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \cdots + \left(\frac{\partial y}{\partial x_n}\right)^2 \sigma_{x_n}^2$$

Evaluated at the mean values of $x_1, \ldots, x_n$.

Example 1: Given that x and y are independent random variables with mean $\mu_x = 0$ and $\mu_y = 3$, and $z = 3x + \sin(x) + y^2$, determine a formula for the variance of z.

$$\frac{\partial z}{\partial x} = 3 + \cos(x) \qquad \frac{\partial z}{\partial y} = 2y$$

Thus,

$$\begin{aligned}
\sigma_z^2 &= (3 + \cos(\mu_x))^2 \sigma_x^2 + (2\mu_y)^2 \sigma_y^2 \\
&= (\cos^2(\mu_x) + 6\cos(\mu_x) + 9)\sigma_x^2 + (4\mu_y^2)\sigma_y^2 \\
&= (\cos^2(0) + 6\cos(0) + 9)\sigma_x^2 + (4(3)^2)\sigma_y^2 \\
&= 16\sigma_x^2 + 36\sigma_y^2
\end{aligned}$$

Example 2: Show that this theory agrees with our previous result that the sum of two independent random variables has a variance that is the sum of their variances.

Letx_1 and x_2 be independent random variables and let y be such that $y = x_1 + x_2$.

Then,

$$\frac{\partial y}{\partial x_1} = 1 \qquad \frac{\partial y}{\partial x_2} = 1$$

And so aplying the variance formula:

$$\begin{aligned}\sigma_y^2 &= (1)^2\sigma_{x_1}^2 + (1)^2\sigma_{x_2}^2 \\ &= \sigma_{x_1}^2 + \sigma_{x_2}^2\end{aligned}$$

So we find that we do indeed recover the previous result.

Example 3: A company manufactures bollards for pedestrian areas by fitting a partially-sheared sphere of radius $t$cm and volume $\frac{5}{4}\pi t^3$ to a cylinder of height $h$cm and radius $r$cm. Thus, the cylinder has volume given by $\pi r^2 h$ and the overall volume of the bollard is:

$$V = \pi r^2 h + \frac{5}{4}\pi t^3$$

The components are manufactured with the variables h, r and t obeying approximately normal distributions with mean: $h = 50$cm, $r = 6$cm and $t = 10$cm, and variances:

$$\sigma_h^2 = 4cm^2 \qquad \sigma_r^2 = 0.5cm^2 \qquad \sigma_t^2 = 1cm^2$$

What is the mean and variance of the volume of the bollards?

The mean is simply:

$$\begin{aligned}\mu_V &= \pi\mu_r^2\mu_h + \frac{5}{4}\pi\mu_t^3 \\ &= \pi(6)^2(50) + \frac{5}{4}\pi(10)^3 \\ &= 9581.85759\ldots \\ &\approx 9580cm^3\end{aligned}$$

Calculating the partial derivatives:

$$\frac{\partial V}{\partial r} = 2\pi rh \qquad \frac{\partial V}{\partial h} = \pi r^2 \qquad \frac{\partial V}{\partial t} = \frac{15}{4}\pi t^2$$

Thus the variance of the volume is given by:

$$\begin{aligned}\sigma_V^2 &= (2\pi\mu_r\mu_h)^2\sigma_r^2 + (\pi\mu_r^2)^2\sigma_h^2 + \left(\frac{15}{4}\pi\mu_t^2\right)^2\sigma_t^2 \\ &= 4\pi^2\mu_r^2\mu_h^2\sigma_r^2 + \pi^2\mu_r^4\sigma_h^2 + \frac{225}{16}\pi^2\mu_t^4\sigma_t^2 \\ &= 4\pi^2(6)^2(50)^2(0.5) + \pi^2(6)^4(4) + \frac{225}{16}\pi^2(10)^4(1) \\ &= 2327341.544\ldots \\ &\approx 2,330,000cm^6\end{aligned}$$

As the variables are normally-distributed, we can then determine boundaries for 68% of the bollards produced - as they will lie within one standard deviation of the mean for a normally-distributed variable.

The standard deviation of the volume is:

$$\sigma_V = \sqrt{2327341.544} = 1525.563 \cdots \approx 1530cm^3$$

And so 68% of all bollards produced will have a volume in the range:

$$9580 \pm 1530cm^3$$

2.5 Rates of change for multivariate functions

Sometimes it is necessary to solve problems in which different quantities have different rates of change - for this we make use of first order partial derivatives.

Second order partial derivatives are used in the solution of partial differential equations, for example in wave theory, thermodynamics (entropy, continuity theorem) and fluid mechanics. They are also used in optimisation problems.

In the previous lecture partial differentiation was introduced for the case where only one variable changes at a time and the other variables are kept constant. In practice, variables may all be changing at the same time.

It can be shown that the rate of change of z wrt t is given by:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}t} + \dots$$

where $u, v, w, \dots$ are variables that z depends on, and we have to consider how each of their rates of change contributes to the rate at which z changes.

For example: if $z = f(x, y)$ and x and y are functions of t (that is, $x = x(t)$ and $y = y(t)$) then z is ultimately a function of t only, and:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

If $w = f(x, y, z)$ and $x = x(t)$, $y = y(t)$ and $z = z(t)$ then w is ultimately a function of t only, and:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

Example 1: The height of a right circular cone is increasing at $3mms^{-1}$ and its radius is decreasing at $2mms^{-1}$. Determine, correct to 3 significant figures, the rate at which the volume is changing (in cm^3s^{-1}) when the height is 3.2 cm and the radius is 1.5 cm.

The volume of a right circular cone is given by $V = \frac{1}{3}\pi r^2 h$. From above, as this formula for V depends on two variables, r and h, the rate of change of volume is:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial V}{\partial h}\frac{\mathrm{d}h}{\mathrm{d}t}$$

Obtaining the partial derivatives of V wrt r and h:

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h, \quad \text{and} \quad \frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$$

Since the height is *increasing* at $3mms^{-1}$, i.e. $0.3cm/s$, then:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = +0.3$$

and since the radius is *decreasing* at $2mms^{-1}$, i.e. $0.2cm/s$, then:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -0.2$$

Substituting both of these into the equation gives:

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\partial V}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial V}{\partial h}\frac{\mathrm{d}h}{\mathrm{d}t} \\
&= \left(\frac{2}{3}\pi r h\right)(-0.2) + \left(\frac{1}{3}\pi r^2\right)(0.3) \\
&= \frac{-0.4}{3}\pi r h + 0.1\pi r^2
\end{aligned}$$

Then to find the rate of change of volume specifically when $h = 3.2cm$ and $r = 1.5cm$:

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{-0.4}{3}\pi r h + 0.1\pi r^2 \\
&= \frac{-0.4}{3}\pi \times 1.5 \times 3.2 + 0.1\pi(1.5)^2 \\
&= -1.304cm^3s^{-1}
\end{aligned}$$

So the volume is decreasing at a rate of $1.30cm^3s^{-1}$ at that particular moment.

Example 2: A rectangular box has sides of length x cm, y cm and z cm. Sides x and z are expanding at rates of $3mms^{-1}$ and $5mms^{-1}$, respectively and side y is contracting at a rate of $2mms^{-1}$. Determine the rate of change of volume when x is 3 cm, y is 1.5 cm and z is 6 cm.

The volume of a cuboid is given by $V = xyz$. Hence, the rate of change of volume is given by the formula:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial V}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial V}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

Partially differentiating V with respect to x, y and z then:

$$\frac{\partial V}{\partial x} = yz, \qquad \frac{\partial V}{\partial y} = xz, \qquad \frac{\partial V}{\partial z} = xy$$

We also know that the rates of change of x, y and z are:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.3, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -0.2, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = 0.5$$

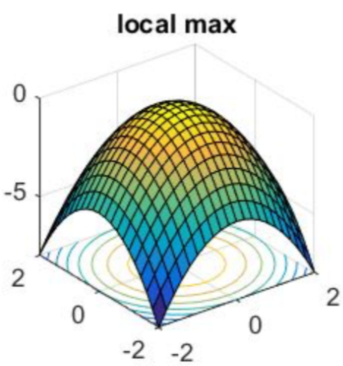
Substituting all of this and the required values of x, y and z into the formula for rate of change of volume yields:

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial V}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial V}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\
&= (yz)(0.3) + (xz)(-0.2) + (xy)(0.5) \\
&= (1.5 \times 6)(0.3) + (3 \times 6)(-0.2) + (3 \times 1.5)(0.5) \\
&= 1.35cm^3s^{-1}
\end{aligned}$$

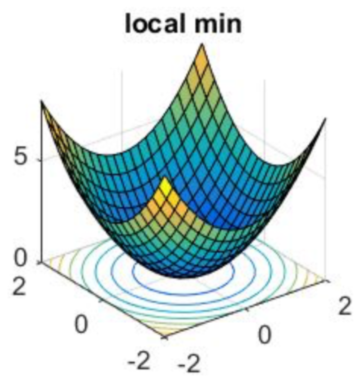
2.6 Maximising and minimising functions of two variables

Recall that we can think of a function of two variables as a surface (rather than a curve). Maximum and minimum heights can occur at stationary points on the surface, where the gradient is zero in all directions and the surface is perfectly flat.

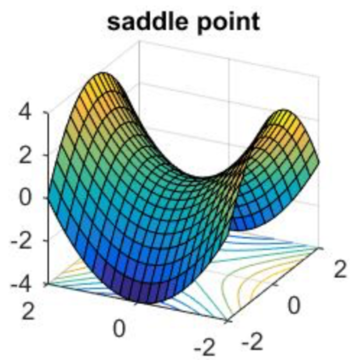
There are three kinds:



(a) Maximum



(b) Minimum



(c) Saddle Point

In order to determine where on a surface stationary points occur and whether these points are maxima, minima or saddle points, we follow the procedure below:

Given a multivariate function $z = f(x, y)$:

1. Determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

2. To determine the location of the stationary points, let $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$. A stationary point is a point on the surface where both slopes in the x and y direction are zero.

3. Solve the pair of simultaneous equations $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ for x and y. This gives us the co-ordinates of the stationary points. (Note that there may be more than one.)

4. Determine the second partial derivatives: $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, and $\frac{\partial^2 z}{\partial y \partial x}$

5. For each of the co-ordinates of the stationary points, substitute the values of x and y into the equation:

$$\Delta = \left(\frac{\partial^2 z}{\partial y \partial x}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)$$

   and evaluate.

6. Apply the following second derivative test. If:

- $\Delta > 0$, then the stationary point is a **saddle point**.
- $\Delta < 0$ and $\frac{\partial^2 z}{\partial x^2} < 0$, then the stationary point is a **maximum point**.
- $\Delta < 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$, then the stationary point is a **minimum point**.

We could replace $\frac{\partial^2 z}{\partial x^2}$ with $\frac{\partial^2 z}{\partial y^2}$ in this test, it doesn't matter which is used.

Example: Find and classify the stationary points of the surface:

$$f(x, y) = x^2 + y^2 - 2x + 4y + 8$$

1. First determining $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$:

$$\frac{\partial f}{\partial x} = 2x - 2 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2y + 4$$

2. Set $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$:

$$2x - 2 = 0 \qquad \text{and} \qquad 2y + 4 = 0$$

3. Solving these for x and y:

$$x = 1 \qquad \text{and} \qquad y = -2$$

So there is only one stationary point in this case, which is located at $(1, -2)$.

4. Determine the second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x - 2) = 2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2y + 4) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2y + 4) = 0$$

5. Substituting these values into the formula for Δ and evaluating:

$$\Delta = \left(\frac{\partial^2 z}{\partial y \partial x}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = 0^2 - 2 \times 2 = -4$$

Thus, since at $(1, -2)$ we have:

$$\Delta = -4 < 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2} > 0$$

then this is a minimum stationary point.

2.7 MATLAB

Partial differentiation is actually the same as regular differentiation in MATLAB, using the `diff` command with two arguments. The only difference is that you will need to remember to declare all variables as symobolic first. For example:

$$\frac{\partial}{\partial x}\big(xy^2 + 3y\sin(x)\big)$$

```
syms x y
diff( x*y*y+3*y*sin(x) , x )
```

3 Revision

So, what should I be able to do at this point in the course?

- Apply the standard techniques of differential calculus:
  - Differentiate simple functions using the table of standard derivatives.
  - Use the chain rule to differentiate functions of functions.
  - Use the product (and/or quotient) rule to differentiate two functions multiplied together.
- Determine the rate of change of a univariate function (i.e. a function of only one variable).
- Locate the maxima and minima of a function by locating the stationary points, and apply the second-derivative test to classify them.
- Calculate the partial derivatives of a function.
- Calculate the second-order partial derivatives of a function.
- Determine the rate of change of a function of two or three variables.
- Calculate the maxima and minima of a function of two variables.
- Calculate the propagation of uncertainty for a multi-variate function.
- Check your results for regular and partial differentiation using MATLAB.
- Plot curves to visualise your results in MATLAB.

Note that for the standard techniques of differentiation, including the product and chain rule, there are numerous extra videos provided to aid your self-guided revision of this topic.