Matrices

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Lecture 1

Today we shall cover:

- Introduction to matrices and vectors.
- Matrix addition and subtraction.
- Scalar multiplication.
- Matrix multiplication.
- The zero matrix and the identity matrix.

Introduction

A matrix is a rectangular array for storing data. We often denote them in mathematics by a capital letter, or underlining.

Matrices are the basis for how computers store information that consists of multiple numbers (similar to a table with multiple dimensions that can only store numerical values).

Particular applications include storing information about the position or movement of an object in space, which computer graphics engines can then use to render an image of the object.

Other applications include cryptography and information security, using very large matrices to encrypt information.

Order

The **order** of a matrix is a description of its dimensions - the number of rows, then the number of columns.

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix}$$ This is a 2×1 matrix. It is also a *vector*.

$$\begin{pmatrix} 1 & -2 & 8 \\ 3 & 1 & 4 \end{pmatrix}$$ This is a 2×3 matrix.

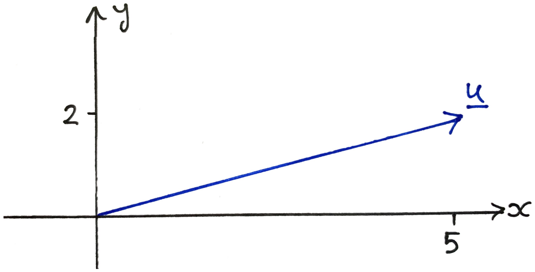
$$\begin{pmatrix} 2 & 0 & -1 & 6 \end{pmatrix}$$ This is a 1×4 matrix.

Vectors

A one-dimensional matrix is also a **vector** (either a row or column vector). These are often used to represent co-ordinates, for example:

$$\underline{\mathbf{u}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

can be drawn as an arrow from the origin $(0, 0)$ to the point $(5, 2)$:



Matrix Addition and Subtraction

Only matrices with the same order can be added or subtracted. If this is satisfied, simply add/subtract the corresponding elements:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \end{pmatrix}$ INVALID as one is 2×2 but the other is 2×1

$$\begin{pmatrix} 4 & -1 & -2 \\ -2 & 3 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 3 \\ 1 & -4 & -6 \end{pmatrix} = \begin{pmatrix} 4-2 & -1-0 & -2-3 \\ -2-1 & 3-(-4) & 5-(-6) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -5 \\ -3 & 7 & 11 \end{pmatrix}$$

Exercise

$$\text{Given} \quad A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$$

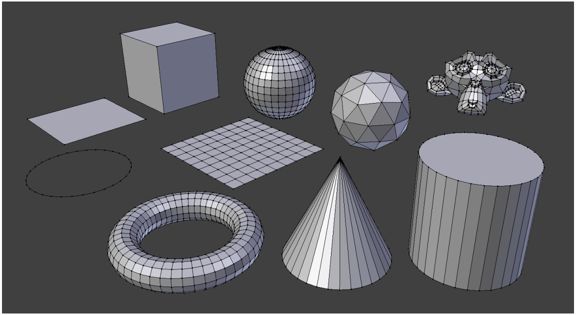
Find (if they exist):

$$A + B, \qquad A + C, \qquad C - A$$

Motivation: Matrix Transformations

One application of this is in 3d graphics engines, such as computer aided design (CAD), or how a video game engine understands the position and movement of objects.

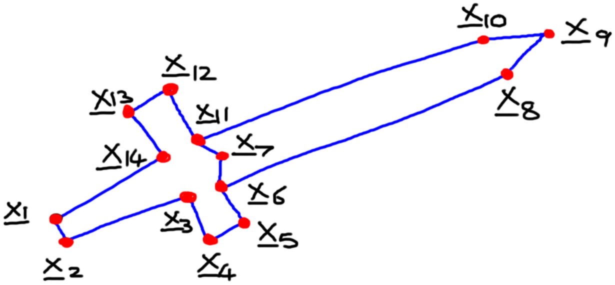
For example, a game understands the position of a sword through vectors $\underline{\mathbf{x}}$ containing the initial co-ordinates of its vertices (corners).



Motivation: Matrix Transformations

Let's say that you have a set of n vectors $\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n$ containing the co-ordinates of the vertices of the two-dimensional image of the sword:

$$\underline{\mathbf{x}}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$



Motivation: Matrix Transformations

Then we can change how the image of the sword appears by applying various matrix operations to its set of vertices:

- To **translate** the image of the sword (to shift it up, down, left or right) we *add* a vector to each vertex.

  For example, adding the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ would move the image 1 unit to the right and 2 units down.

- To **stretch** the image in all directions (i.e. to resize it without distortion), we can multiply the vertices by a scaling factor α. This is called *scalar multiplication*.

Scalar Multiplication

To multiply a matrix by a scalar (a real or complex *number*, rather than a vector or matrix) simply by multiplying ("scaling") each element of the matrix by that scalar.

Scalar Multiplication

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

For example:

$$-3 \begin{pmatrix} 2 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} -3 \times 2 \\ -3 \times 8 \\ -3 \times -5 \end{pmatrix} = \begin{pmatrix} -6 \\ -24 \\ 15 \end{pmatrix}$$

Motivation: Matrix Transformations

The action of swinging the sword to a new position may change the orientation of the image, rotating, distorting and stretching it in more complicated ways. This action can be encoded in a matrix A, and applying the matrix multiplication to all the position vectors associated with the sword is how the game figures out where it moves to.

This transformation can be expressed in matrix form as:

$$\underline{\mathbf{y}} = A\underline{\mathbf{x}}$$

Matrix multiplication determines the new position $\underline{\mathbf{y}}$.

Matrix Multiplication

Matrix multiplication is a **non-commutative** operation. This means that $A \times B$ is *not* equivalent to $B \times A$ and does not necessarily yield the same result.

The order of matrix multiplication *can not be changed*.

In fact, one order might not even exist whilst the other does!

Matrix Multiplication

The number of columns in the first matrix must match the number of rows in the second.

If this is satisfied, the order of the result is given by the remaining dimensions - the same number of rows as the first matrix and columns as the second matrix.

Then imagine setting the rows of the first matrix upon the columns of the second:

$$\underset{2 \times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \quad \underset{2 \times 1}{\begin{pmatrix} 5 \\ 6 \end{pmatrix}} \quad = \quad \begin{pmatrix} 1 \times 5 + 2 \times 6 \\ 3 \times 5 + 4 \times 6 \end{pmatrix} \quad = \quad \underset{2 \times 1}{\begin{pmatrix} 17 \\ 39 \end{pmatrix}}$$

Matrix Multiplication: Example 1

Let,

$$B = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$$

Calculate BC and CB if they exist.

As B is a 2×1 and C is a 2×2 matrix, BC does not exist as the columns of B do not match the number of rows of C.

However, CB does exist, and the result will be another 2×1 matrix:

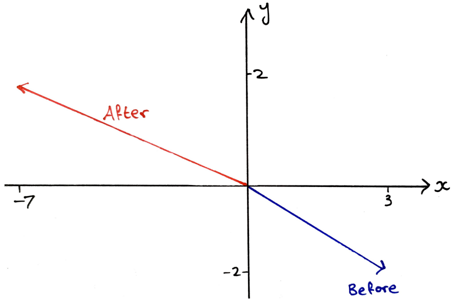
$$CB = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \times 3 + 2 \times -2 \\ 4 \times 3 + 5 \times -2 \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$$

Effect of Matrix Multiplication on Vectors

Let's plot the vector before and after the "action" of matrix C on a 2-dimensional graph:

$$\begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$$

Matrix multiplication has stretched *and* rotated the input vector.



Matrix Multiplication: Example 2

Let,

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

Calculate AD if it exists.

The result will be another 2×2 matrix:

$$\begin{aligned} AD &= \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 \times 3 + 0 \times 0 & 2 \times 1 + 0 \times (-2) \\ -1 \times 3 + 1 \times 0 & -1 \times 1 + 1 \times (-2) \end{pmatrix} \\ &= \begin{pmatrix} 6 + 0 & 2 + 0 \\ -3 + 0 & -1 - 2 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ -3 & -3 \end{pmatrix} \end{aligned}$$

Exercise

Given

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \quad E = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad F = \begin{pmatrix} 1 & -3 \\ 0 & 4 \\ -2 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

Find (if they exist):

$$-4A, \quad 3E, \quad AE, \quad EA, \quad AF, \quad FA, \quad DA$$

Transpose

The **transpose** of a matrix is obtained by swapping the rows and columns (you can think of this as reflecting across the diagonal axis). It is denoted by a superscript T.

For example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

Elements on the diagonal are the only ones that are *not* changed.

Symmetric matrices

If a **square** matrix A is such that:

$$A^T = A$$

then it is called **symmetric** as it has reflective symmetry about the leading diagonal.

On the other hand, if it is such that:

$$A^T = -A$$

then it is called **antisymmetric**. In this case all of the diagonal elements must be zero!

Symmetric matrices

Which of the following matrices are symmetric, antisymmetric, both or neither?

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 3 \\ -3 & -3 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & -5 & -4 \\ 3 & -4 & 19 \end{pmatrix} \quad \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix}$$

Special matrices

The **zero matrix** is a square matrix where every entry is zero.

$$\underline{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \underline{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It acts like the number 0 in matrix addition and matrix multiplication, so:

$$A\underline{O} = \underline{O} = \underline{O}A \quad \text{for any matrix } A \text{ of suitable order.}$$

and

$$A + \underline{O} = A = \underline{O} + A \quad \text{for any matrix } A \text{ of suitable order.}$$

Special matrices

For each positive integer n, the $n \times n$ **identity matrix** consists of one's on the diagonal entries and zeroes elsewhere.

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

This is the only matrix which satisfies, for a matrix A of appropriate dimensions,

$$AI = IA = A$$

So it acts like a matrix version of the number "1" when it comes to multiplication.

Special matrices

Consider the 2×2 and 3×3 identity matrices:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix operations in MATLAB

Addition, subtraction and multiplication are carried out for matrices just like for any other variable: using $+, -, *$ etc.

However, we need to be able to declare a matrix in MATLAB. To do this, start from the top row, separating columns with a space. Move on to the next row with a semi-colon.

For example:

```
A = [ 1 2 3; 4 5 6 ]
```

stores

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Summary

After today, you should be able to ...

- Add, subtract, and multiply matrices by hand.
- Declare a matrix in MATLAB and check these calculations.
- Identify and explain the special properties of a zero matrix and an identity matrix.

This week ...

In this week's tutorial, we shall practice:

- Performing these elementary matrix operations by hand.
- Checking the results with MATLAB.

Next week, we will learn how to calculate the determinants and inverses of square matrices.

For more information on how matrices are used for visualisation in the video game *Dark Souls*, watch:

`https://youtu.be/EV16R0aHVfo?t=6472`

starting at 1:48 (most relevant is 1:49-1:50).