MMaD: Lecture 11 handout

Fourier series of a periodic function

For a periodic function $f(t)$ that has period T and angular frequency $\omega = \frac{2\pi}{T}$:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

where

$$a_0 = \frac{2}{T}\int_0^T f(t)\mathrm{d}t$$

$$a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega t)\mathrm{d}t$$

$$b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega t)\mathrm{d}t$$

Essential integrals

If α is a real constant, then the following integrals (with respect to t) hold:

$$\int \alpha \,\mathrm{d}t = \alpha t$$

$$\int \cos(\alpha t)\,\mathrm{d}t = \frac{1}{\alpha}\sin(\alpha t)$$

$$\int \sin(\alpha t)\,\mathrm{d}t = \frac{-1}{\alpha}\cos(\alpha t)$$

Odd and even functions

- An **odd** function is one where $f(-x) = -f(x)$. The graph has rotational symmetry of $180°$ about the origin.
- An **even** function is one where $f(-x) = f(x)$. The graph has reflective symmetry about the vertical axis.
- In the Fourier series of functions that are purely **odd**, we can eliminate the a_k terms.
- In the Fourier series of functions that are purely **even**, we can eliminate the b_k terms.

Because the sine wave is an odd function, and the cosine wave is an even function:

$$\cos(-x) = \cos(x) \quad \text{and} \quad \sin(-x) = -\sin(x)$$

Example: Square wave

This is one example of a periodic wave:

$$f(t) = \begin{cases} 17 & \text{for} \quad \frac{-\pi}{2} < t < \frac{\pi}{2} \\ 0 & \text{for} \quad \frac{-3\pi}{2} < t < \frac{-\pi}{2} \end{cases}$$

which repeats every 2π units, written as:

$$f(t) = f(t + 2\pi)$$