Matrices

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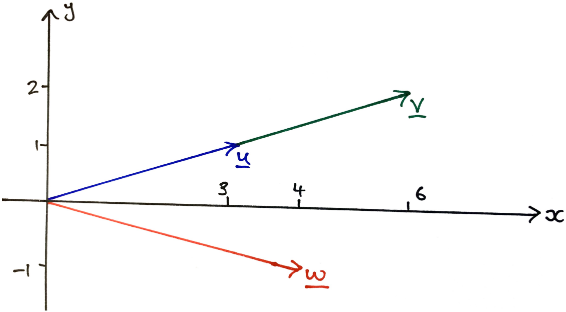
Today we shall cover:

- Magnitude and direction of vectors.
- Unit vectors.
- An introduction to the idea of eigenvalues and eigenvectors.
- Some of their applications and properties.

Vectors

Recall that a **vector** is a one-dimensional matrix.

It can also be thought of as having both **magnitude** and **direction**.



$$\underline{\mathbf{u}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \underline{\mathbf{v}} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 2\underline{\mathbf{u}}, \quad \underline{\mathbf{w}} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Magnitude of a vector

Vectors have an associated **magnitude**.

In a graphical interpretation, this is the **length** of the arrow. It is denoted by vertical lines and calculated using Pythagoras' theorem.

e.g. The magnitude of:

$$\underline{\mathbf{u}} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

is

$$|\underline{\mathbf{u}}| = \sqrt{(2)^2 + (-1)^2 + (3)^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

Unit Vectors

It is sometimes conventional to use **unit** vectors when discussing eigenvectors. This is the vector in the direction of the set of eigenvectors, but specifically with a magnitude equal to one.

We can convert a vector to a unit vector by "normalising", which means dividing it by it's own magnitude.

Unit vector

Given any vector, $\underline{\mathbf{v}}$ we can find the unit vector in the same direction by:

$$\hat{\underline{\mathbf{v}}} = \frac{\underline{\mathbf{v}}}{|\underline{\mathbf{v}}|}$$

Example: Unit Vectors

Consider the vector:

$$\underline{\mathbf{x}} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

It has magnitude:

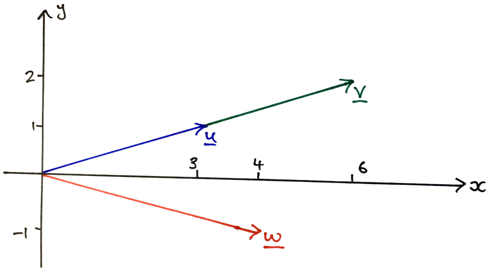
$$|\underline{\mathbf{x}}| = \sqrt{(1)^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$$

So a unit vector in the same direction as $\underline{\mathbf{x}}$ is:

$$\hat{\underline{\mathbf{x}}} = \frac{\underline{\mathbf{x}}}{|\underline{\mathbf{x}}|} = \frac{1}{\sqrt{(1)^2 + (-3)^2}} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Direction of a vector

In addition to **magnitude**, vectors have an associated **direction**.



$$\underline{\mathbf{u}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \underline{\mathbf{v}} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 2\underline{\mathbf{u}}, \quad \underline{\mathbf{w}} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Here, $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$ have the *same* direction, as they are scalar multiples of each other: their x and y co-ordinates have the same ratio.

Motivation: Computer-Aided Design

In computer-aided design (e.g. SolidWorks), a graphical model of an object can be represented by a wire-frame diagram, with the co-ordinates of each mesh point stored as a **vector**.

When you manipulate the object (e.g. rotating it), the new co-ordinates are calculated by applying a matrix transformation to this set of vectors:

$$\underline{\mathbf{y}} = A\underline{\mathbf{x}}$$

where $\underline{\mathbf{x}}$ is the initial co-ordinate of a point, $\underline{\mathbf{y}}$ is the co-ordinate it gets mapped to, and A is a matrix that encodes the transformation.

Thus, we use matrix multiplication to determine the output $\underline{\mathbf{y}}$.

Motivation

The following transformation matrix is applied to a 2-d model:

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

Using matrix multiplication, what is the effect of the transformation on points with the following co-ordinates?

1. $$\underline{\mathbf{x}}_1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix},$$
2. $$\underline{\mathbf{x}}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Motivation

1.

$$\underline{\mathbf{y}}_1 = A\underline{\mathbf{x}}_1 = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 15-8 \\ 6-8 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

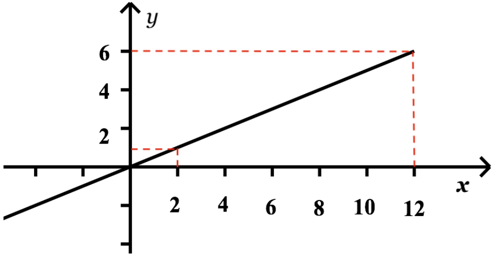
2.

$$\underline{\mathbf{y}}_2 = A\underline{\mathbf{x}}_2 = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 10+2 \\ 4+2 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Motivation

In (2), the output vector has the **same direction** as the input. In fact, *any* vector in this direction will *not* be rotated by A. The magnitude may change, but the direction is preserved.



Any point which lies on this line will be mapped to a point on the same line after the transformation.

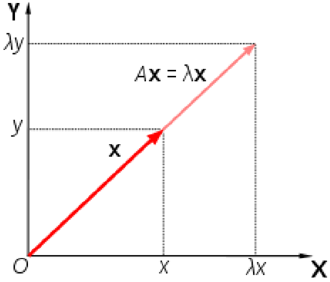
Motivation - Eigenvalues and Eigenvectors

When a square matrix A acts on a vector $\underline{\mathbf{x}}$, we obtain a new vector $A\underline{\mathbf{x}}$ that may be stretched and rotated in some way.

It is often useful to find solutions to:

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \quad \text{where} \quad \lambda \quad \text{is a scalar.}$$

These are vectors (eigenvectors) $\underline{\mathbf{x}}$ whose direction is **preserved** when we multiply by matrix A. They are magnified by a scaling/magnification factor (eigenvalue) λ.



Applications

So the key idea, is what vectors are there, such that when the they are multiplied by the matrix, we obtain a vector with *the same direction as the old one*. That is an eigenvector, and how much bigger/smaller it's magnitude is, is the corresponding eigenvalue.

Eigenvalues are useful in many areas of mathematical modelling of real (biological, mechanical, electronic) systems:

- Determining the *stability* of points of equilibria.
- Identifying resonant frequencies of coupled oscillating bodies.
- Determining the final states of "Markov processes" governed by repeated applications of probabilities.

Let's consider some of their general properties...

Properties of the eigenvalues and eigenvectors

- For an $n \times n$ square matrix A, there are n eigenvalues (although some may be the same).
- Every eigenvalue has a family of infinitely-many eigenvectors associated with it. They all have the **same direction**, but can be of **any magnitude**.
- This means that if $\underline{\mathbf{e}}_i$ is an eigenvector of A with eigenvalue λ_i, then so is *any scalar multiple* of $\underline{\mathbf{e}}_i$.
- We will use this to help us find the "easiest" example of an eigenvector in our examples.

Properties of the eigenvalues

The eigenvalues of a matrix have relationships with certain other properties.

For a square matrix, the **trace** is the sum of the diagonal values.

For a square matrix . . .

The **sum** of the eigenvalues is equal to the trace.

The **product** of the eigenvalues is equal to the determinant.

Calculating eigenvalues and eigenvectors

The solutions to many engineering problems can be found from the eigenvalues and eigenvectors of an associated matrix.

Next week, we will learn how to calculate the eigenvalues of a 2×2 matrix.

For now, let's consider one application of why eigenvalues are useful in an engineering model: the linear stability analysis of equilibria.

Application: Linear stability analysis (1/4)

Two components $x(t), y(t)$ of a vibrating mechanical system have motion governed by the following pair of ODEs:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + 2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x - 4y$$

Recall that an ODE tells us how the rates of change of the variables behave.

- If $x > 0$ and $y > 0$, will $x(t)$ increase or decrease over time?
- If $x < 0$ and $y > 0$, will $y(t)$ increase or decrease over time?

Application: Linear stability analysis (2/4)

This system is called "linear" because, like the linear simultaneous equations we saw last week, there is only an x and y term (no xy or x^2 etc.) on the right.

At the origin, when $x = 0$ and $y = 0$, this system becomes:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

So there is no rate of change in either direction. We say the system has an **equilibrium** at $(0, 0)$. If $x = 0$ and $y = 0$ exactly, then the system is stuck there forever.

What would happen if x or y were very small but non-zero: would the system get sucked back to $(0, 0)$, or would it take off and move away from the equilibrium? This is the **stability** question.

Application: Linear stability analysis (3/4)

To solve this, we represent the ODE system as a matrix problem:

$$\underline{\dot{x}} = A\underline{x}$$

where

$$\underline{\dot{x}} = \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Now, the square matrix A is called the *Jacobian matrix*, and we shall calculate next week that this particular example has eigenvalues:

$$\lambda_1 = -5, \qquad \lambda_2 = 2$$

The eigenvalues of the Jacobian matrix are important because they determine the stability of the system...

Application: Linear stability analysis (4/4)

Stability criterion:

The equilibrium of a linear autonomous ODE system at $(0, 0)$ is:

- *Stable* if all of the eigenvalues of the Jacobian matrix have negative real part.
- *Unstable* otherwise.

In this example, as one of the eigenvalues is $\lambda_2 = 2$, which is real and positive, the equilibrium at $(0, 0)$ is **unstable**.

This means that any small disturbances will be amplified rather than dissipated - probably not what you would want!

Typically when designing a system (e.g. a bridge's vibrations), engineers want to ensure the desirable equilibria are stable.

Exercises

Given $A = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$, calculate $A\underline{\mathbf{x}}$ for each vector $\underline{\mathbf{x}}$ given below.

Which ones have their direction preserved under the transformation?

For each of those, also determine the "magnification factor".

1. $\underline{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
2. $\underline{\mathbf{x}} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
3. $\underline{\mathbf{x}} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$
4. $\underline{\mathbf{x}} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$
5. $\underline{\mathbf{x}} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
6. $\underline{\mathbf{x}} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

Can you make a general statement about what vectors are preserved by this matrix?