Fourier Series (Part II)

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Aims for this week

- Briefly revise the integration of constants, sine and cosine functions.
- Learn to identify odd and even functions.
- Learn how to calculate the formulae for all Fourier coefficients of a periodic signal, by integrating manually (rather than using MATLAB).

Recap I: standard integration

If α is any real constant, then the following integrals hold:

$$\int \alpha \, \mathrm{d}t = \alpha t$$

$$\int \cos(\alpha t) \, \mathrm{d}t = \frac{1}{\alpha} \sin(\alpha t)$$

$$\int \sin(\alpha t) \, \mathrm{d}t = \frac{-1}{\alpha} \cos(\alpha t)$$

Prerequisite: Odd and Even functions

Functions can be classified as:

- **odd**
- **even**
- both (in some very trivial cases, like $f(x) = 0$)
- neither

Being able to recognise an odd or even function will enable us to take shortcuts when calculating Fourier Series.

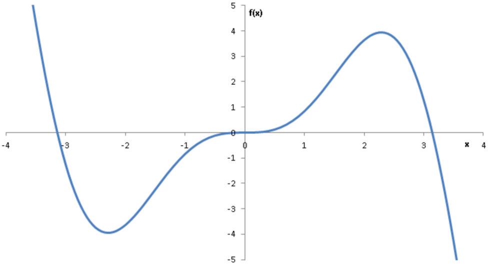
Odd functions

Odd Functions

An odd function is one where $f(-x) = -f(x)$.

The graph has rotational symmetry of 180° about the origin.

Example:

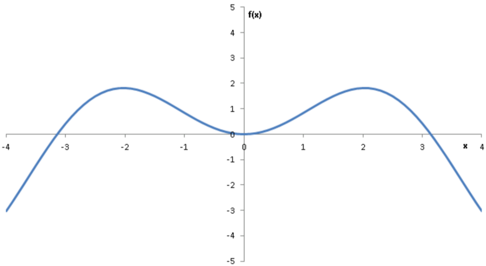


Even Functions

An even function is one where $f(-x) = f(x)$.

The graph has reflective symmetry about the vertical axis.

Example:



Odd and Even functions

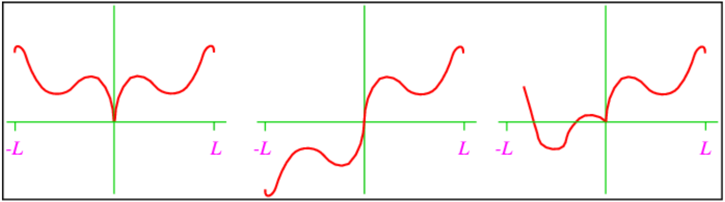
- Examples of odd functions include:

$$x, \quad x^3, \quad x^5, \quad \text{and} \quad \sin(mx)$$

- Examples of even functions include:

$$17, \quad x^2, \quad x^4, \quad \text{and} \quad \cos(mx)$$

- Which of the functions below are odd, even, or neither?



Application to Fourier Series

Fourier series of odd or even functions

If we have functions that are purely **odd**, then we can eliminate a_0 and all the a_k terms.

If we have functions that are purely **even**, then we can eliminate all the b_k terms.

A useful consequence of **cosine being even**, and **sine being odd**, is that for any value of x:

Sine and Cosine

$$\cos(-x) = \cos(x) \quad \text{and} \quad \sin(-x) = -\sin(x)$$

Recap II: Fourier Series

Any periodic function $f(t)$ with angular frequency ω can be written as a (potentially infinite) combination of sine and cosine waves:

Fourier Series for a general periodic function

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + \sum_{k=1}^{\infty} b_k \sin(k\omega t)$$

We need to find the values of the coefficients (numbers) a_0, a_1, a_2, ... and b_1, b_2, b_3, ...

Fourier analysis consists of determining these constants by the following integrals...

Fourier coefficients

Integral formulae for Fourier coefficients

$$a_0 = \frac{2}{T}\int_0^T f(t)\,\mathrm{d}t$$

$$a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega t)\,\mathrm{d}t$$

$$b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega t)\,\mathrm{d}t$$

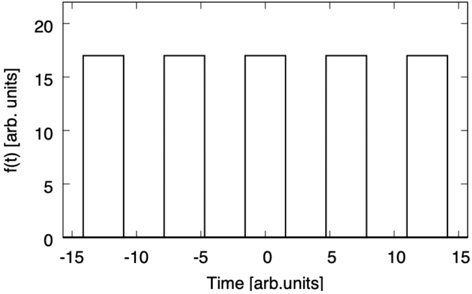
where ω is the angular frequency of $f(t)$.

Example: Square Wave

A common example is this square wave, given by:

$$f(t) = \begin{cases} 17 & \text{for} \quad \frac{-\pi}{2} < t < \frac{\pi}{2} \\ 0 & \text{if} \quad \frac{\pi}{2} < t < \frac{3\pi}{2} \end{cases}$$

And it repeats with period 2π.



Example: Square Wave

Last week, we saw how to calculate the first few coefficients using MATLAB. In this way we could find that:

$$a_1 = \frac{34}{\pi}$$

$$a_2 = 0$$

$$a_3 = \ldots$$

This would allow us to obtain an approximation to the Fourier series, called the **Fourier partial sum**.

However, by using the integral formulae for general k, we can calculate the infinite Fourier series.

Example: Square Wave

As the period is $T = 2\pi$, the angular frequency is therefore:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Note that this is a **piecewise** function, meaning that it behaves in two different ways during different regions of a single cycle:

$$f(t) = \begin{cases} 17 & \text{for} \quad \frac{-\pi}{2} < t < \frac{\pi}{2} \\ 0 & \text{for} \quad \frac{\pi}{2} < t < \frac{3\pi}{2} \end{cases}$$

We will therefore have to split the integrals for a_0, a_k and b_k up and consider these different regions separately (multiple integrals).

Example: Square Wave

We previously noted that we can choose *any range* for our integrals as long as they span a width equal to the period, which in this case is $T = 2\pi$.

For this example, rather than integrating over $0 < t < 2\pi$, let's integrate over $\frac{-\pi}{2} < t < \frac{3\pi}{2}$ so that they must be split into just two, rather than three, integrals each time.

So for example, instead of:

$$a_0 = \frac{2}{2\pi} \int_0^{2\pi} f(t)\,\mathrm{d}t$$

Let's calculate:

$$a_0 = \frac{2}{2\pi} \int_{-\pi/2}^{3\pi/2} f(t)\,\mathrm{d}t$$

Example: Square Wave

Calculating the DC level first:

$$a_0 = \frac{2}{2\pi}\int_{-\pi/2}^{3\pi/2} f(t)\,\mathrm{d}t$$

Then splitting the range in two:

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(t)\,\mathrm{d}t + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} f(t)\,\mathrm{d}t$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 17\,\mathrm{d}t + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} 0\,\mathrm{d}t$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 17\,\mathrm{d}t$$

as the integral of 0 is simply 0!

Example: Square Wave

So there is only one integral we actually need to evaluate here:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 17\,\mathrm{d}t \\ &= \frac{1}{\pi}\Big[17t\Big]_{-\pi/2}^{\pi/2} \\ &= \frac{1}{\pi}\left\{17\left(\frac{\pi}{2}\right) - 17\left(\frac{-\pi}{2}\right)\right\} \\ &= 17 \end{aligned}$$

Example: Square Wave

So we have found that:

$$a_0 = 17$$

The DC level is then:

$$\frac{a_0}{2} = \frac{17}{2} = 8.5$$

This is the **average value of the function** over one cycle, which can be an easy alternative method to use to find the DC level.

Next, we obtain the formula for a general a_k:

$$\begin{aligned}
a_k &= \frac{2}{2\pi}\int_{-\pi/2}^{3\pi/2} f(t)\cos(kt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 17\cos(kt)\,\mathrm{d}t + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} 0\cdot\cos(kt)\,\mathrm{d}t \\
&= \frac{17}{\pi}\left[\frac{1}{k}\sin(kt)\right]_{-\pi/2}^{\pi/2} \\
&= \frac{17}{k\pi}\left\{\sin\left(\frac{k\pi}{2}\right) - \sin\left(-\frac{k\pi}{2}\right)\right\}
\end{aligned}$$

Example: Square Wave

But earlier we saw that sine is an **odd** function, and so:

$$\sin\left(-\frac{k\pi}{2}\right) = -\sin\left(\frac{k\pi}{2}\right)$$

Thus,

$$\begin{aligned} a_k &= \frac{17}{k\pi}\left\{\sin\left(\frac{k\pi}{2}\right) - \sin\left(-\frac{k\pi}{2}\right)\right\} \\ &= \frac{17}{k\pi}\left\{\sin\left(\frac{k\pi}{2}\right) + \sin\left(\frac{k\pi}{2}\right)\right\} \\ &= \frac{34}{k\pi}\sin\left(\frac{k\pi}{2}\right) \end{aligned}$$

Example: Square Wave

If k is even, then $\sin\left(\frac{k\pi}{2}\right) = 0$, so $a_k = 0$ for any even k.

If k is odd,

$$k = 1 \implies \sin\left(\frac{k\pi}{2}\right) = 1$$

$$k = 3 \implies \sin\left(\frac{k\pi}{2}\right) = -1$$

$$k = 5 \implies \sin\left(\frac{k\pi}{2}\right) = 1$$

So this will give a pattern of $0, 1, 0, -1, \ldots,$ multiplied by $\frac{34}{k\pi}$

Example: Square Wave

Fortunately b_k is easier:

$$\begin{aligned}
b_k &= \frac{2}{2\pi}\int_{-\pi/2}^{3\pi/2} f(t)\sin(kt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 17\sin(kt)\,\mathrm{d}t + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} 0\cdot\sin(kt)\,\mathrm{d}t \\
&= \frac{17}{\pi}\left[\frac{-1}{k}\cos(kt)\right]_{-\pi/2}^{\pi/2} \\
&= \frac{-17}{k\pi}\left\{\cos\left(\frac{k\pi}{2}\right) - \cos\left(-\frac{k\pi}{2}\right)\right\}
\end{aligned}$$

Example: Square Wave

But earlier we saw that cosine is an **even** function, and so:

$$\cos\left(-\frac{k\pi}{2}\right) = \cos\left(\frac{k\pi}{2}\right)$$

Thus,

$$\begin{aligned} b_k &= \frac{-17}{k\pi}\left\{\cos\left(\frac{k\pi}{2}\right) - \cos\left(-\frac{k\pi}{2}\right)\right\} \\ &= \frac{-17}{k\pi}\left\{\cos\left(\frac{k\pi}{2}\right) - \cos\left(\frac{k\pi}{2}\right)\right\} \\ &= \frac{-17}{k\pi} \times 0 \\ &= 0 \qquad \text{for any integer } k \end{aligned}$$

Example: Square Wave

We have obtained formulae for all the Fourier coefficients for this square wave. The general formula of the Fourier series:

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + \sum_{k=1}^{\infty} b_k \sin(k\omega t)$$

Will become in this specific case:

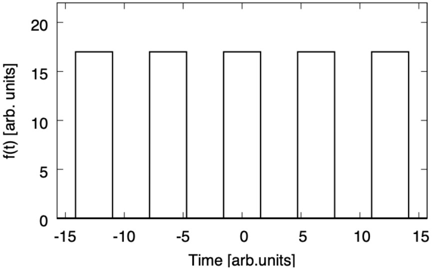
$$\begin{aligned} f(t) &= \frac{17}{2} + \sum_{k=1}^{\infty} \frac{34}{k\pi} \sin\left(\frac{k\pi}{2}\right) \cos(kt) \\ &= \frac{17}{2} + \frac{34}{\pi}\cos(t) - \frac{34}{3\pi}\cos(3t) + \frac{34}{5\pi}\cos(5t) - \dots \end{aligned}$$

Summary

We have carried out a Fourier analysis of the square wave.

We saw that it consists **only** of a_k terms.

This is because the square wave we drew was an **even** function (it has reflective symmetry about the y-axis), so we could have realised $b_k = 0$ (for all k) right at the start!



Exercise

A periodic waveform is given by:

$$f(t) = \begin{cases} -2 & \text{for} \quad -\pi < t < -\frac{\pi}{2} \\ 0 & \text{for} \quad \frac{-\pi}{2} < t < \frac{\pi}{2} \\ 2 & \text{if} \quad \frac{\pi}{2} < t < \pi \end{cases}$$

and this function repeats every 2π, which is denoted by

$$f(t) = f(t + 2\pi)$$

1. Sketch this function over at least three periods.
2. Determine the Fourier Series of $f(t)$.