MMaD: Matrices Lecture 6 Handout

Calculating eigenvalues and eigenvectors

To obtain the eigenvalue-eigenvector pairs of a square matrix A:

1. First find the eigenvalues by solving the **characteristic polynomial** for λ:

   $$\det\big(A - \lambda I\big) = 0$$

   where I is the identity matrix with the same order as A.

2. Then for each eigenvalue $\lambda = \lambda_1, \lambda_2, \ldots,$ we obtain a corresponding eigenvector $\underline{\mathbf{x}} = \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \ldots$

   We can do this by substituting in the eigenvalue and solving:

   $$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \qquad \text{for the column vector} \qquad \underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$$

Example: 3×3 Matrix

Consider the following 3×3 matrix A.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

We will calculate the three eigenvalues and associated eigenvectors for this matrix.

Linear stability analysis

Given a system of **linear** ODEs of the form:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = ax_1 + bx_2 + cx_3$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = dx_1 + ex_2 + fx_3$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = gx_1 + hx_2 + ix_3$$

where $a, b, c, d, e, f, g, h, i$ are all constants, we can write it in matrix form as:

$$\dot{X} = AX$$

where:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \dot{X} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}$$

and

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is the **Jacobian matrix**.

Stability criterion:

The equilibrium of such a linear ODE system is:

- *Stable* if <u>all</u> of the eigenvalues of the Jacobian matrix have negative real part.
- *Unstable* otherwise.

Example

Consider a process governed by the differential equations:

$$\dot{x} = x - y$$

$$\dot{y} = -x + 2y - z$$

$$\dot{z} = -y + z$$