Maths for Materials and Design

# Fourier Analysis

# Contents

# 1 Lecture 1: Fourier series by MATLAB

## 1.1 Objectives

- Learn about the underlying idea of **Fourier Series**.
- Recognise a **periodic** function and determine its angular frequency.
- Represent a piecewise function using **Heaviside step functions**.
- Calculate the Fourier coefficients using MATLAB.

**Jean-Baptiste Joseph Fourier (1768-1830)**

“[Mathematics] brings together phenomena the most diverse, and discovers the hidden analogies which unite them”

*The Analytical Theory of Heat*

## 1.2 Introduction

**Fourier series:**

Fourier series is a technique by which **periodic** functions may be represented or approximated by combinations of **simple sine and cosine waves**. We can then determine how important each frequency is to the overall function.

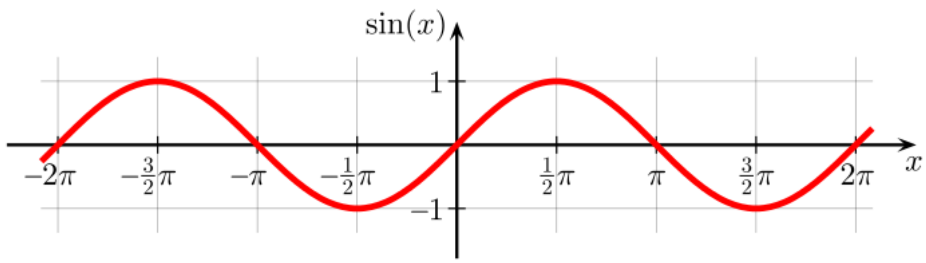
This technique finds applications in many areas of engineering where you might wish to analyse a signal:

- image processing
- audio compression
- seismic wave analysis
- x-ray crystallography
- material spectroscopy

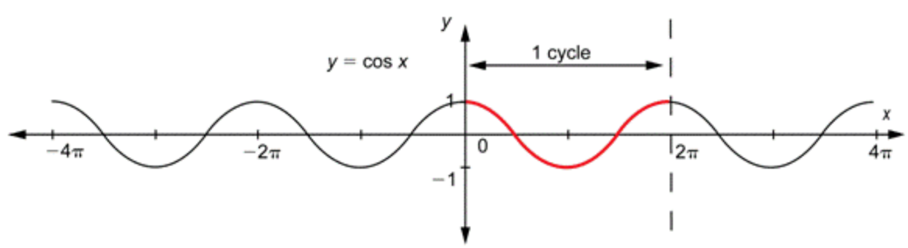
## 1.3 Periodic functions

### 1.3.1 Trigonometry

Sine, cosine and tangent are examples of **periodic** functions, meaning that they repeat a pattern forever.




The **period** $T$ of both sine and cosine is $2\pi$ as this is the minimum time required before the pattern begins to repeat.




**Note:** We will **never** use degrees to measure the angular input to sine and cosine.

We will **always** use radians - make sure your calculator is set to it!

So for sine and cosine...

- Instead of a full period every $360°$, a full period is $2\pi$
- Instead of a half-period every $180°$, a half-period is $\pi$.

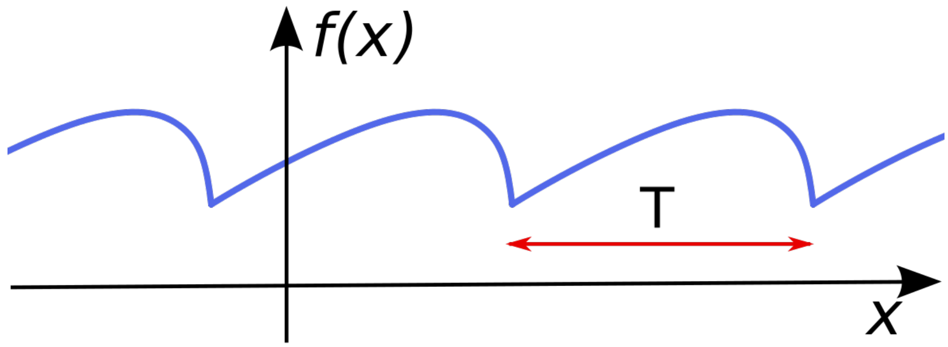
### 1.3.2 Periodicity

**Periodic function**

A function $f(t)$ is periodic with period $T$ if for all values of $t$, and for any integer $m$:

$$f(t + mT) = f(t)$$

The minimum time required for one full cycle is the **period** $T$.




The number of full cycles per unit of time (usually seconds) is called the **frequency** and given by $f = T^{-1}$.

However it is often useful to use the **angular frequency** $\omega$, measured in radians per second.

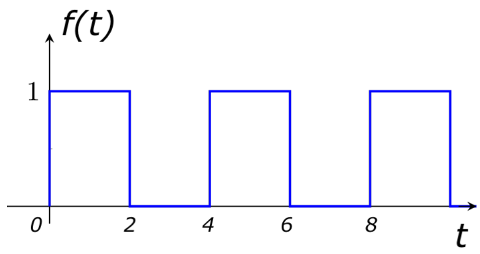
**Angular Frequency:**

$$\omega = \frac{2\pi}{T} \qquad \text{or} \qquad T = \frac{2\pi}{\omega}$$

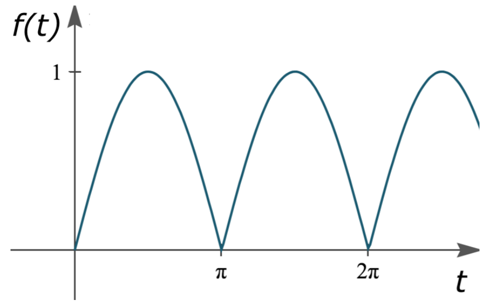
As the period of oscillation $T$ increases, the frequency (both angular and regular) decreases and vice versa.

**Exercise: Periodic functions**

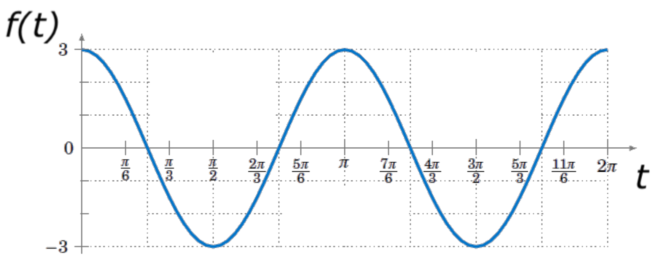
Determine the period and angular frequency of the following periodic functions:




(a) Pulse wave




(b) $|\sin(t)|$




(c) $3\cos(2t)$

### 1.3.3 Periodic functions: pure sine and cosine waves

Note that if we have a sine or cosine wave of the form:

$$f(t) = a\sin(mt) \quad \text{or} \quad f(t) = a\cos(mt)$$

where $a$ and $m > 0$ are constants,

Then the angular frequency is just:

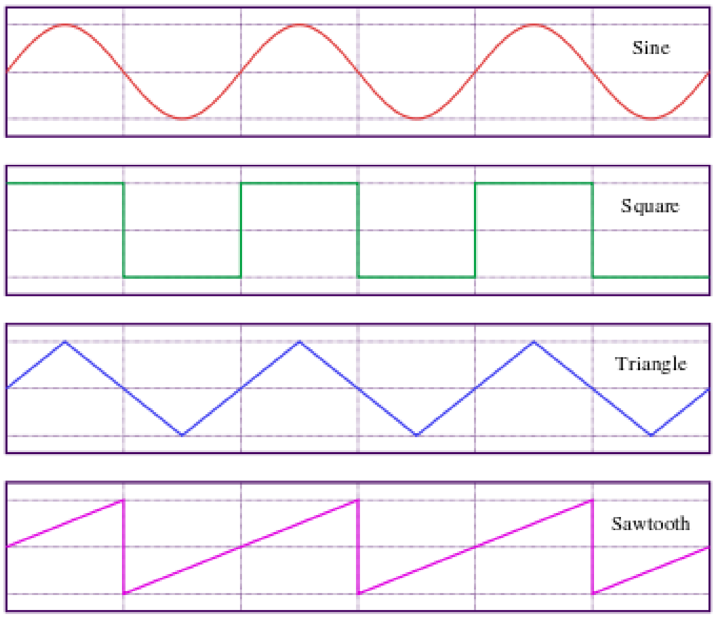
$$\omega = m$$

## 1.4 Fourier series

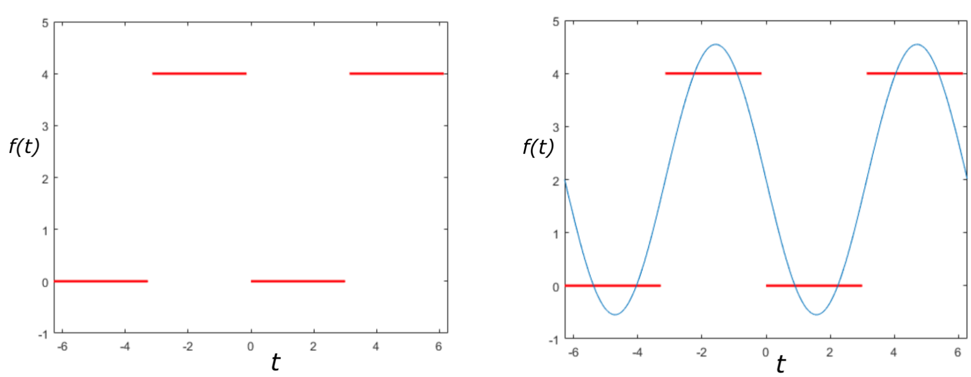
### 1.4.1 Motivation for Fourier Series

So what about other kinds of signals?

Often when analysing audio, electrical, or other signals, we encounter (possibly discountinuous) periodic waveforms:




To analyse the frequencies present it can be useful to approximate these signals by fitting a continuous curve:



This is the core idea of Fourier Series:

We can represent (almost) any periodic function by some **combination of sine and cosine waves of different frequencies**.

**Fourier series:**

We can construct any **periodic** function $f(t)$ with period $T$ (angular frequency $\omega = 2\pi/T$) by adding the right amount sine and cosine waves together:

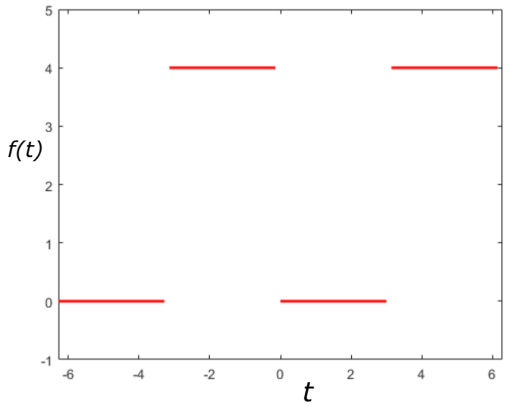
$$\begin{aligned} f(t) = \frac{1}{2}a_0 \quad &+ \quad a_1\cos(\omega t) + a_2\cos(2\omega t) + a_3\cos(3\omega t) + \dots \\ &+ \quad b_1\sin(\omega t) + b_2\sin(2\omega t) + b_3\sin(3\omega t) + \dots \end{aligned}$$

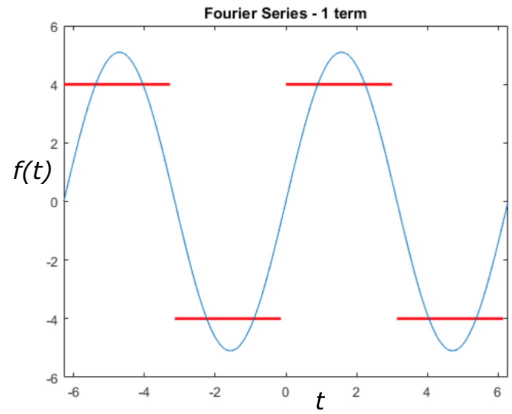
This is now (in general) an infinite sum. We can use sigma notation to write it as:

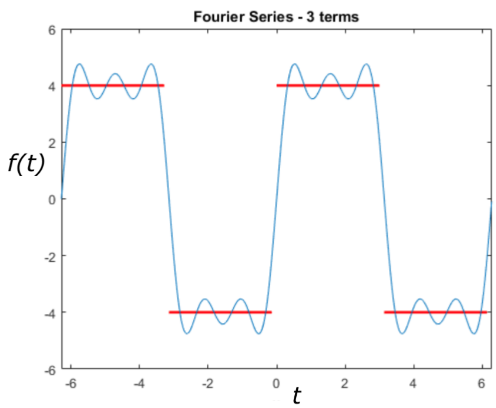
**Fourier Series for a general periodic function:**

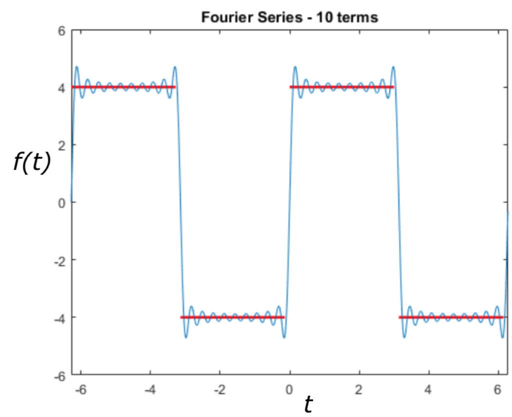
$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega t) + \sum_{n=1}^{\infty} b_n\sin(n\omega t)$$

As more terms in the Fourier Series are calculated (with higher frequencies), a more accurate approximation is found:




To determine the Fourier series of our signal $f(t)$, in addition to the angular frequency $\omega$ we will need to know how much of each frequency is required. So we need to find the values of the **Fourier coefficients** (numbers) $a_0, a_1, a_2, \ldots$ and $b_1, b_2, b_3, \ldots$

From applications in electronics, $\frac{a0}{2}$ is called the DC level. In simple cases, it can be found by calculating the average value of the graph of $f(t)$.

But how do we determine these constants in general? They are given by integrals that we will calculate using MATLAB.

### 1.4.2 Calculating the Fourier coefficients

**Fourier Series for a general periodic function:**

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + \sum_{k=1}^{\infty} b_k \sin(k\omega t)$$

$\omega$ is the angular frequency of the function $f(t)$. It is sometimes written as $\omega_0$ and called the **fundamental frequency**.

$$a_0 = \frac{2}{T}\int_0^T f(t)\mathrm{d}t$$

$$a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega t)\mathrm{d}t$$

$$b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega t)\mathrm{d}t$$

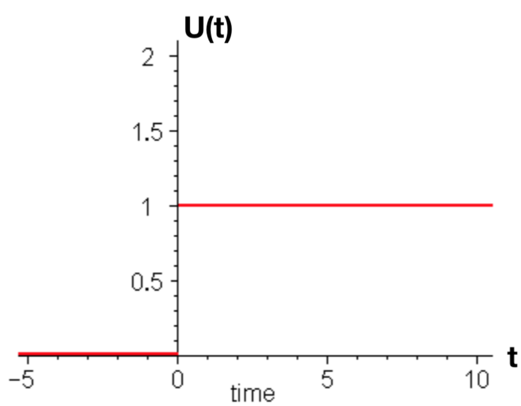
## 1.5 Heaviside Step Function

To describe a piecewise function into MATLAB, we can construct them using a combination of **step functions** that "switch" the constitutent parts of the functions behaviour "on" and "off" at the necessary times.

The **Heaviside step function** $H(t)$ (also known as the unit step function $U(t)$ is defined by:

**Heaviside Step Function $H(t)$:**

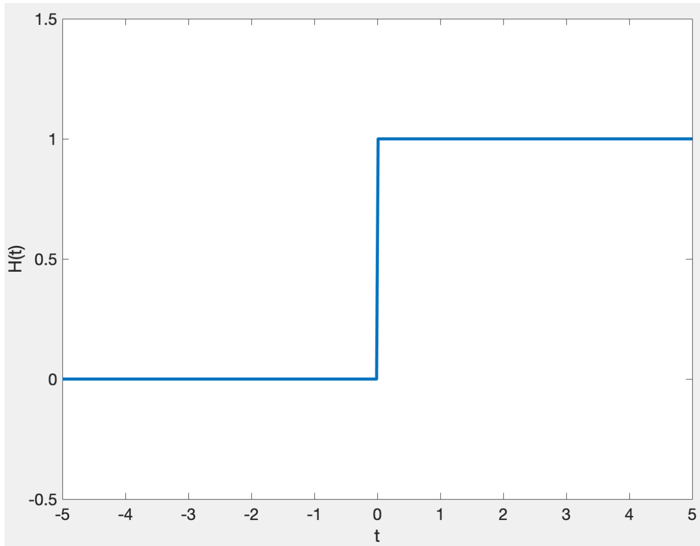
$$H(t) = \begin{cases} 0 & \text{if } t < 0; \\ 1 & \text{if } t > 0. \end{cases}$$




To input this in MATLAB:

```
syms t;
h=heaviside(t);
fplot(t,h);
```

Which results in:



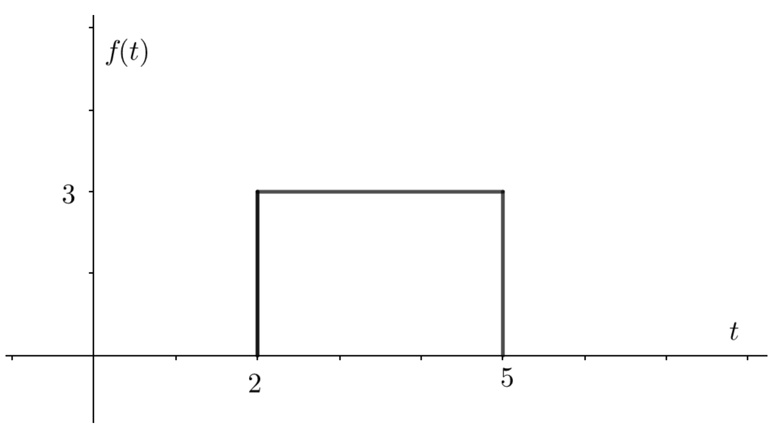

### 1.5.1 Using step functions to construct piecewise periodic functions

We can combine multiple step functions to switch signals on and off.

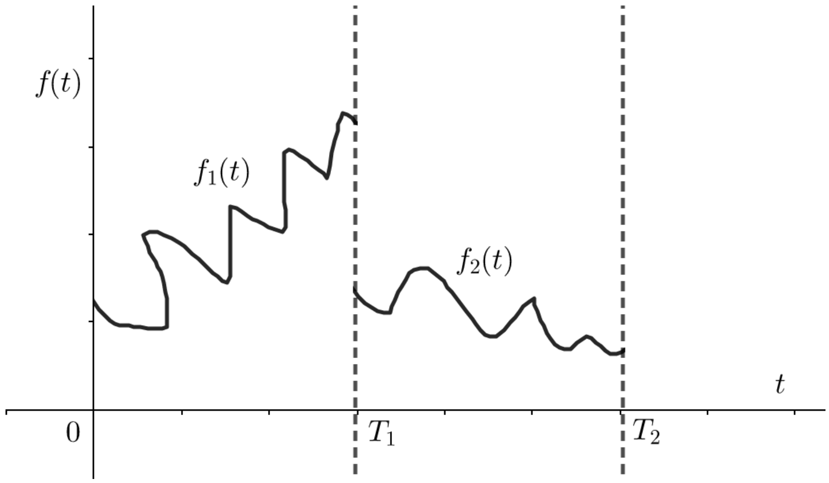
**Example:**

$$\begin{aligned} f(t) &= 3H(t-2) - 3H(t-5) \\ &= 3\Big(H(t-2) - H(t-5)\Big) \end{aligned}$$

- Signal of constant value 3.
- Begins at time $t = 2$
- Ends at time $t = 5$



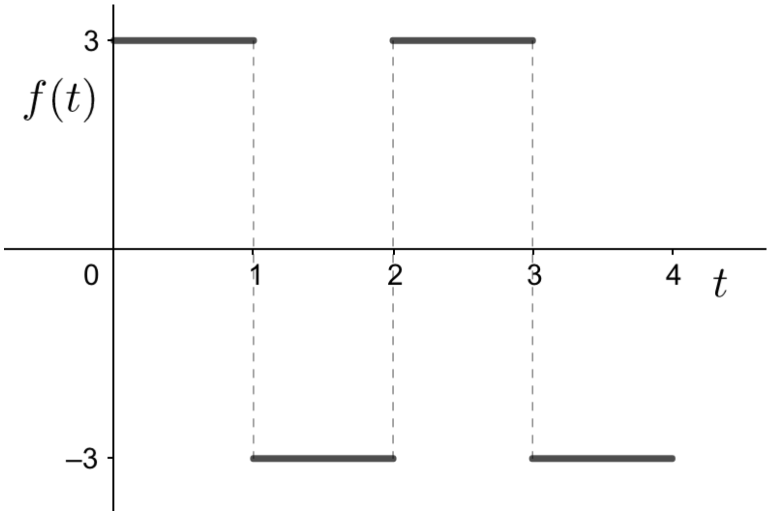
For a general function $f$ that behaves like $f_1$ for the interval $[0, T_1]$, then changes to act like $f_2$ during the next interval $[T_1, T_2]$ before switching off:




$$f(t) = f_1(t)\Big(H(t) - H(t - T_1)\Big) + f_2(t)\Big(H(t - T_1) - H(t - T_2)\Big)$$

### 1.5.2 Example: Square Wave

Consider the following square wave signal.




What is the period and angular frequency?

**Solution:**

The period is the shortest interval required before the pattern repeats. This is equal to:

$$T = 2$$

Hence the angular frequency is given by:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

(Note: this function has rotation symmetry about the origin, called being "odd", this means that $a_k = 0$ for all $k$. We can also see that the DC level must be zero.)

Now to determine the Fourier series in MATLAB.

First, you need to describe this function using Heaviside functions so we can declare it in MATLAB.

In the first period $0 < t < 2$, this function is described by:

$$f(t) = 3\Big(H(t) - H(t-1)\Big) + (-3)\Big(H(t-1) - H(t-2)\Big)$$

Open the worksheet:
`Lecture10ArbFourierExample.mlx`

Define the function:

```
f = 3 * (heaviside(t) - heaviside(t - 1)) +
    (-3) * (heaviside(t - 1) - heaviside(t - 2))
```

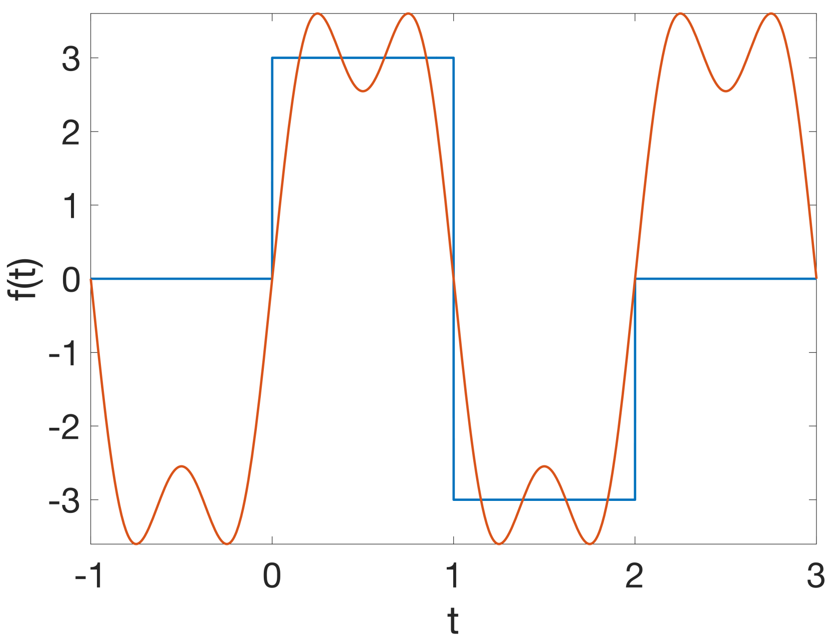
Input $T$ and $\omega$:

```
T = 2     w = pi
```

We need to use the new formula for the integals, e.g.

```
a1 = int(f * cos(w*t), t, 0, T)*2/T
b3 = int(f * sin(3 * w * t), t, 0, T)*2/T

FourApprox = a0/2 + a1 * cos(w * t) +
  a2 * cos(2 * w * t) + a3 * cos(3 * w * t) + ...
```



If we were to evaluate the integrals for general $k$, we would find:

$$a_k = 0 \quad \text{for all integers } k$$

and

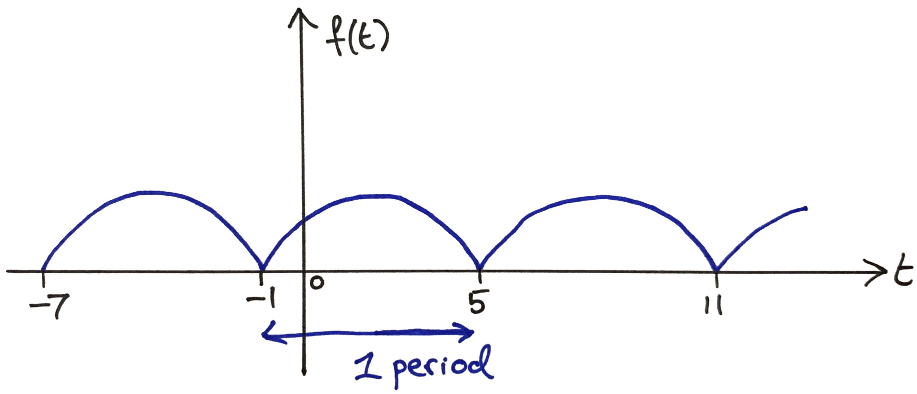
$$b_k = \frac{6}{k\pi}\{1 - (-1)^k\}$$

Thus the (infinite) Fourier series is given by:

$$\begin{aligned} f(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \\ &= \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\big(1 - (-1)^n\big)\sin(n\pi t) \end{aligned}$$

## 1.6 Arbitrary limits

**Note:** In many cases, the function does not start and end its behaviour nicely at $t = 0$ and $t = T$. For example:




In this case, it would clearly be easier to integrate over the period $[-1, 5]$, rather than $[0, 6]$. Fortunately, **we are allowed to choose any interval** of length $T$ to integrate over!

This means that in the square wave example we have just done, we could have integrated over the range $[-1, 1]$ or $[-2, 0]$, or $[1, 2]$ etc.

Any interval of length equal to $T = 2$ would suffice.

It doesn't *have* to be $[0, 2]$.

# 2 Lecture 2: Fourier series by hand

## 2.1 Objectives

- Briefly revise integration of constants, sine waves, and cosine waves.
- Learn to identify odd and even functions.
- Learn how to calculate the formulae for all Fourier coefficients of a periodic signal.

## 2.2 Recap

### 2.2.1 I: Standard integration

If $\alpha$ is a real constant, then the following integrals (with respect to $t$) hold:

$$\int \alpha \, \mathrm{d}t = \alpha t$$

$$\int \cos(\alpha t) \, \mathrm{d}t = \frac{1}{\alpha} \sin(\alpha t)$$

$$\int \sin(\alpha t) \, \mathrm{d}t = \frac{-1}{\alpha} \cos(\alpha t)$$

### 2.2.2 II: Fourier Series

A periodic function $f(t)$ can be written as a combination of sine and cosine waves:

**Fourier Series for a general periodic function:**

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + \sum_{k=1}^{\infty} b_k \sin(k\omega t)$$

We need to find the values of the coefficients (numbers) $a_0, a_1, a_2, \ldots$ and $b_1, b_2, b_3, \ldots$

Fourier analysis consists of determining these constants by the following integrals. . .

## 2.3 Odd and Even functions

Functions can be classified as:

- odd
- even
- both (in some very trivial cases, like $f(x) = 0$)
- neither

Being able to recognise such functions will enable us to take shortcuts when calculating Fourier Series.

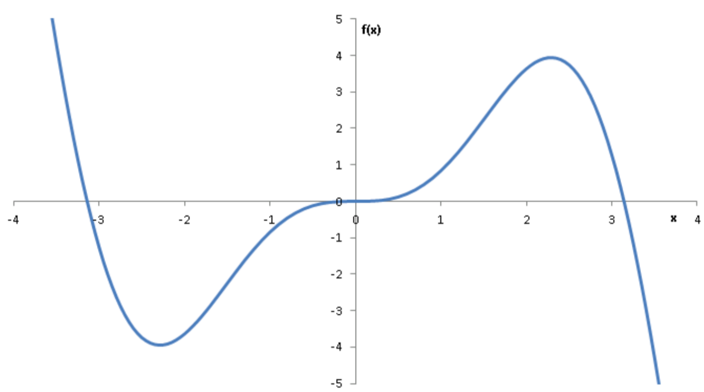
### 2.3.1 Odd functions

**Odd Functions:**

An odd function is one where $f(-x) = -f(x)$.

The graph has rotational symmetry of 180° about the origin.

**Example:**



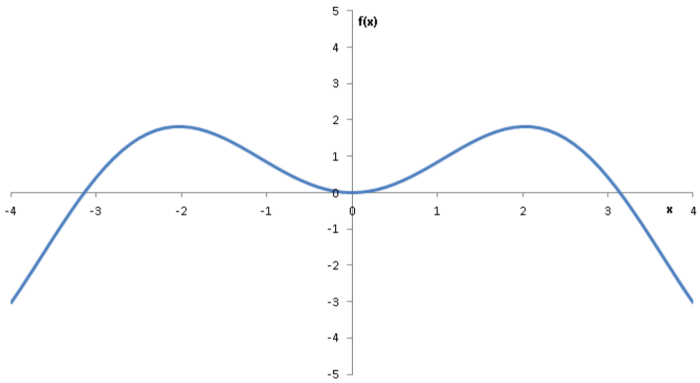

### 2.3.2 Even functions

**Even Functions:**

An even function is one where $f(-x) = f(x)$.

The graph has reflective symmetry about the vertical axis.

**Example:**




### 2.3.3 Odd and Even functions

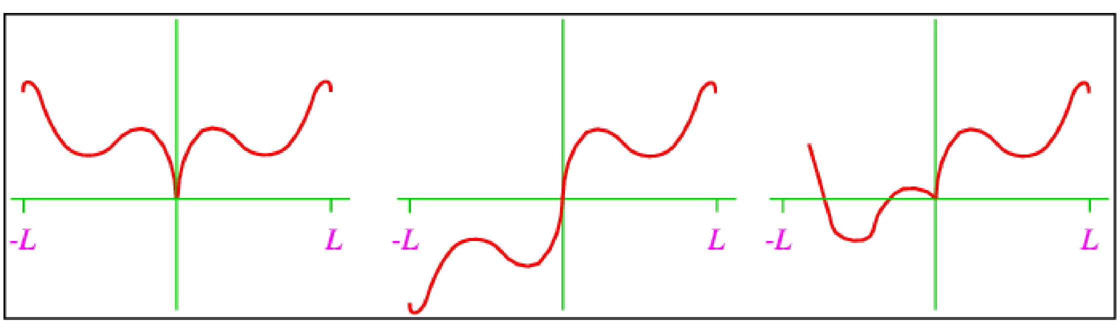
- Examples of odd functions include:

$$x, \quad x^3, \quad x^5, \quad \text{and} \quad \sin(mx)$$

- Examples of even functions include:

$$17, \quad x^2, \quad x^4, \quad \text{and} \quad \cos(mx)$$

- Which of the functions below are odd, even, or neither?



### 2.3.4 Application to Fourier Series

**Fourier series of odd and even functions:**

If we have functions that are purely **odd**, then we can eliminate $a_0$ and all the $a_k$ terms.

If we have functions that are purely **even**, then we can eliminate all the $b_k$ terms.

A very useful consequence of **cosine being even**, and **sine being odd**, is that for **any value of** $x$:

**Sine and Cosine:**

$$\cos(-x) = \cos(x) \quad \text{and} \quad \sin(-x) = -\sin(x)$$

We will use these facts to help us in the following examples.

## 2.4 Fourier coefficients

**Integral formulae for Fourier coefficients:**

$$a_0 = \frac{2}{T}\int_0^T f(t)\,\mathrm{d}t$$

$$a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega t)\,\mathrm{d}t$$

$$b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega t)\,\mathrm{d}t$$

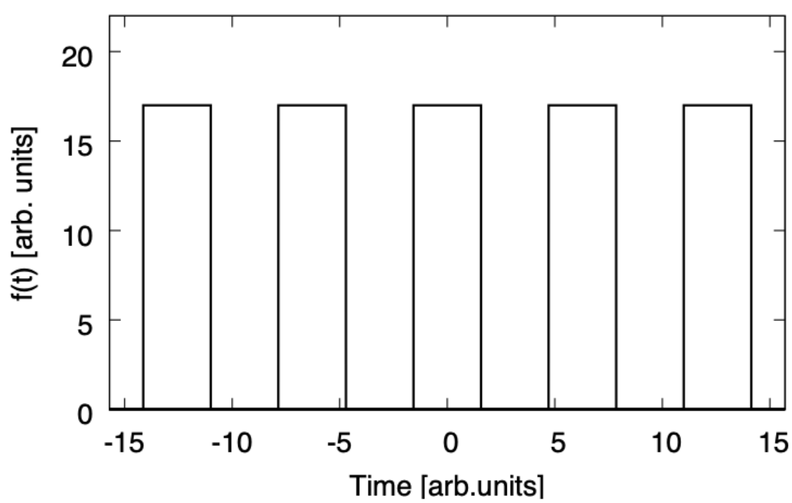
where $\omega$ is the angular frequency of $f(t)$.

### 2.4.1 Example: Square Wave

A common example is this square wave, given by:

$$f(t) = \begin{cases} 17 & \text{for} \quad \frac{-\pi}{2} < t < \frac{\pi}{2} \\ 0 & \text{if} \quad \frac{\pi}{2} < t < \frac{3\pi}{2} \end{cases}$$

And it repeats with period $2\pi$.



**Solution:**

In the previous lecture, we saw how to calculate the first few coefficients using MATLAB:

$$\begin{aligned} a_1 &= \frac{34}{\pi} \\ a_2 &= 0 \\ a_3 &= \ldots \end{aligned}$$

This would allow us to obtain an approximation to the Fourier series, called the **Fourier partial sum**.

However, by using the integral formulae for general $k$, we can calculate the infinite Fourier series.

As the period is $T = 2\pi$, the angular frequency is therefore:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Note that this is a **piecewise** function, meaning that it behaves in two different ways during different regions of a single cycle:

$$f(t) = \begin{cases} 17 & \text{for} \quad \frac{-\pi}{2} < t < \frac{\pi}{2} \\ 0 & \text{for} \quad \frac{\pi}{2} < t < \frac{3\pi}{2} \end{cases}$$

We will therefore have to split the integrals for $a_0$, $a_k$ and $b_k$ up and consider these different regions separately (multiple integrals).

Last week, we noted that we can choose *any range* for our integrals as long as they span a width equal to the period, which in this case is $T = 2\pi$.

For this example, rather than integrating over $0 < t < 2\pi$, let's integrate over $\frac{-\pi}{2} < t < \frac{3\pi}{2}$ so that they must be split into just two, rather than three, integrals each time.

For example, instead of:

$$a_0 = \frac{2}{2\pi} \int_0^{2\pi} f(t) \, \mathrm{d}t$$

Let's choose:

$$a_0 = \frac{2}{2\pi} \int_{-\pi/2}^{3\pi/2} f(t) \, \mathrm{d}t$$

Calculating the DC level first:

$$\begin{aligned}
a_0 &= \frac{2}{2\pi} \int_{-\pi/2}^{3\pi/2} f(t) \, \mathrm{d}t \quad \text{Then splitting the range in two:} \\
&= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(t) \, \mathrm{d}t + \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} f(t) \, \mathrm{d}t \\
&= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 17 \, \mathrm{d}t + \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} 0 \, \mathrm{d}t \\
&= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 17 \, \mathrm{d}t \quad \text{as the integral of 0 is simply 0!}
\end{aligned}$$

So there is only one integral we actually need to evaluate here:

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 17 \, \mathrm{d}t \\
&= \frac{1}{\pi} \Big[ 17t \Big]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{\pi} \left\{ 17\left(\frac{\pi}{2}\right) - 17\left(\frac{-\pi}{2}\right) \right\} \\
&= 17
\end{aligned}$$

Thus, we have found that:

$$a_0 = 17$$

The DC level is then:

$$\frac{a_0}{2} = \frac{17}{2} = 8.5$$

This is the **average value of the function** over one cycle, which can be an easy alternative method to use to find the DC level.

Next, we obtain the formula for a general $a_k$:

$$
\begin{aligned}
a_k &= \frac{2}{2\pi}\int_{-\pi/2}^{3\pi/2} f(t)\cos(kt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 17\cos(kt)\,\mathrm{d}t + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} 0\cdot\cos(kt)\,\mathrm{d}t \\
&= \frac{17}{\pi}\left[\frac{1}{k}\sin(kt)\right]_{-\pi/2}^{\pi/2} \\
&= \frac{17}{k\pi}\left\{\sin\left(\frac{k\pi}{2}\right) - \sin\left(-\frac{k\pi}{2}\right)\right\}
\end{aligned}
$$

But earlier we saw that sine is an **odd** function, and so:

$$\sin\left(-\frac{k\pi}{2}\right) = -\sin\left(\frac{k\pi}{2}\right)$$

Thus,

$$
\begin{aligned}
a_k &= \frac{17}{k\pi}\left\{\sin\left(\frac{k\pi}{2}\right) - \sin\left(-\frac{k\pi}{2}\right)\right\} \\
&= \frac{17}{k\pi}\left\{\sin\left(\frac{k\pi}{2}\right) + \sin\left(\frac{k\pi}{2}\right)\right\} \\
&= \frac{34}{k\pi}\sin\left(\frac{k\pi}{2}\right)
\end{aligned}
$$

If $k$ is even, then $\sin\left(\frac{k\pi}{2}\right) = 0$ and $\sin\left(\frac{-k\pi}{2}\right) = 0$, so $a_k = 0$.

If $k$ is odd,

$$k = 1 \quad \Longrightarrow \quad \sin\left(\frac{k\pi}{2}\right) - \sin\left(\frac{-k\pi}{2}\right) = 2$$

$$k = 3 \quad \Longrightarrow \quad \sin\left(\frac{k\pi}{2}\right) - \sin\left(\frac{-k\pi}{2}\right) = -2$$

$$k = 5 \quad \Longrightarrow \quad \sin\left(\frac{k\pi}{2}\right) - \sin\left(\frac{-k\pi}{2}\right) = 2$$

So this will give a pattern of $0, 2, 0, -2, \ldots$, multiplied by $\frac{17}{k\pi}$

Fortunately $b_k$ is easier:

$$\begin{aligned} b_k &= \frac{2}{2\pi}\int_{-\pi/2}^{3\pi/2} f(t)\sin(kt)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 17\sin(kt)\,\mathrm{d}t + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} 0\cdot\sin(kt)\,\mathrm{d}t \\ &= \frac{17}{\pi}\left[\frac{-1}{k}\cos(kt)\right]_{-\pi/2}^{\pi/2} \\ &= \frac{-17}{k\pi}\left\{\cos\left(\frac{k\pi}{2}\right) - \cos\left(-\frac{k\pi}{2}\right)\right\} \end{aligned}$$

But earlier we saw that cosine is an **even** function, and so:

$$\cos\left(-\frac{k\pi}{2}\right) = \cos\left(\frac{k\pi}{2}\right)$$

Thus,

$$
\begin{aligned}
b_k &= \frac{-17}{k\pi}\left\{\cos\left(\frac{k\pi}{2}\right) - \cos\left(-\frac{k\pi}{2}\right)\right\} \\
&= \frac{-17}{k\pi}\left\{\cos\left(\frac{k\pi}{2}\right) - \cos\left(\frac{k\pi}{2}\right)\right\} \\
&= \frac{-17}{k\pi} \times 0 \\
&= 0 \qquad\qquad \text{for any integer } k
\end{aligned}
$$

We have obtained formulae for all the Fourier coefficients for this square wave. The general formula of the Fourier series:

$$
f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + \sum_{k=1}^{\infty} b_k \sin(k\omega t)
$$

Will become in this specific case:

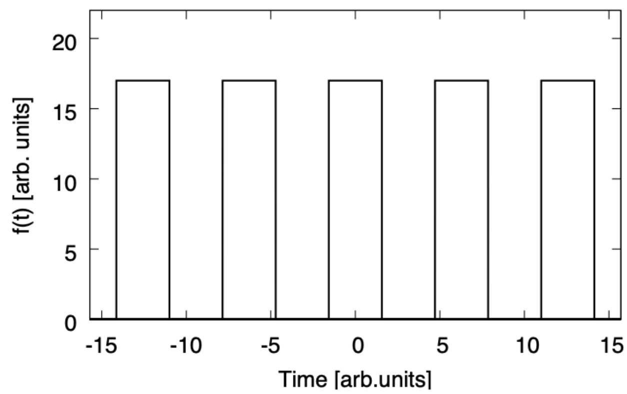
$$
\begin{aligned}
f(t) &= \frac{17}{2} + \sum_{k=1}^{\infty} \frac{34}{k\pi} \sin\left(\frac{k\pi}{2}\right) \cos(kt) \\
&= \frac{17}{2} + \frac{34}{\pi}\cos(t) - \frac{34}{3\pi}\cos(3t) + \frac{34}{5\pi}\cos(5t) - \dots
\end{aligned}
$$

### 2.4.2 Summary

We have carried out a Fourier analysis (or "Fourier decomposition") of the square wave.

We saw that it consists **only** of $a_k$ terms.

This is because the square wave we drew was an **even** function (it has reflective symmetry about the $y$-axis).




### 2.4.3 Exercise

A periodic waveform is given by:

$$f(t) = \begin{cases} -2 & \text{for} \quad -\pi < t < -\frac{\pi}{2} \\ 0 & \text{for} \quad \frac{-\pi}{2} < t < \frac{\pi}{2} \\ 2 & \text{if} \quad \frac{\pi}{2} < t < \pi \end{cases}$$

and this function repeats every $2\pi$, which is denoted by

$$f(t) = f(t + 2\pi)$$

1. Sketch this function over at least three periods.
2. Determine the Fourier Series of $f(t)$.

# 3 Lecture 3: Fourier transform of discrete data

## 3.1 Objectives

- Learn how to apply the **discrete Fourier transform** to sampled data.

- Interpret the resulting **frequency spectrum** to recover the continuous signal being sampled.

## 3.2 The Fourier Transform

The forward Fourier transform $F(\omega)$ is a special operation that turns a time-signal $f(t)$ into a frequency spectrum:

**Fourier transforms:**

The **(forward) Fourier transform**:

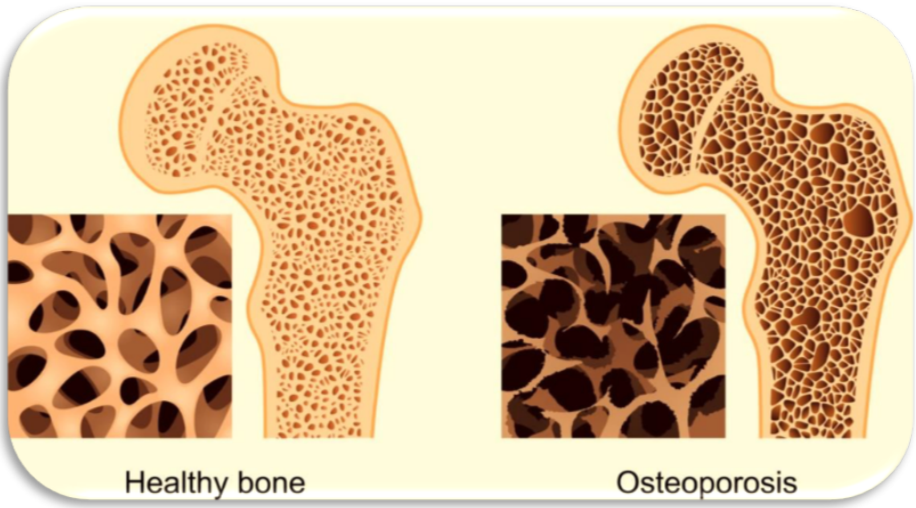
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-j\omega t}\, \mathrm{d}t$$

and the **(inverse) Fourier transform**:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{j\omega t}\, \mathrm{d}\omega$$

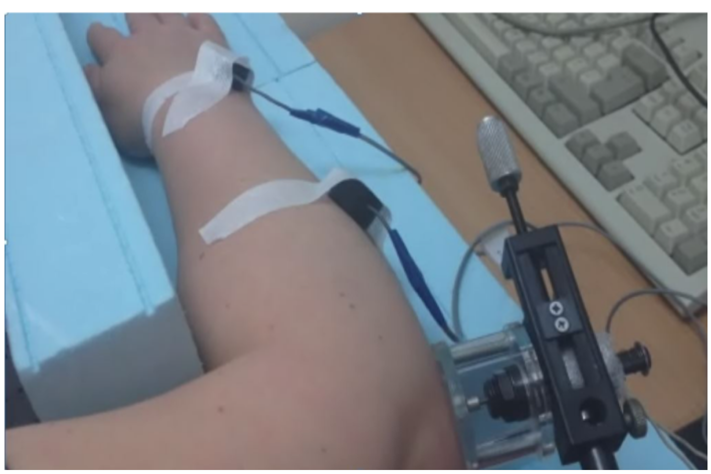
### 3.3 Application: bone density

This example involves analysing the vibrations of a patient's bones in order to detect osteoporosis.

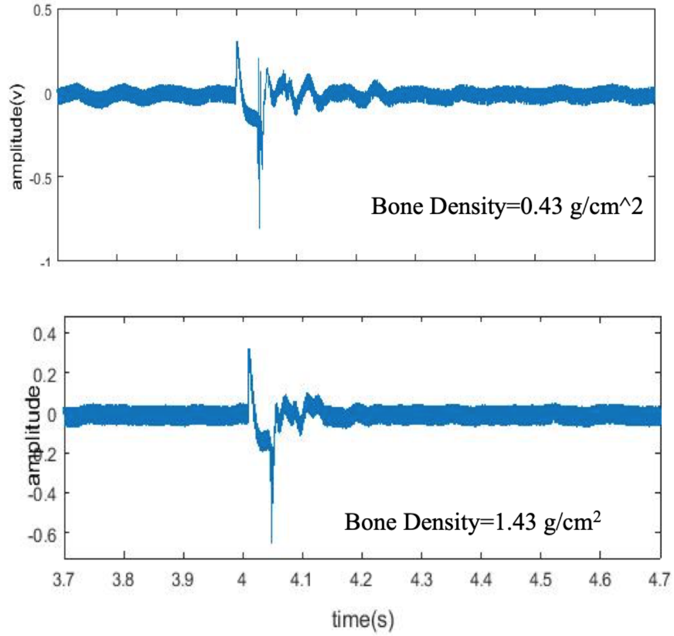



When we detect sounds, we are measuring the amplitude of the displacement of air over time.

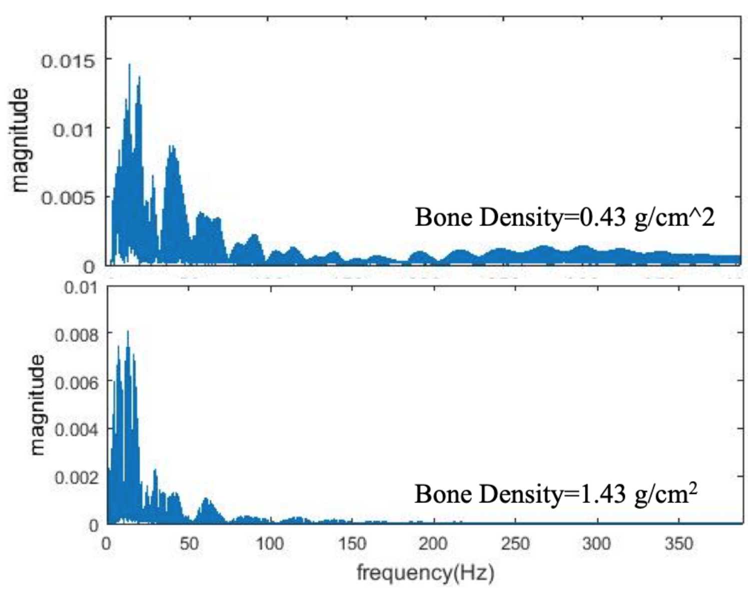
The density of bones affects the sound they produce when vibrating. In a project at Sheffield Hallam, researchers subject the bones to vibration, and record the response:



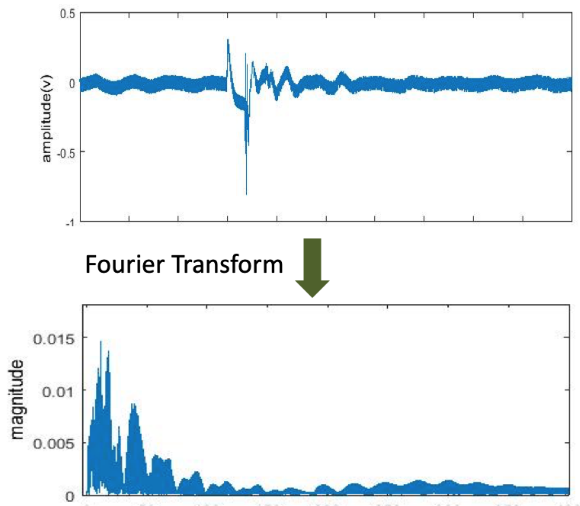
The amplitudes of the vibrations **over time** between healthy and affected samples are hard to distinguish:




But the Fourier transform tells us what **frequencies** are present in the sound of the vibrating bones, and with what amplitude.



## 3.4 Fourier Transform of Discrete Data




The data in the top figure is not actually continuous, but consists of a set of points (**discrete data**). Collecting *any* real data can only ever give discrete samples at every minute, second, cm, etc.

**How do we find the Fourier transform of discrete data?**

If we have a set of $N$ data points $\{f_n\}$ observed at time $\{t_n\}$, the discrete Fourier transform takes us to the frequency domain:

$$(t_n, f_n) \xrightarrow{Fourier \;\; transform} (\omega_k, F_k)$$

### 3.4.1 Notation

Note that some notation has a different meaning here to other parts of the module:

- $t_n$ are the observation times of our data.
- $f_n$ are the corresponding strengths of the signal observed at time $t_n$.
- $\omega_k$ here refers to frequencies in hertz (*not* angular frequency).

- $F_k$ is the magnitude of corresponding frequency $\omega_k$ in the Fourier transform.
- $T$ will refer to the total time interval over which the signal was observed.
- $N$ is the number of samples taken in time $T$.

### 3.4.2 Method: Fourier Transform of Discrete Data

There are $N$ integer values of $k$, ranging between $-\frac{N}{2}$ and $\frac{N-1}{2}$

For each value of $k$, there is a specific frequency $\omega_k$, and associated with it a complex number $F_k$.

This sequence of numbers $F_k$, which are independent of the observation time $t_n$, are the Fourier transform of the sequence $\{f_n\}$.

They are determined by:

$$F_k = \sum_{n=0}^{N-1} f_n \, \mathrm{e}^{-j2\pi nk/N}$$

How do we obtain the $y$-axis ($F_k$)?

This can be done by an in-built MATLAB function.
"fft" stands for the "Fast Fourier Transform" algorithm.

For example, to take the Fourier Transform of a sequence $1, 2, 1, 0$:

```
x = [ 1 2 1 0 ];
F = fft(x);
```

And we get the result $F = \{\ 4,\ -2j,\ 0,\ 2j\ \}$.

To determine what frequencies these values correspond to, we consider at what times the original data points were observed.

Now, how do we obtain the $x$-axis ($\omega_k$)?

The frequencies $\omega_k$ start at zero, and increase by the frequency spacing:

$$\Delta f = \frac{s_r}{N} = \frac{1}{T}$$

where $s_r$ is the sampling rate (number of samples per second).

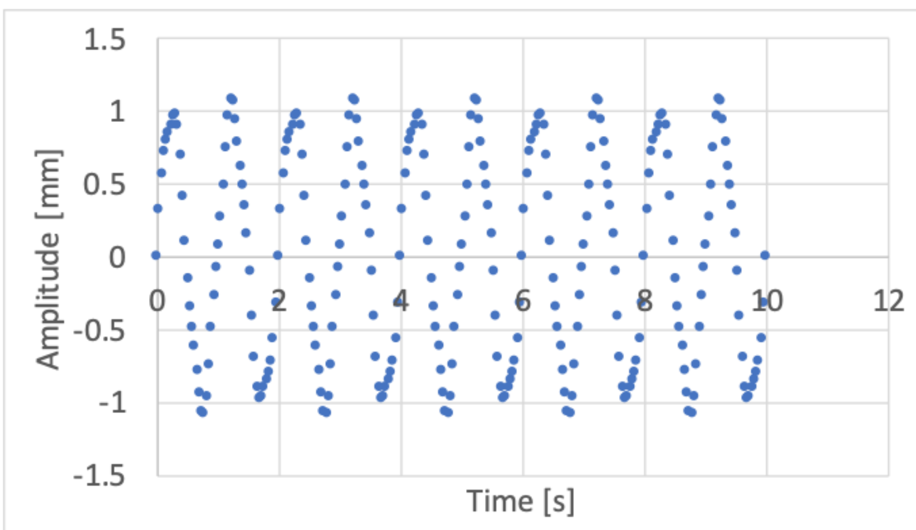
The maximum frequency on the $x$-axis is the Nyquist frequency:

$$\frac{N}{2T}$$

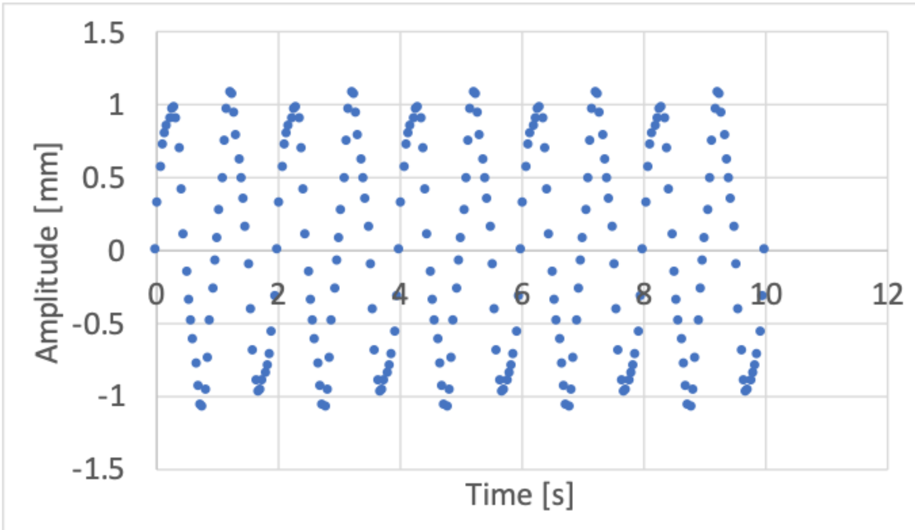
we will discuss this more later.

### 3.4.3 Example

This signal is sampled 256 times over 10 seconds:




As there are $N = 256$ samples over $T = 10$ seconds, the sampling rate is $s_r = 256/10 = 25.6s^{-1}$




From looking at the graph, there appears to be an oscillation with period of about 1 $s$ and hence a 1 $Hz$ frequency.

But is that everything?

Take the Fourier transform of the dependent data (the values of $f$), and then we want the **magnitudes** of these complex numbers:

```
F = fft(f)        This takes the Fourier transform

m = abs(F)        ``abs'' for absolute value
```

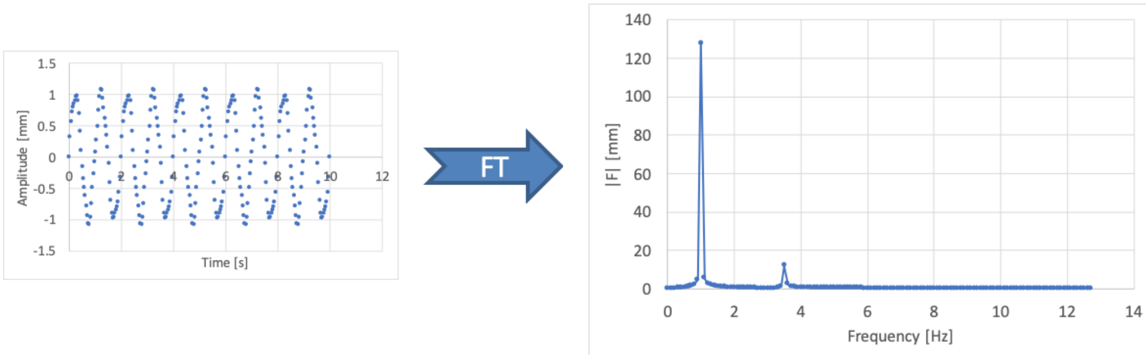
Plot the frequency spectrum of $(N/2) - 1 = 127$ values:

- On the $x$-axis, we want 127 frequencies that start at 0 and increase by the frequency spacing:

$$s = \frac{1}{T} = \frac{1}{10} = 0.1 \; Hz$$

- On the $y$-axis, we want the first 127 values of magnitude $m$.

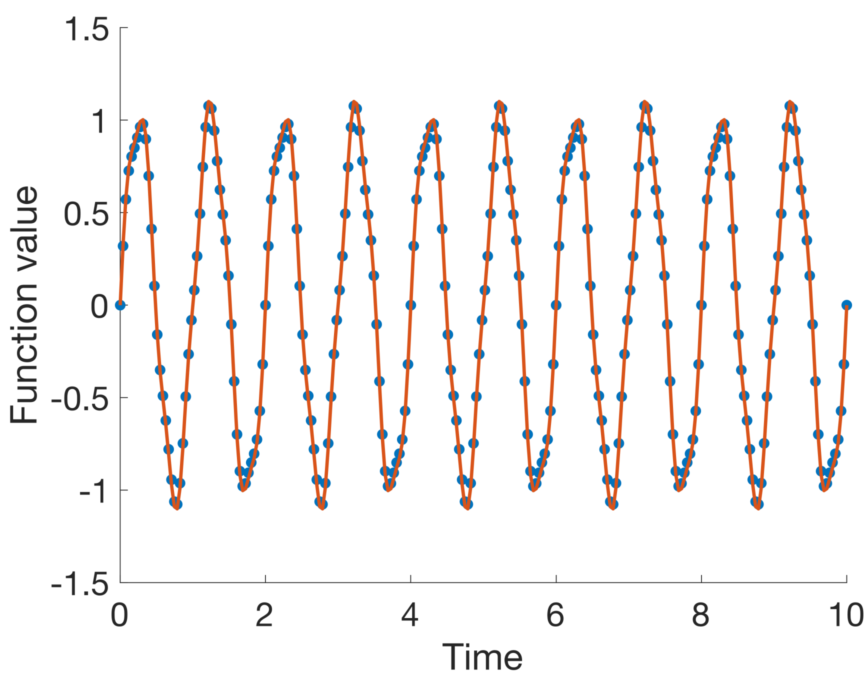
Plotting this results in the frequency spectrum:




We can now see **two** peaks, where there are important frequencies present in the signal:

- One at 1 $Hz$, as expected, with magnitude 127.5.
- Another at 3.5 $Hz$, with magnitude 12.06.

By scaling the two sine waves appropriately, we can then reconstruct the actual waveform that this data was sampled from:




$$f(t) = \frac{A}{127.50}\Big(127.50\sin(2\pi \times 1t) + 12.06\sin(2\pi \times 3.5t)\Big)$$

where $A = 1.0278$ is the estimated amplitude of the "main" oscillation that we can measure from the scatter graph.

To see this yourself, open the worksheet:

```
Lecture12_DiscreteFourierTransformExample.mlx
```

## 3.5 The Nyquist Frequency

The **maximum frequency we can resolve** (detect) is related to the total amount of time $T$ that we have sampled the signal over.

This is known as the **Nyquist frequency**:

$$\frac{N}{2T}$$

The point is that high frequency signals cannot be detected if we do not sample with sufficient frequency. For example, if you only sampled a regular sine wave $\sin(t)$ every $2\pi$, you would appear to be observing a constant signal!

To explain this concept in another context, imagine you wanted to know the patterns governing how many people use this classroom at any given time.

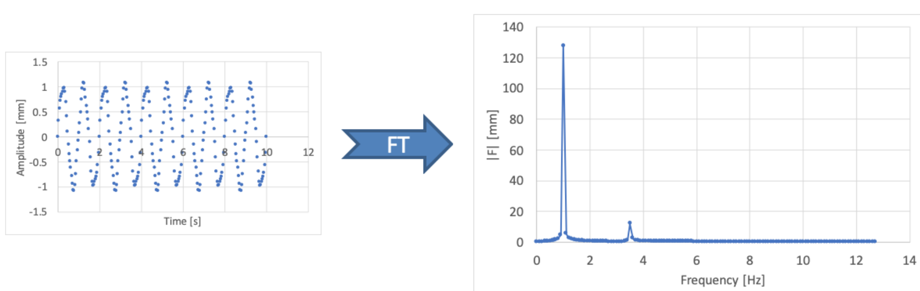
One big pattern will be that there a lot of people in the room during the daytime, but very few at night.

How often must you look into the room to detect this pattern?

If you only sample the usage once per day at 10am, this pattern will be impossible to detect, it is “between” the data. The Nyquist frequency quantifies this idea for Fourier analysis of sampled data.

**Example: Nyquist Frequency**

Returning to our example:




In this case the Nyquist frequency is:

$$\frac{N}{2T} = \frac{256}{2 \times 10} = 12.8\ Hz$$

so this is the upper limit of the $x$-axis on the frequency spectrum.

## 3.6 Summary

Fourier transforms produce the **frequency spectrum** of a signal, which shows us what frequencies are present in a potentially complicated-looking waveform.

Real data usually consists of **discrete samples** of the continuous function. To deal with this, and approximate the "true" continuous Fourier transform, we can use the **discrete Fourier transform**.

We will be practicing this in both EXCEL and MATLAB in the final tutorial.

## 3.7 Real Applications of Fourier Analysis

If earthquake vibrations can be separated into vibrations of different speeds and amplitudes, buildings can be designed to avoid interacting with the strongest ones.

If sound waves can be separated into bass and treble frequencies, we can boost the parts we care about, and hide the ones we don't. The crackle of random noise can be removed.

If computer data can be represented with oscillating patterns, perhaps the least-important ones can be ignored. This can drastically shrink file sizes (and why JPEG and MP3 files are much smaller than raw .bmp or .wav files).