Matrices

Dr Gavin M Abernethy

Lecture 3

Today we shall cover:

- Using matrices to solve systems of simultaneous linear equations.

Motivation

Many engineering problems can be modelled as a system of simultaneous equations.

For example, let's say that there are two materials A and B, whose densities are unknown. You have two samples of different composites of these: one is 15% A and 85% B and has a density of $1kgm^{-3}$, while the other is 40% A and 60% B but twice as dense. This could be written as:

$$0.15A + 0.85B = 1$$
$$0.4A + 0.6B = 2$$

We wish to determine the densities of the constituents A and B.

Introduction

This is an example of a pair of simultaneous linear equations.
Another example:

$$3x + 2y = 16$$
$$-x + 4y = 7$$

You may have learned to solve them (i.e. find the unique values of x and y for which both equations are true) using two methods:

- Elimination
- Substitution

However, they can also be solved using a matrix method.

Method (1)

Given a pair of simultaneous equations:

$$ax + by = p$$
$$cx + dy = q$$

1. Write the pair of equations as a matrix equation:

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

$$AX = B$$

Method (2)

1. So the square matrix of coefficients is $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the vector X contains x and y which we want to find.
2. Calculate the inverse matrix A^{-1}
3. Pre-multiply both sides by the inverse matrix to obtain X:

$$AX = B \quad \Longrightarrow \quad A^{-1}AX = A^{-1}B \quad \Longrightarrow \quad X = A^{-1}B$$

4. From the entries in vector X, read off the values of x and y.
5. (Optional) Substitute the values of x and y back into the original equations to verify solutions.

Example 1

Solve for x and y:

$$5x + 2y = 10$$
$$4x - 3y = 14$$

Re-writing this as a matrix equation,

$$\begin{pmatrix} 5 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

so we have $AX = B$, where

$$A = \begin{pmatrix} 5 & 2 \\ 4 & -3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

Example 1

Then,

$$A^{-1} = \frac{1}{(5)(-3) - (2)(4)} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix} = \frac{-1}{23} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix}$$

and so

$$X = A^{-1}B = \frac{-1}{23} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 10 \\ 14 \end{pmatrix} = \begin{pmatrix} 58/23 \\ -30/23 \end{pmatrix}$$

Thus we find $x = 58/23$ and $y = -30/23$.

Example 2

Solve for x and y:

$$\begin{aligned} 3x &= 7 + 5y \\ 4y + 2x &= 20 \end{aligned}$$

Example 2

First, re-write both of these in a consistent format:

$$3x - 5y = 7$$
$$2x + 4y = 20$$

Re-writing this as a matrix equation,

$$\begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 20 \end{pmatrix}$$

so we have $AX = B$, where

$$A = \begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 7 \\ 20 \end{pmatrix}$$

Example 2

Then,

$$A^{-1} = \frac{1}{(3)(4) - (-5)(2)} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix}$$

and so

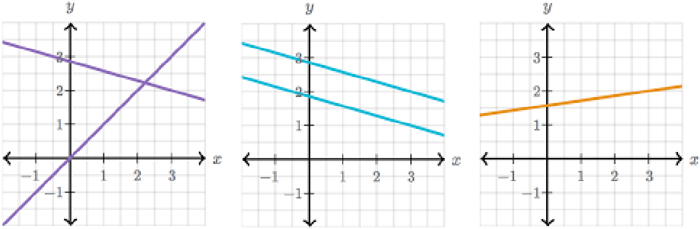
$$X = A^{-1}B = \frac{1}{22} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ 20 \end{pmatrix} = \begin{pmatrix} 64/11 \\ 23/11 \end{pmatrix}$$

Thus we find $x = 64/11$ and $y = 23/11$.

Special cases

A linear equation $ax + by = d$ can be re-written in the form $y = mx + c$. In other words, we have been trying to find the co-ordinates of the point where two straight lines intersect.

What if the lines are parallel or the same?



In these cases (zero solutions or infinitely many solutions), the matrix of coefficients will be **uninvertible** (its determinant $= 0$).

Special cases

If the matrix of coefficients has determinant $= 0$, examine the two equations and determine if they are the same equation (infinitely-many solutions), or if they are contradictory (zero solutions).

$$x - 3y = 10$$
$$2x - 6y = 20$$

$$-2x + y = 3$$
$$4x - 2y = 17$$

The first pair are the **same**, and the second pair are **contradictory**.

MATLAB commands

To solve the following simultaneous equations:

$$5x + 2y = 10$$
$$4x - 3y = 14$$

```
A = [5 2; 4 -3];

B = [10; 14];

X = inv(A)*B
```

MATLAB commands

Note that this will provide decimal answers. To obtain the precise fractions that we found when solving by hand, we simply ask Matlab to convert the answer to a symbolic variable:

```
sym(X)
```

Application

If we already knew how to use substitution or elimination, why is the matrix method useful?

It can be easily computed for much larger systems.

You have probably heard this general rule before:

If you have n unknown variables, and you want to find a unique solution for each of them, then you will need at least n distinct equations describing their relationships.

Application

Imagine you had thirty variables $x_1, x_2, \ldots, x_{30}$ and you knew thirty linear equations that approximately decribed their relationships.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,30}x_{30} = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,30}x_{30} = b_2$$
$$\ldots$$
$$a_{30,1}x_1 + a_{30,2}x_2 + \cdots + a_{30,30}x_{30} = b_{30}$$

You could just declare the 30×30 matrix A of coefficients and the 30×1 vector B, then `X = inv(A)*B` will immediately solve it.

Solving this system by a non-matrix method would be extremely tedious, and is not so easy to program.

Respite

This session is perhaps slightly shorter, so let's take a few moments to recap what we've done so far. These first three weeks constitute a standard introduction to matrices, which might have been familiar to some students depending on your previous experiences.

We've studied matrix operations including **multiplication**, **determinants** and **inverses** - some of which we will need to use in the second half of the topic, which will be new to everyone: eigenvalues and eigenvectors.

- Do you have any questions on the matrix topics so far?
- Are there any processes you are unsure of?

Now is a great time to ask!

Exercises

Use the matrix method to solve the following systems of simultaneous equations:

(a)
$$7x + 2y = 4$$
$$3x - 5y = 6$$

(b)
$$5x = 10 + 2y$$
$$3x + 4y = 6$$

(c)
$$2 + \frac{8y}{x} = \frac{12}{x}$$
$$-2y = 3x + 12$$