MMaD: Lecture 10 handout

Fourier series of a periodic function

For a periodic function $f(t)$ that has period T and angular frequency $\omega = \frac{2\pi}{T}$:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

where

$$a_0 = \frac{2}{T}\int_0^T f(t)\mathrm{d}t$$

$$a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega t)\mathrm{d}t$$

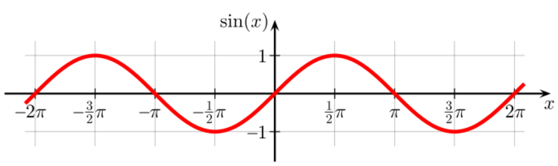
$$b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega t)\mathrm{d}t$$

For our particular function $f(t)$, we need to know the values of the Fourier coefficients (numbers) $a_0, a_1, a_2, \ldots$ and $b_1, b_2, b_3, \ldots$

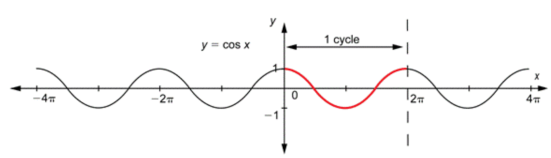
Determining the values of these constants is the problem of Fourier analysis.

Periodic functions

Sine and cosine are examples of periodic functions - they repeat a pattern every 2π radians.



(a) $\sin(x)$



(b) $\cos(x)$

For a periodic function $f(t)$, the minimum time required for one full cycle is the **period** T.

Such a function can be written as:

$$f(t + T) = f(t)$$

The number of full cycles per unit of time (usually seconds) is called the **frequency** f:

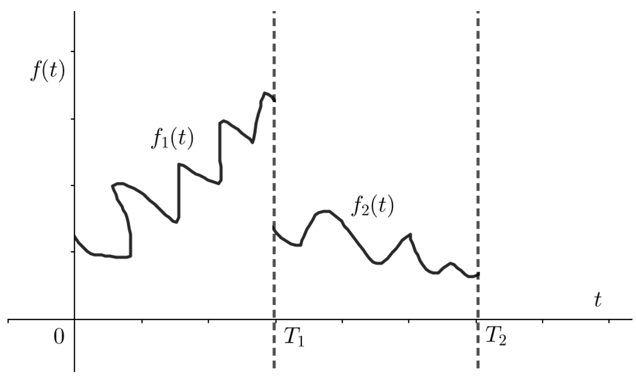
$$f = \frac{1}{T}$$

It is often useful to consider the **angular frequency** ω, measured in radians per second:

$$\omega = \frac{2\pi}{T} \qquad \text{or} \qquad T = \frac{2\pi}{\omega}$$

Constructing piecewise functions using Heaviside functions

Consider a function f that behaves like f_1 for the interval $[0, T_1]$, then changes to act like f_2 during the next interval $[T_1, T_2]$ before switching off.



We can write this as:

$$f(t) = f_1(t)\Big(H(t) - H(t - T_1)\Big) + f_2(t)\Big(H(t - T_1) - H(t - T_2)\Big)$$

Example: Square wave

This is one example of a periodic wave:

$$f(t) = \begin{cases} 3 & \text{if } 0 < t < 1 \\ -3 & \text{if } 1 < t < 2 \end{cases}$$

which repeats every 2 units, written as:

$$f(t) = f(t + 2)$$