The Fourier Transform for Discrete Data

Dr Gavin M Abernethy

Aims for this week

- Learn how to apply the **discrete Fourier transform** to sampled data.
- Interpret the resulting **frequency spectrum** to recover the continuous signal being sampled.

The forward Fourier transform $F(\omega)$ is a special operation that turns a time-signal $f(t)$ into a frequency spectrum:

Fourier transforms

The **(forward) Fourier transform**:

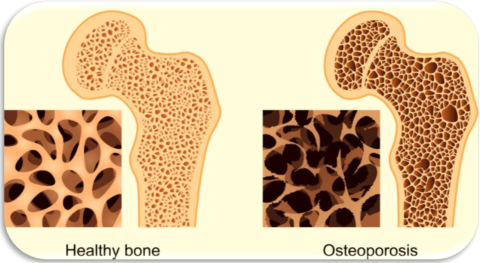
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, \mathrm{d}t$$

and the **(inverse) Fourier transform**:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, \mathrm{d}\omega$$

Application: bone density

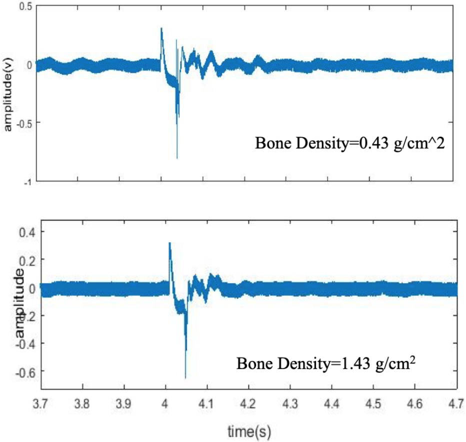
This example involves analysing the vibrations of a patient's bones in order to detect osteoporosis.



When we detect sounds, we are measuring the amplitude of the displacement of air over time. The density of bones affects the sound they produce when vibrating. In an experiment researchers subject the bones to vibration, and record the response.

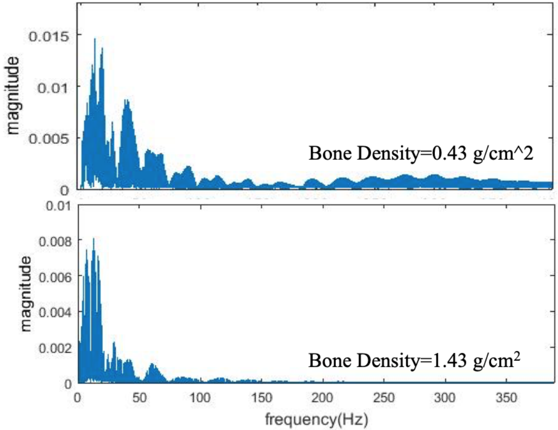
Bone density application

The amplitudes of the vibrations **over time** between healthy and affected samples are hard to distinguish:

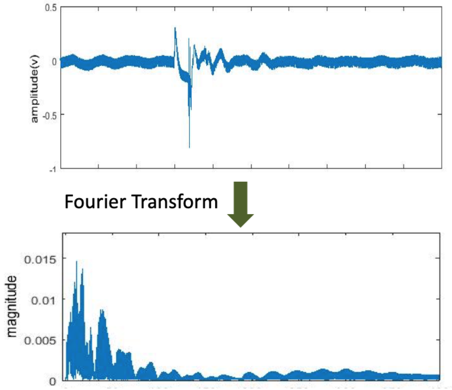


Bone density application

But the Fourier transform tells us what **frequencies** are present in the sound of the vibrating bones, and with what amplitude.



Fourier Transform of Discrete Data



The data in the top figure is not actually continuous, but consists of a set of points (**discrete data**). Collecting *any* real data can only ever give discrete samples at every minute, second, cm, etc.

How do we find the Fourier transform of discrete data?

Fourier Transform of Discrete Data

If we have a set of N data points $\{f_n\}$ observed at time $\{t_n\}$, the discrete Fourier transform takes us from the time-domain (where we have the strength of the signal over time) to the frequency domain (where we have the relative importance of different frequencies):

$$(t_n, f_n) \xrightarrow{Fourier \quad transform} (\omega_k, F_k)$$

Notation

Note that some notation has a different meaning here to other parts of the module:

- t_n are the observation times of our data.
- f_n are the corresponding strengths of the signal observed at time t_n.
- ω_k here refers to frequencies in hertz (*not* angular frequency).
- F_k is the magnitude of corresponding frequency ω_k in the Fourier transform.
- T will refer to the total time interval over which the signal was observed.
- N is the number of samples taken in time T.

Fourier Transform of Discrete Data

There are N integer values of k, ranging between $-\frac{N}{2}$ and $\frac{N-1}{2}$

For each value of k, there is a specific frequency ω_k, and associated with it a complex number F_k.

This sequence of numbers F_k, which are independent of the observation time t_n, are the Fourier transform of the sequence $\{f_n\}$.

They are determined by:

$$F_k = \sum_{n=0}^{N-1} f_n \, \mathrm{e}^{-j2\pi nk/N}$$

The y-axis (F_k)

This can be done by an in-built MATLAB function.
"fft" stands for the "Fast Fourier Transform" algorithm.

For example, to take the Fourier Transform of a sequence $1, 2, 1, 0$:

```
x = [ 1 2 1 0 ];
F = fft(x);
```

And we get the result $F = \{ \, 4, \; -2j, \; 0, \; 2j \, \}$.

To determine what frequencies these values correspond to, we consider at what times the original data points were observed.

The x-axis (ω_k)

The frequencies ω_k start at zero, and increase by the frequency spacing:

$$\Delta f = \frac{s_r}{N} = \frac{1}{T}$$

where s_r is the sampling rate (number of samples per second).

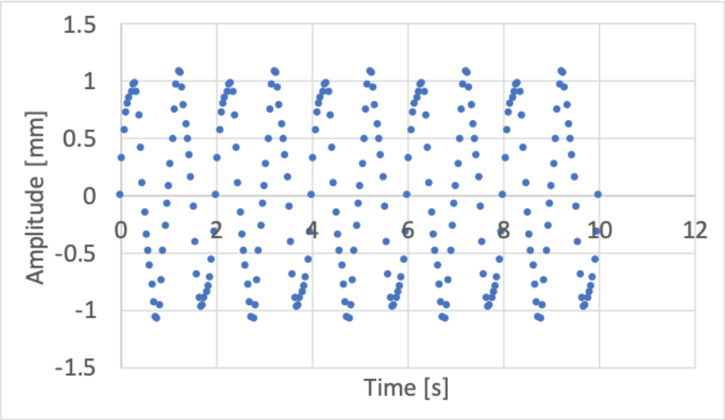
The maximum frequency on the x-axis is the Nyquist frequency:

$$\frac{N}{2T}$$

we will discuss this more later.

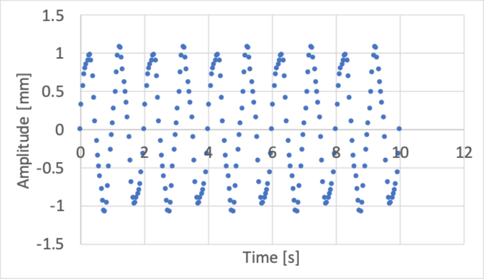
Example

This signal is sampled 256 times over 10 seconds:



Example: inspecting the data

As there are $N = 256$ samples over $T = 10$ seconds, the sampling rate is $s_r = 256/10 = 25.6s^{-1}$



From looking at the graph, there appears to be an oscillation with period of about 1 s and hence a 1 Hz frequency.

But is that everything?

Example: MATLAB procedure

Take the Fourier transform of the dependent data (the values of f), and then we want the **magnitudes** of these complex numbers:

```
F = fft(f)        This takes the Fourier transform

m = abs(F)        ``abs'' for absolute value
```

Plot the frequency spectrum of $(N/2) - 1 = 127$ values:

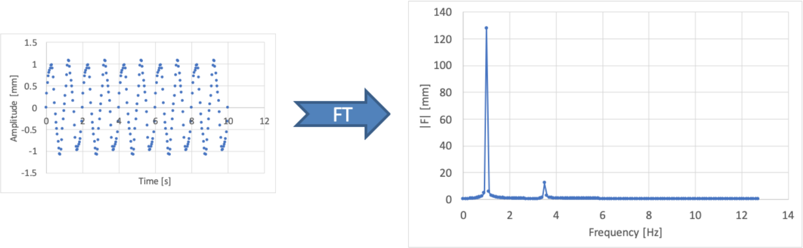
- On the x-axis, we want 127 frequencies that start at 0 and increase by the frequency spacing:

$$s = \frac{1}{T} = \frac{1}{10} = 0.1\ Hz$$

- On the y-axis, we want the first 127 values of magnitude m.

Example: results

Plotting this results in the frequency spectrum:

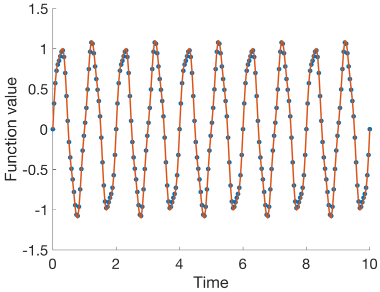


We can now see **two** peaks, where there are important frequencies present in the signal:

- One at 1 Hz, as expected, with magnitude 127.5.
- Another at 3.5 Hz, with magnitude 12.06.

Example: results

By scaling the two sine waves appropriately, we can then reconstruct the actual waveform that this data was sampled from:



$$f(t) = \frac{A}{127.50}\left(127.50\sin(2\pi \times 1t) + 12.06\sin(2\pi \times 3.5t)\right)$$

where $A = 1.0278$ is the estimated amplitude of the "main" oscillation that we can measure from the scatter graph.

Example:

Open the worksheet:

```
Lecture12_DiscreteFourierTransformExample.mlx
```

The Nyquist Frequency

The **maximum frequency we can resolve** (detect) is related to the total amount of time T that we have sampled the signal over.

This is known as the **Nyquist frequency**:

$$\frac{N}{2T}$$

The point is that high frequency signals cannot be detected if we do not sample with sufficient frequency. For example, if you only sampled a regular sine wave $\sin(t)$ every 2π, you would appear to be observing a constant signal!

The Nyquist Frequency

To explain this concept in another context, imagine you wanted to know the patterns governing how many people use this classroom at any given time.

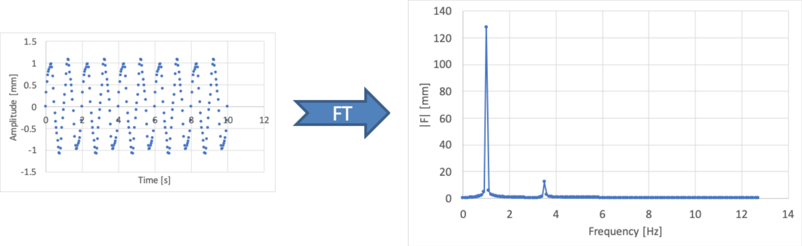
One big pattern will be that there a lot of people in the room during the daytime, but very few at night.

How often must you look into the room to detect this pattern?

If you only sample the usage once per day at 10am, this pattern will be impossible to detect, it is “between” the data. The Nyquist frequency quantifies this idea for Fourier analysis of sampled data.

Example: Nyquist Frequency

Returning to our example:



In this case the Nyquist frequency is:

$$\frac{N}{2T} = \frac{256}{2 \times 10} = 12.8 \; Hz$$

so this is the upper limit of the x-axis on the frequency spectrum.

Summary

Fourier transforms produce the **frequency spectrum** of a signal, which shows us what frequencies are present in a potentially complicated-looking waveform.

Real data usually consists of **discrete samples** of the continuous function. To deal with this, and approximate the "true" continuous Fourier transform, we can use the **discrete Fourier transform**.

We will be practicing this in both EXCEL and MATLAB in the final tutorial.

Real Applications of Fourier Analysis

If earthquake vibrations can be separated into vibrations of different speeds and amplitudes, buildings can be designed to avoid interacting with the strongest ones.

If sound waves can be separated into bass and treble frequencies, we can boost the parts we care about, and hide the ones we don't. The crackle of random noise can be removed.

If computer data can be represented with oscillating patterns, perhaps the least-important ones can be ignored. This can drastically shrink file sizes (and why JPEG and MP3 files are much smaller than raw .bmp or .wav files).