Matrices

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Lecture 2

Today we shall cover:

- Determinants.
- Inverse matrices.

Determinants

Square matrices (with dimensions $n \times n$) have a property called the **determinant**. This is a number (i.e. a scalar) associated with the matrix that is somewhat analogous to magnitude.

The determinant of matrix A can be denoted by $det(A)$ or $|A|$.

what does the determinant *mean*?

Recall that last lecture we said matrix multiplication could transform an object in a 3d graphics engine.

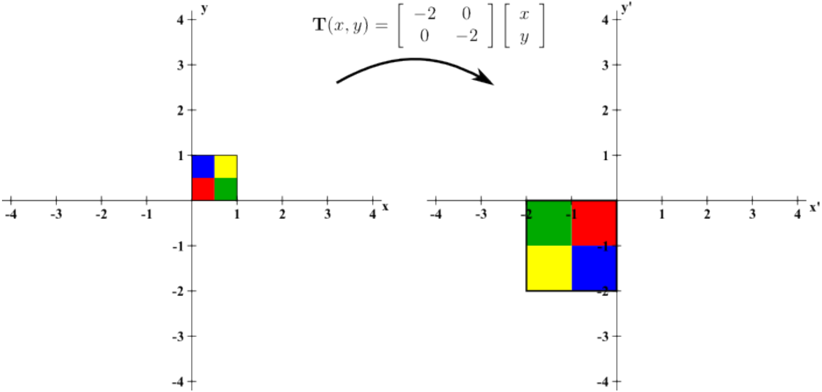
If the matrix A encodes this action of rotating and stretching an object, then the determinant of A represents the **scaling factor**:

- If the absolute value is greater than 1, the matrix encodes an area-expanding transformation and the image is larger than the original object.
- If $|\det(A)| = 1$ the matrix is area-preserving.
- If the absolute value is less than 1 (that is $-1 < \det(A) < 1$) then the matrix encodes an area-contracting transformation that shrinks the image and pulls the vertices closer together.

A negative determinant further indicates that the orientation of the object is flipped (so it undergoes a reflection as well as rotation).

Application of the determinant

In this example, when every point in the square is multiplied by the matrix with determinant 4, the transformed object is *four times as large* the original area:



Determinant of a 2×2 matrix

For a 2×2 matrix A, the determinant is very simple to calculate by multiplying the diagonal entries:

Determinant of a 2×2 matrix:

$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: determinant of a 2×2 matrix

Given the square matrix

$$B = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$$

The determinant is given by:

$$\begin{aligned} det(B) &= 3 \times 2 - (-1) \times 4 \\ &= 6 + 4 \\ &= 10 \end{aligned}$$

Determinant of a 3×3 matrix

The procedure is a bit more involved for 3×3 matrices.

Determinant of a 3 × 3 matrix

Work across the **top row** and multiply each entry by the determinant of the corresponding 2×2 co-matrix of the rows and columns that the current entry is *not* in.

Then **change the sign of the middle entry**.

Determinant of a 3 × 3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

Example: Determinant of a 3×3 matrix

Find the determinant of

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} det(A) &= \begin{vmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} \\ &= 3\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} - (0)\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 3\big(0 \times 1 - (-2) \times 1\big) - 0 + 2\big(2 \times 1 - 0 \times 0\big) \\ &= 3(0+2) + 2(2-0) = 10 \end{aligned}$$

Exercise

Find the determinant of:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

Properties of the Determinant

Given two square matrices A and B of the same size...

Product rule

$$\det(AB) = \det(A)\det(B)$$

Transpose rule

$$\det\left(A^T\right) = \det(A)$$

If two rows or columns of A are identical:

$$\det(A) = 0$$

Application of the inverse matrix

Now, let's say we have transformed the vertices of an object (e.g. a sword) by the action of matrix A (e.g. swinging the sword). What if we wanted to *reverse* that action and return the object to its original position?

Mathematically speaking, what if we had already done:

$$\underline{\mathbf{y}} = A\underline{\mathbf{x}}$$

and we need to find $\underline{\mathbf{x}}$?

This can be achieved by multiplying by the *inverse matrix* of A:

$$\underline{\mathbf{x}} = A^{-1}\underline{\mathbf{y}}$$

This is the matrix equivalent of division.

Inverse matrix

For a **square** matrix A, there may exist an inverse matrix A^{-1}

Inverse Matrix

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

So an inverse matrix is analagous to the reciprocal of a number - it's what you multiply by to get back to 1 (or the identity):

$$5 \times \frac{1}{5} = 1$$

$$A \times A^{-1} = I$$

Calculating the inverse of a 2×2 matrix

For a general 2×2 square matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

Inverse of a 2×2 matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{or} \quad A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If the determinant of a square matrix is equal to zero, then that matrix has no inverse!

Example: Inverse of a 2×2 matrix

To find (if it exists) the inverse of 2×2 square matrix A:

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

First obtain the determinant:

$$det(A) = (1)(2) - (-1)(0) = 2$$

Then as the determinant is non-zero, the inverse exists and is:

$$A^{-1} = \frac{1}{det(A)} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}$$

Exercise: Inverse of a 2×2 matrix

For the following square matrices, find the determinant and the inverse matrix if it exists:

$$B = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Solution: Inverse of a 2×2 matrix

$$B = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

$$B^{-1} = \frac{1}{(1)(2) - (0)(-3)} \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3/2 & 1/2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$det(C) = (1)(-1) - (1)(-1) = 0$$

Hence C has zero determinant - Inverse does not exist.

MATLAB commands

Calculating the determinant, and especially the inverse, of larger matrices such as 3×3's is much more complicated to perform by hand. We shall instead use MATLAB:

```
A = [1 2; 3 -4];
```
(first declare the matrix A)

```
C = inv(A);
```
(C is the inverse matrix of A)

```
d = det(A);
```
(d is the determinant of A)

MATLAB commands

Remember that the inverse **does not exist for a non-square matrix**. If you attempt this, you will receive the error:

```
Error using inv. Matrix must be square.
```

and the script will fail.

Also remember that the inverse does not exist if **the determinant is zero**, and you should always check this first. If you ask Matlab for the inverse of such a matrix it *will not fail*, although you will receive the message:

```
Warning: Matrix is singular to working precision.
```

It will be up to you to realise that this means you can't proceed.

Summary

After today, you should be able to . . .

- Find the determinant of a 2×2 and a 3×3 matrix by hand.
- Find the inverse of a 2×2 matrix by hand.
- Identify when the inverse of a square matrix does not exist.
- Obtain these properties using MATLAB, potentially for matrices with greater dimensions (3×3, 4×4, etc.)